The area of reduced spherical polygons

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Abstract: We confirm two conjectures of Lassak on the area of reduced spherical polygons. The area of every reduced spherical non-regular \( n \)-gon is less than that of the regular spherical \( n \)-gon of the same thickness. Moreover, the area of every reduced spherical polygon is less than that of the regular spherical odd-gons of the same thickness and whose number of vertices tends to infinity.

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1 Introduction

We focus on the reduced convex bodies introduced by Heil in [2]. Reduced convex bodies are helpful for solving various extremal problems concerning the minimal width of convex bodies. Some basic properties of the reduced convex bodies in two-dimensional Euclidean space \( E^2 \) are introduced by Lassak in [3]. Lassak [7] demonstrates that in \( E^2 \) the area of every reduced non-regular \( n \)-gon is less than that of the regular \( n \)-gon of the same thickness.

The notions about reduced convex bodies are extended to the \( d \)-dimensional unit sphere \( S^d \) in [4]. Lassak [4, 5] discusses the properties of reduced convex bodies on \( S^d \); he [6] further characterizes reduced convex polygons on \( S^2 \) and proposes the following conjectures:

(1) The area of every reduced spherical polygon is less than that of the regular spherical odd-gons of the same thickness and whose number of vertices tends to infinity.

(2) The area of every reduced spherical non-regular \( n \)-gon is less than that of the regular spherical \( n \)-gon of the same thickness.

In Section 2, we present the necessary notions of reduced spherical convex bodies and review some results in the literature. Several useful lemmas are established in Section 3. Then Section 4 aims to confirm the above two conjectures.

2 Preliminaries

Let \( S^2 \) be the unit sphere in \( E^3 \) centered at the origin. In this paper, all the notions are discussed in \( S^2 \). A great circle is the intersection of \( S^2 \) with any two-dimensional subspace of \( E^3 \). A pair of antipodes are the intersection of \( S^2 \) with any one-dimensional subspace of \( E^3 \).

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Clearly, if different points $a, b \in S^2$ are not antipodes, then there is exactly one great circle containing them; denote by $arc\ ab$, shortly $ab$, the shorter part of the great circle containing them. The spherical distance $|ab|$, or shortly distance of $a$ and $b$ is the length of $ab$.

A set $C \subset S^2$ is called convex, if it does not contain any pair of antipodes of $S^2$ and for arbitrary points $a, b \in C$, it is true that $arc\ ab \subseteq C$. The convex body is a closed convex set with non-empty interior.

By a spherical disk of radius $r \in (0, \pi/2]$ and center $k \in S^2$ we mean the set $B = \{p : |pk| \leq r, p \in S^2\}$; and the boundary $bd(B)$ is called a spherical circle. The spherical disk of radius $\pi/2$ is called a hemisphere. If hemispheres $G$ and $H$ are different and their centers are not antipodes, then $L = G \cap H$ is called a lune of $S^2$. The parts of $bd(G)$ and $bd(H)$ contained in $G \cap H$ are denoted by $G/H$ and $H/G$, respectively. We define the thickness $\Delta(L)$ of the lune $L = G \cap H$ as the distance of the centers of $G/H$ and $H/G$. We recall some notions in [4]. We say that a hemisphere $H$ supports a convex body $C \subseteq S^2$ at point $p$ if $C \subseteq H$ and $p \in bd(H) \cap C$. For any convex body $C \subseteq S^2$ and any hemisphere $K$ supporting $C$, we define the width of $C$ determined by $K$ as the minimum thickness of a lune $K \cap K^*$ over all hemispheres $K^* \neq K$ supporting $C$ and we denote it by $width_K(C)$; the thickness of $C$ is defined by

$$\Delta(C) = \min \{width_K(C) : K \text{ is a supporting hemisphere of } C\}.$$  

The thickness of $C$ is nothing else but the minimum thickness of a lune containing $C$. A convex body $C \subset S^2$ is said to be reduced if $\Delta(R) < \Delta(C)$ for each convex body $R$ being a proper subset of $C$.

