Boundedness of Certain Bilinear Operators on Vanishing Generalized Morrey Spaces

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Abstract In this article, we consider the bilinear operator \( T \) satisfying that there exists a positive constant \( C(T) \), depending on \( T \), such that, for any measurable functions \( f \) and \( g \) with compact support, \( t \in \mathbb{R} \) with \( 0 < |t| \leq 1 \), and \( x \in \mathbb{R}^n \) with \( 0 \notin \text{supp}(f(x-t)) \cap \text{supp}(g(x-\cdot)) \),

\[
|T(f, g)(x)| \leq C(T) \int_{\mathbb{R}^n} \frac{|f(x-ty)g(x-y)|}{|y|^n} \, dy.
\]

We investigate the boundedness of \( T \) on vanishing generalized Morrey spaces \( V_0^p,\phi(\mathbb{R}^n) \) and \( V^\infty_\omega L^p,\phi(\mathbb{R}^n) \), and the boundedness of the subbilinear maximal operator \( M \) on the vanishing generalized Morrey space \( V_\omega^* L^p,\phi(\mathbb{R}^n) \), and their applications to some classical (sub)bilinear operators in harmonic analysis. As a byproduct, we also show that \( T \) is bounded on generalized Morrey spaces \( L^p,\phi(\mathbb{R}^n) \). Some typical examples for the main results of this paper are also included.

1 Introduction

The theory of Morrey-type spaces play important roles in the study of harmonic analysis and partial differential equations. In 1938, Morrey [34] introduced the notion of Morrey spaces which are useful to the regularity results for solutions to various partial differential equations, especially for quasilinear elliptic systems. Later, Campanato [8] and Peetre [35] systematically investigated the theory of Morrey spaces. Since then, it attracts a lot of attention from many mathematicians. We just list a few references due to the limitation of our knowledge. For applications of the theory of Morrey spaces to harmonic analysis, please see the articles [36, 31, 38, 39, 2] and monographs [42, 1]. We refer the reader to papers [23, 29, 30, 31, 28] and monographs [40, 41, 26] for the applications of the theory of Morrey spaces to potential analysis and partial differential equations. The theory of Morrey spaces was also applied to the interpolation theory, see, for instance, [32, 16, 17, 33]. The boundedness of some classical multilinear operators on Morrey spaces can be found in [12, 19, 20, 21]. For the applications of the theory of generalized Morrey-type spaces, please see [6, 7, 10, 15] and the references therein. To see the recent development in the theory of vanishing (generalized) Morrey spaces, we refer the reader to [37, 5, 4, 3, 14].

In what follows, for any \( p \in (0, \infty) \), denote by \( L^p_{\text{loc}}(\mathbb{R}^n) \) the set of all locally \( p \)-integrable functions on \( \mathbb{R}^n \). Now, we recall the notion of classical Morrey spaces from [34]. For any \( p \in (0, \infty) \), \( \lambda \in (0, n) \), the classical Morrey spaces \( L^{p, \lambda}(\mathbb{R}^n) \) is defined to be the set of all functions

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\]

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\( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) such that
\[
\|f\|_{L^p(\mathbb{R}^n)} := \sup_{(x,t) \in \mathbb{R}^n \times (0,\infty)} \left[ \mathfrak{M}_{p,t}(f; x, r) \right]^\frac{1}{p} < \infty,
\]
where, for any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \),
\[
\mathfrak{M}_{p,t}(f; x, r) := \frac{1}{r^t} \int_{B(x, r)} |f(y)|^p \, dy
\]
and \( B(x, r) := \{ y \in \mathbb{R}^n : |x - y| < r \} \). The \textit{generalized Morrey space} \( L^{p,\varphi}(\mathbb{R}^n) \) (see [11]) is defined as \( L^{p,\varphi}(\mathbb{R}^n) \) with \( r^t \) replaced by a general measurable function \( \varphi \) from \( \mathbb{R}^n \times (0, \infty) \) to \( (0, \infty) \), and, for any \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), \( \mathfrak{M}_{p,\varphi}(f; x, r) \) replaced by
\[
\mathfrak{M}_{p,\varphi}(f; x, r) := \frac{1}{\varphi(x, r)} \int_{B(x, r)} |f(y)|^p \, dy.
\]
The \textit{norm} of a function \( f \) in \( L^{p,\varphi}(\mathbb{R}^n) \) is defined by setting
\[
\|f\|_{L^{p,\varphi}(\mathbb{R}^n)} := \sup_{(x,t) \in \mathbb{R}^n \times (0,\infty)} \left[ \mathfrak{M}_{p,\varphi}(f; x, r) \right]^\frac{1}{p}.
\]

Fan and Zhao [12] introduced the following bilinear operator \( T \) satisfying that there exists a positive constant \( C_T \), depending on \( T \), such that, for any measurable functions \( f \) and \( g \) with compact support, \( t \in \mathbb{R} \) with \( 0 < |t| \leq 1 \), and \( x \in \mathbb{R}^n \) with \( 0 \notin \text{supp}(f(x - t \cdot)) \cap \text{supp}(g(x + \cdot)) \),
\[
|T(f, g)(x)| \leq C_T \int_{\mathbb{R}^n} \frac{|f(x - ty)g(x - y)|}{|y|^n} \, dy,
\]
which is a generalization of some classical bilinear operators in harmonic analysis, such as the bilinear Hilbert transform, bilinear oscillatory Hilbert transform and their maximal operators, and the first Calderón commutator etc. Fan and Zhao [12] then established the boundedness of \( T \) on Morrey spaces under the assumption that \( T \) is bounded on Lebesgue spaces.