We recall some definitions given in [6]. Let $p$ be a point in a hemisphere different from its center and let $l$ be the great circle bounding this hemisphere. The projection of $p$ on $l$ is the point $t$ such that $|pt| = \min \{|pc| : c \in l\}$. If $C$ is a subset of a convex set of $S^2$, then the intersection of all convex sets containing $C$ is called a convex hull of $C$. The convex hull of $k \geq 3$ points on $S^2$ such that each of them does not belong to the convex hull of the remaining points is called a spherically convex $k$-gon. If $V$ is a spherically convex $k$-gon, we denote by $v_1, \ldots, v_k$ the vertices of $V$ in the counterclockwise order. A spherically convex polygon with sides of equal length and interior angles of equal measure is called a regular spherical polygon.

**Lemma 2.1** [9, Theorem 41.2] Let $V$ be a convex $n$-gon in the unit sphere with angles $\gamma_1, \gamma_2, \ldots, \gamma_n$, then area$(V) = \gamma_1 + \ldots + \gamma_n - (n - 2)\pi$.

For a convex odd-gon $V = v_1v_2 \cdots v_n$, by the opposite side to the vertex $v_i$ we mean the side $v_{i+(n-1)/2}v_{i+(n+1)/2}$, the indices are taken modulo $n$.

**Lemma 2.2** [6, Theorem 3.2] Every reduced spherical polygon is an odd-gon of thickness at most $\frac{\pi}{2}$. A spherically convex odd-gon $V$ with $\Delta(V) < \frac{\pi}{2}$ is reduced if and only if the projection of every its vertices on the great circle containing the opposite side belongs to the relative interior of this side and the distance of this vertex from this side is $\Delta(V)$.

**Lemma 2.3** [6, Corollary 3.3] Every regular spherical odd-gon of thickness at most $\frac{\pi}{2}$ is reduced.
3 Working lemmas

We recall a few formulas of spherical geometry in [8] which are the basic method for the research. Consider the right spherical triangle with hypotenuse $c$ and legs $a, b$, we use $A, B$ and $C$ to represent the corresponding angles of edges $a, b$ and $c$, respectively. Then

\[
\cos B = \cos b \sin A, \tag{1}
\]
\[
\cos A = \tan b \cot c, \tag{2}
\]
\[
\sin b = \sin c \sin B. \tag{3}
\]

In a reduced spherical polygon $V = v_1 v_2 \cdots v_n$, according to Lemma 2.2, we give some related notations. Denote by $t_i$ the projection of $v_i$ in the opposite side $v_{i+(n-1)/2} t_{i+(n+1)/2}$. Denote by $o_i$ the intersection of $v_i t_i$ and $v_{i+(n+1)/2} t_{i+(n+1)/2}$; put $\alpha_i = \angle v_i v_{i+(n+1)/2} o_i$, $\beta_i = \angle t_i v_i v_{i+(n+1)/2}$, and $\phi_i = \angle v_i o_i t_{i+(n+1)/2} = \angle t_i o_i v_{i+(n+1)/2}$, where $i \in \{1, 2, \ldots, n\}$. For example, Fig. 1 presents some notations in a reduced spherical pentagon.

![Fig. 1: Some notations](image)

**Lemma 3.1** [6, Corollary 3.9] If $V = v_1 v_2 \cdots v_n$ is a reduced spherical polygon with $\Delta(V) < \pi/2$, then $\beta_i \leq \alpha_i$.

Actually, we can gain the following lemma by Corollary 3.6 in [6]. Here we prove it in a different way.

**Lemma 3.2** For every reduced spherical polygon $V = v_1 v_2 \cdots v_n$ with $\Delta(V) < \pi/2$, the spherical triangles $v_i o_i t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} o_i t_i$ are congruent, where $i \in \{1, 2, \ldots, n\}$.

**Proof** Lemma 2.2 shows that $|v_i t_i| = |v_{i+(n+1)/2} t_{i+(n+1)/2}| = \Delta(V)$, where $i \in \{1, \ldots, n\}$. From this and $|v_i v_{i+(n+1)/2}| = |v_{i+(n+1)/2} t_i|$, we find that the right spherical triangles $v_i t_i v_{i+(n+1)/2}$ and $v_{i+(n+1)/2} t_i v_i$ are congruent. Then we acquire

\[
\alpha_i + \beta_i = \angle v_i v_{i+(n+1)/2} v_{i+(n+1)/2} = \angle v_i v_{i+(n+1)/2} t_i \tag{4}
\]
and

\[ \beta_i = \angle t_i v_i v_{i+(n+1)/2} = \angle v_i v_{i+(n+1)/2} t_{i+(n+1)/2}. \]  

(5)

From (4) and (5), we obtain that \( \alpha_i = \angle t_i+(n+1)/2 v_i o_i = \angle t_i v_i+(n+1)/2 o_i. \) Applying this and \( \angle v_i o_i t_{i+(n+1)/2} = \angle t_i o_i v_{i+(n+1)/2} = \beta_i \), it turns out that the right spherical triangles \( v_i o_i t_{i+(n+1)/2} \) and \( v_{i+(n+1)/2} o_i t_i \) are congruent. \( \square \)

Lemma 3.3 All the vertices of a regular spherical polygon are contained in a spherical circle.

**Proof** Let \( V = v_1 v_2 \cdots v_n \) be a regular spherical polygon. Denote by \( \gamma \) the length of the side of \( V \) and by \( \theta \) the interior angle of \( V \). Let \( D \) be a spherical circle passing through \( v_1, v_2 \) and \( v_3 \), and denote by \( o \) the center of the spherical disk whose boundary is \( D \). Connect \( o \) with \( v_i \), where \( i \in \{1,2,\ldots,n\} \).

In the spherical triangles \( ov_1 v_2 \) and \( ov_2 v_3 \), we have \( |ov_1| = |ov_2| = |ov_3| \). By the definition of regular spherical polygon, we have \( |v_1 v_2| = |v_2 v_3| = \gamma \), then \( ov_1 v_2 \) and \( ov_2 v_3 \) are congruent. Since \( |ov_1| = |ov_2| = |ov_3| \), we have \( \angle ov_1 v_3 = \angle ov_3 v_1 v_i = \phi / 2 \), where \( i = 1,2 \).

In the spherical triangles \( ov_1 v_2 \) and \( ov_3 v_4 \). From \( |ov_1| = |ov_3|, \angle ov_1 v_2 = \angle ov_3 v_4 = \phi / 2 \) and \( |v_1 v_2| = |v_3 v_4| \), we obtain that \( ov_1 v_2 \) and \( ov_3 v_4 \) are congruent. Hence we have \( |ov_1| = |ov_2| = |ov_3| = |ov_4| \).

Similarly, the spherical triangles \( ov_1 v_2 \) and \( ov_j v_{j+1} \) are congruent, where \( j \in \{4,\ldots,n\} \) and the indices are taken modulo \( n \). Hence \( |ov_1| = |ov_2| = |ov_3| = |ov_{j+1}| \) with \( j \in \{4,\ldots,n\} \). Consequently, we have \( |ov_1| = |ov_2| = \cdots = |ov_n| \), and thus \( v_1, v_2, \ldots, v_n \) are all contained in \( D \). \( \square \)

Fact 3.4 For every reduced spherical polygon \( V = v_1 v_2 \cdots v_n \) with \( \Delta(V) < \pi / 2 \), we have \( 0 < \varphi_i < \pi / 2 \), where \( i \in \{1,\ldots,n\} \).

**Proof** Consider the right spherical triangle \( v_i v_{i+(n+1)/2} \) for every \( i \in \{1,\ldots,n\} \). From (1), we obtain \( \cos(\alpha_i + \beta_i) = \cos(\beta_i) \sin(\alpha_i) \). As \( |v_i v_{i+(n+1)/2}| = \Delta(V) < \pi / 2 \), we get \( \alpha_i + \beta_i < \pi / 2 \).

Applying this and Lemma 3.1, it follows that \( 2\beta_i \leq \alpha_i + \beta_i < \pi / 2 \).

It is obvious that the area of the spherical triangle \( o_i v_i v_{i+(n+1)/2} \) is nonnegative. Hence by Lemma 2.3 and (5), we get that \( \beta_i + \beta_1 + (\pi - \varphi_i) - \pi \geq 0 \). Therefore, \( \varphi_i \leq 2\beta_i < \pi / 2 \). The proof is complete. \( \square \)

By a rotation of a set \( C \subseteq S^2 \) around a point \( p \in S^2 \), we mean the rotation of \( C \) around the straight line through \( p \) and the center of \( E^3 \). For any two points \( a, b \in bd(C) \), \( ab \) is called a chord of \( C \subseteq S^2 \).

Lemma 3.5 For any reduced spherical polygon \( V = v_1 v_2 \cdots v_n \) with \( \Delta(V) < \pi / 2 \), we have \( \sum_{i=1}^{n} \varphi_i \geq \pi \).

**Proof** Lemma 2.2 shows that \( V \) is an odd-gon. We use a similar technic as that of Lemma in [7] to show the statement. Let \( B_i = v_i o_i v_{i+(n+1)/2} \cup v_{i+(n+1)/2} o_i t_i \), then \( B_i \subseteq V \), where \( i \in \{1,2,\ldots,n\} \). Thus we have \( B_1 \cup \cdots \cup B_n \subseteq V \).

We intend to show \( V \subseteq B_1 \cup \cdots \cup B_n \). We present every \( B_i \) as the union of chords of \( V \) which pass through \( o_i \). All the chords of successively \( B_1, B_{1+(n+1)/2}, \ldots, B_{(n+1)/2} \) are
in big circles which step by step rotate changing the centers of rotation; those centers successively are $o_1, o_1 + (n+1)/2, \ldots, o_1 + (n+1)/2$. We assume that all the above chords in $B_i$ are oriented with the origins in $v_i t_{i+(n+1)/2}$. For any point $p \in V$, we assume that $p$ is in the left hand side of $v_i t_1$. When we start from $v_i t_1$, after total rotation by $\varphi_1 + \cdots + \varphi_n$, we arrive at $t_1 v_1$, which has the opposite direction. Now $p$ is in the right hand side of the oriented chord $v_i t_1$. Since the described changes of $v_i t_1$ are continuous, there is a position such that the chord contains $p$. Hence $p \in B_1 \cup \cdots \cup B_n$ and then $V \subseteq B_1 \cup \cdots \cup B_n$

Consequently, $V = B_1 \cup \cdots \cup B_n$.

**Claim 1:** If $V$ is a non-regular reduced spherical polygon, then $\sum_{i=1}^n \varphi_i \geq \pi$.

From Lemmas 2.1 and 3.2, the area of $V$ is $S_V = 2(\alpha_1 + \cdots + \alpha_n) - (n-2)\pi$. The area of $B_i$ is $S_{B_i} = 2(\varphi_i + \alpha_i - \frac{\pi}{2})$. From $V = B_1 \cup \cdots \cup B_n$, we have $S_V \leq \sum_{i=1}^n S_{B_i}$, that is

\[
2 \sum_{i=1}^n \alpha_i - (n-2)\pi \leq 2 \sum_{i=1}^n (\varphi_i + \alpha_i - \frac{\pi}{2}),
\]

and then $\sum_{i=1}^n \varphi_i \geq \pi$.

**Claim 2:** If $V$ is a regular spherical polygon, then $\sum_{i=1}^n \varphi_i = \pi$ and $\varphi_i = \frac{\pi}{n}$, where $i \in \{1, 2, \ldots, n\}$.

In this case, by Lemma 3.3, we have $\alpha_1 = \cdots = \alpha_n = 0$ (the notation $o$ is described in Lemma 3.3). Then, $B_i = v_i o t_{i+(n+1)/2} \cup v_i t_{i+(n+1)/2} t_i$, where $i \in \{1, 2, \ldots, n\}$. Clearly, for arbitrary $i, j \in \{1, 2, \ldots, n\}$, $B_i$ and $B_j$ are congruent, and their interiors satisfy $\text{int}(B_i) \cap \text{int}(B_j) = \emptyset$. Thus we have $S_V = \sum_{i=1}^n S_{B_i}$ and then (6) becomes

\[
2 \sum_{i=1}^n \alpha_i - (n-2)\pi = 2 \sum_{i=1}^n (\varphi_i + \alpha_i - \frac{\pi}{2}).
\]

Hence we obtain that $\sum_{i=1}^n \varphi_i = \pi$ and $\varphi_i = \frac{\pi}{n}$, where $i \in \{1, 2, \ldots, n\}$. 

In the following lemmas, we investigate the monotonicity and concavity of two kinds of functions, respectively, which are needed in Section 4.

**Lemma 3.6** Let $f_1(x) = \arccos \frac{x \sqrt{1 + \lambda^2}}{\lambda x}$ and $f_2(x) = \arccos \frac{x (1 + \lambda x)}{\lambda x}$. Then $\frac{f_1(x)}{f_2(x)}$ is a decreasing function of $x$, where $\lambda \in (0, +\infty)$ and $x \in (0, \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda})$.

**Proof** Set $f(x) = \frac{f_1(x)}{f_2(x)}$. Let us show that $f'(x) < 0$. The derivative of $f(x)$ is

\[
f'(x) = \frac{\lambda \cdot h(x)}{(\lambda - x) \sqrt{1 + x^2} \sqrt{-\lambda^2 x^2 - 2\lambda x + \lambda^2 f_2^2(x)}},
\]

where

\[
h(x) = - \frac{1}{\sqrt{1 + \lambda^2}} \frac{x \sqrt{1 + \lambda^2}}{\lambda x} (1 + x^2) f_2(x) + (-x^2 + 2\lambda x + 1) f_1(x).
\]

From $x < \frac{-1 + \sqrt{1 + \lambda^2}}{\lambda} < \lambda$, and if $h(x) < 0$, then $f'(x) < 0$. The first derivative of $h(x)$ is

\[
h'(x) = - \frac{x \sqrt{1 + \lambda^2}}{\sqrt{1 + x^2}} f_2(x) + 2(\lambda - x) f_1(x).
\]

The second derivative of $h(x)$ is

\[
h''(x) = - \frac{\lambda \sqrt{1 + \lambda^2}}{\sqrt{-\lambda^2 x^2 - 2\lambda x + \lambda^2}} \frac{2\lambda - 3x - x^3}{(1 + x^2)(\lambda - x)} - \frac{\sqrt{1 + \lambda^2}}{1 + x^2} f_2(x) - 2f_1(x).
\]

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Since $x \in (0, \frac{1 + \sqrt{1 + \lambda^2}}{\lambda})$, it follows that $\lambda - x > 0$ and $2\lambda - 3x - x^3 > 0$. Also we have $f_1(x) > 0$ and $f_2(x) > 0$. Hence $h^0(x) < 0$ and then $h'(x) > h'(\frac{1 + \sqrt{1 + \lambda^2}}{\lambda}) = 0$. Thus from $h'(x) > 0$, we get $h(x) < h(\frac{1 + \sqrt{1 + \lambda^2}}{\lambda}) = 0$. Therefore, $f'(x) < 0$. \hfill $\Box$

**Lemma 3.7** Let $F(x) = \arcsin \frac{g(x)\sqrt{1 + \lambda^2}}{\lambda^2(x)}$ and $g(x) = \frac{-(1 + \cos x) + \sqrt{(1 + \cos x)^2 + 4\lambda^2 \cos x}}{2\lambda}$, where $\lambda \in (0, +\infty)$ and $x \in (0, \frac{\pi}{2})$. Then $F'(x) < 0$ and $F''(x) < 0$.

**Proof** For convenience, set $r(x) = \sqrt{(1 + \cos x)^2 + 4\lambda^2 \cos x}$. We find the first derivative of $F(x)$ is

$$F'(x) = -\frac{\lambda \sqrt{2 + 2\lambda^2 \sin x}}{r(x)\sqrt{1 - \cos x} \sqrt{1 + 2\lambda^2 + \cos x - r(x)}}.$$

Therefore, we obtain $F'(x) < 0$. The second derivative of $F(x)$ is

$$F''(x) = \frac{\lambda \sqrt{2 + 2\lambda^2 \sin^2 \frac{x}{2} \left(-2(1 + \cos x)^2 - 8\lambda^2 + 2(1 + \cos x)r(x)\right)}}{(1 - \cos x)^2 r^3(x) \sqrt{1 + 2\lambda^2 + \cos x - r(x)}}.$$

We can check that $-2(1 + \cos x)^2 - 8\lambda^2 + 2(1 + \cos x)r(x) < 0$ and $(1 - \cos x)^2 r^3(x) > 0$ in the domain $x \in (0, \frac{\pi}{2})$. Hence $F''(x) < 0$ and then $F(x)$ is a concave function of $x$. \hfill $\Box$

## 4 The area of reduced spherical polygons

This section aims to prove the conjectures mentioned in the introduction. For ease of notations, we use $\omega$ to replace the thickness of a reduced spherical polygon in this part. Here we have $\omega \in (0, \pi/2)$. Let $\lambda = \tan \omega$, then $\lambda \in (0, +\infty)$. Denote by $S$ the area of a reduced spherical polygon.

Let us define several functions which are needed in the following theorems. Set

$$f(x) = \arcsin \frac{x \sqrt{1 + \lambda^2}}{\lambda - x}, f_2(x) = \arccos \frac{x(1 + \lambda x)}{\lambda - x},$$

$$f_1(x) = \frac{\pi}{2} - f(x) = \arccos \frac{x \sqrt{1 + \lambda^2}}{\lambda - x},$$

where $x \in (0, (-1 + \sqrt{1 + \lambda^2})/\lambda)$. Set

$$g(\varphi) = \frac{-(1 + \cos \varphi) + \sqrt{(1 + \cos \varphi)^2 + 4\lambda^2 \cos \varphi}}{2\lambda},$$

where $\varphi \in (0, \pi/2)$. And thus $g(\varphi) \in (0, (-1 + \sqrt{1 + \lambda^2})/\lambda)$.

**Lemma 4.1** For a reduced spherical polygon $V = v_1 \cdots v_n$ with $\omega \in (0, \frac{\pi}{2})$, the area is $S = 2 \sum_{i=1}^{n} f(y_i) - (n - 2)\pi$, where $y_i = g(\varphi_i)$.

**Proof** For every $i \in \{1, 2, \ldots, n\}$, we focus on the right spherical triangle $o_i t_i v_{i+(n+1)/2}$. Put $|o_i t_i| = b_i$ and $|o_i v_{i+(n+1)/2}| = c_i$. By Lemma 3.2 we have $|o_i t_i| + |o_i v_{i+(n+1)/2}| = b_i + c_i = \omega$. Here we have $b_i < \omega$. From (2), we obtain

$$\cos \varphi_i = \frac{\tan b_i}{\tan c_i} = \frac{\tan b_i}{\tan(\omega - b_i)} = \frac{\tan b_i(1 + \tan \omega \tan b_i)}{\tan \omega - \tan b_i}. \tag{7}$$

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By a simple calculation, we get that \( \tan b_i = g(\varphi_i) \), for simplicity, we denote \( g(\varphi_i) \) by \( y_i \). Hence \( b_i = \text{arctan} y_i \). Fact 3.4 shows that \( 0 < \varphi_i < \frac{\pi}{2} \). From this and \( b_i < \omega \), it follows that \( y_i \in (0, \frac{1 + \sqrt{1 + \lambda^2}}{\lambda}) \).

Hence (7) becomes \( \cos \varphi_i = \frac{y_i(1 + \lambda y_i)}{\lambda - y_i} \) and then \( \varphi_i = f_2(y_i) = \text{arccos} \frac{y_i(1 + \lambda y_i)}{\lambda - y_i} \). Moreover, we have \( \tan c_i = \frac{\tan b_i}{\cos \varphi_i} = \frac{\lambda - y_i}{1 + \lambda y_i} \) and thus \( c_i = \text{arctan} \frac{\lambda - y_i}{1 + \lambda y_i} \).

From (3), we obtain

\[
\sin \alpha_i = \frac{\sin b_i}{\sin c_i} = \frac{\sin \text{arctan} y_i}{\sin \text{arctan} \frac{\lambda - y_i}{1 + \lambda y_i}} = \frac{y_i \sqrt{1 + \lambda^2}}{\lambda - y_i},
\]

and thus \( \alpha_i = f(y_i) = \text{arcsin} \frac{y_i \sqrt{1 + \lambda^2}}{\lambda - y_i} \). Then Lemmas 2.1 and 3.2 imply that the area of \( V \) is

\[
S = 2 \sum_{i=1}^{n} \alpha_i - (n - 2)\pi = 2 \sum_{i=1}^{n} f(y_i) - (n - 2)\pi = (8).
\]

\( \square \)

**Theorem 4.2** The regular spherical \( n \)-gon has the maximum area among all regular spherical \( k \)-gons of fixed thickness, with odd numbers \( k, n \) and \( 3 \leq k \leq n \).

**Proof** Let \( V = v_1 v_2 \cdots v_k \) be a regular spherical odd-gon. Lemma 2.3 shows that \( V \) is reduced, then we use the same notations as that in Lemma 4.1. By Claim 2 in Lemma 3.5, we have \( \varphi_1 = \cdots = \varphi_k = \frac{\pi}{k} \). Thus \( y_1 = \cdots = y_k = g(\frac{\pi}{k}) \), where

\[
g(\frac{\pi}{k}) = \frac{-1 + \cos \frac{\pi}{k}}{2} + \frac{1 + \cos \frac{\pi}{k}}{2} \cdot \frac{1 + \lambda^2}{\lambda} = \frac{1 + \cos \frac{\pi}{k}}{2} \cdot \frac{1 + \lambda^2}{\lambda} = \frac{1 + \cos \frac{\pi}{k}}{2} \cdot \frac{1 + \lambda^2}{\lambda}.
\]

and \( g(\frac{\pi}{k}) \in \left(0, \frac{1 + \sqrt{1 + \lambda^2}}{\lambda}\right) \). For simplicity, we denote \( \varphi_i \) and \( y_i \) by \( \varphi \) and \( y \), respectively, for every \( i \in \{1, 2, \ldots, k\} \). From the proof process of Lemma 4.1, we can easily see that \( k = \frac{\pi}{\varphi} = \frac{\pi}{f_2(y)} \), where \( f_2(y) = \varphi = \text{arccos} \frac{y(1 + \lambda y)}{\lambda - y} \).

From \( k = \frac{\pi}{f_2(y)} \) and \( f_1(y) + f(y) = \frac{\pi}{2} \), it follows that (8) becomes \( S = -2\pi f_1(y) + 2\pi \).

By Lemma 3.6, we get that \( \frac{f_1(y)}{f_2(y)} \) is a decreasing function of \( y \). Since \( y = g(\frac{\pi}{k}) \), one can check that \( y \) is an increasing function of \( k \). Consequently, \( \frac{f_1(y)}{f_2(y)} \) is a decreasing function of \( k \).

Because \( S = -2\pi f_1(y) + 2\pi \), the above analysis implies that \( S \) is an increasing function of \( k \). This completes the proof. \( \square \)

**Corollary 4.3** The area of the regular spherical odd-gon with thickness \( \omega \in (0, \frac{\pi}{2}) \) is \( 2(1 - \cos \frac{\omega}{2})\pi \) when the number of vertices tends to infinity.

**Proof** Let \( V = v_1 v_2 \cdots v_n \) be a regular spherical odd-gon. Since \( \lambda = \tan \omega \), we get

\[
\cos \frac{\omega}{2} = \cos \frac{\arctan \lambda}{2} = \sqrt{1 + \cos \arctan \lambda} = \sqrt{\frac{1 + \lambda^2 + 1}{2\lambda^2 + 1}}.
\]
By Theorem 4.2, we get that the area of $V$ is $S = -2\pi f_1(y) + 2\pi$, where $y = g(\frac{\pi}{n}) = \frac{1}{2} \sqrt{(1+\cos \frac{\pi}{n})^2+4\lambda^2 \cos \frac{\pi}{n}}$. When $n$ tends to infinity, $y$ tends to $t = \frac{1+\sqrt{\lambda^2+1}}{\lambda}$. Then from $\lim_{y \to t} f_1(y) = 0$ and $\lim_{y \to t} f_2(y) = 0$, by using L'Hospital rule, we get

$$\lim_{y \to t} \frac{f_1(y)}{f_2(y)} = \lim_{y \to t} \frac{\sqrt{1+\lambda^2 \sqrt{1+y^2}}}{-y^2 + 2\lambda y + 1} = \sqrt{\frac{\lambda^2 + 1}{2\lambda^2 + 1}}.$$ 

Consequently, $\lim_{n \to +\infty} S = 2(1 - \cos \frac{\pi}{2}) \pi$. 

The next theorem shows that the second conjecture mentioned in the introduction is true.

**Theorem 4.4** The area of every reduced spherical non-regular $n$-gon is less than that of the regular spherical $n$-gon of the same thickness.

**Proof** Let $V = v_1 \cdots v_n$ be a reduced spherical odd-gon. By Lemma 4.1, the area of $V$ is $S = 2\sum_{i=1}^{n} F(\varphi_i) - (n-2)\pi$, where $F(\varphi_i) = f(g(\varphi_i)) = \arcsin \frac{g(\varphi_i)\sqrt{1+\lambda^2}}{\lambda - g(\varphi_i)}$ and by Fact 3.4, we have $\varphi_i \in (0, \frac{\pi}{2})$.

By Lemma 3.7, we obtain that $F(x)$ is a concave function of $x$. Thus from Jensen’s inequality [1], we have

$$\frac{F(\varphi_1) + \cdots + F(\varphi_n)}{n} \leq F\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right),$$

the equality holds when $\varphi_1 = \cdots = \varphi_n$. Then the area of $V$ satisfies

$$S = 2n\left(\frac{F(\varphi_1) + \cdots + F(\varphi_n)}{n}\right) - (n-2)\pi \leq 2nF\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right) - (n-2)\pi.$$

**Case 1.** If $V$ is a regular spherical polygon, then by Claim 2 in Lemma 3.5 we have $\varphi_1 + \cdots + \varphi_n = \pi$ and $\varphi_1 = \cdots = \varphi_n = \frac{\pi}{n}$. In this case, $S = 2nF\left(\frac{\pi}{n}\right) - (n-2)\pi$.

**Case 2.** If $V$ is a non-regular spherical polygon, then by Claim 1 in Lemma 3.5 we have $\varphi_1 + \cdots + \varphi_n \geq \pi$.

By Lemma 3.7, we obtain that $F(x)$ is a decreasing function of $x$. Since $\varphi_1 + \cdots + \varphi_n \geq \pi$, it follows that $F\left(\frac{\varphi_1 + \cdots + \varphi_n}{n}\right) \leq F\left(\frac{\pi}{n}\right)$. In this case, we have $S \leq 2nF\left(\frac{\pi}{n}\right) - (n-2)\pi \leq 2nF\left(\frac{\pi}{n}\right) - (n-2)\pi$.

The above two cases show that the area of $V$ always satisfies $S \leq 2nF\left(\frac{\pi}{n}\right) - (n-2)\pi$, which is exactly the area of the regular spherical $n$-gon. This completes the proof. 

By Theorems 4.2, 4.4 and Corollary 4.3 we obtain the following corollary which shows that the first conjecture is true.

**Corollary 4.5** The area of every reduced spherical polygon $V$ is less than $2(1 - \cos \frac{\Delta(V)}{n}) \pi$, which is the limit value for the area of the regular spherical odd-gons whose number of vertices tends to infinity.
References