We now recall the following three closed subspaces of \( L^{p,\varphi}(\mathbb{R}^n) \) from [3, 27] for any \( p \in (0, \infty) \) and general measurable function \( \varphi : \mathbb{R}^n \times (0, \infty) \rightarrow (0, \infty) \). The \textit{space} \( V_0L^{p,\varphi}(\mathbb{R}^n) \) is defined to be the set of all functions \( f \in L^{p,\varphi}(\mathbb{R}^n) \) satisfying the \textit{vanishing property at zero} (for short, the \textit{V} \textit{0 property}):
\[
\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\varphi}(f; x, r) = 0.
\]
As in [37], we assume that the function \( \varphi \) in the definition of \( V_0L^{p,\varphi}(\mathbb{R}^n) \) satisfies that
\[
\lim_{r \to 0^+} \frac{r^n}{(\varphi(x, r))^p} = 0
\]
and
\[
\inf_{r \in (1, \infty)} \sup_{x \in \mathbb{R}^n} \varphi(x, r) > 0.
\]
The space \( V_\infty L^{p,\varphi}(\mathbb{R}^n) \) is defined to be the set of all functions \( f \in L^{p,\varphi}(\mathbb{R}^n) \) satisfying the vanishing property at \( \infty \) (for short, the \( V_\infty \) property):

\[
\lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{\varphi}(f; x, r) = 0.
\]

As in [27], we suppose that the function \( \varphi \) in the definition of \( V_\infty L^{p,\varphi}(\mathbb{R}^n) \) satisfies (1.3) and

\[
\lim_{r \to \infty} \frac{1}{r^\alpha} \sup_{x \in \mathbb{R}^n} \varphi(x, r) = 0.
\]

We also recall the space \( V^{(s)} L^{p,\varphi}(\mathbb{R}^n) \) introduced by Liu and Fu [27], which consists of all functions \( f \in L^{p,\varphi}(\mathbb{R}^n) \) satisfying the \( V^{(s)} \) vanishing property (for short, the \( V^{(s)} \) property):

\[
\lim_{N \to \infty} \mathfrak{A}_{N,p}(f) := \lim_{N \to \infty} \sup_{x \in \mathbb{R}^n} \int_{\mathbb{B}(x, 1)} |f(y)|^p \chi_N(y) \, dy = 0,
\]

where, for any \( N \in \mathbb{N} \),

\[
\chi_N := \chi_{\mathbb{R}^n \setminus \mathbb{B}(0,N)}.
\]

Motivated by [3, 12], in this article, we obtain the boundedness of the bilinear operator \( T \) on the generalized Morrey spaces \( L^{p,\varphi}(\mathbb{R}^n) \) under the assumption of the boundedness of \( T \) on Lebesgue spaces. We also show that the \( V_0 \) and \( V_\infty \) properties are preserved under the action of \( T \), and the \( V^{(s)} \) property is preserved under the action of the subbilinear maximal operator \( \mathcal{M} \) defined as (5.3) below. The main novelty of this article is that we establish the boundedness of \( T \) on \( V_0 L^{p,\varphi}(\mathbb{R}^n) \) and \( V_\infty L^{p,\varphi}(\mathbb{R}^n) \), and the boundedness of \( \mathcal{M} \) on \( V^{(s)} L^{p,\varphi}(\mathbb{R}^n) \), which are applied to some classical (sub)bilinear operators in harmonic analysis. Some specific examples for the main results of this paper are also included.

The organization of this article is as follows.

In Section 2, we show the boundedness of \( T \) from \( L^{p,\varphi_1}(\mathbb{R}^n) \times L^{p,\varphi_2}(\mathbb{R}^n) \) to \( L^{p,\varphi}(\mathbb{R}^n) \) (see Theorem 2.3 below), whose proof is based on some ideas from [12]. In Section 3, we establish a preliminary result (see Theorem 3.1 below) which is necessary to the proof of the main results of this paper. We prove that the \( V_0 \) and \( V_\infty \) properties are preserved under the action of \( T \) in Section 4 (see, respectively, Theorems 4.1 and 4.3 below) via estimates of the modules \( \mathfrak{M}_{\varphi}(T(f, g); x, r) \) established in Section 3 and some ideas in the proof of [37, Theorem 5.1]. In Section 5, we prove that the space \( V^{(s)} \) property is invariant under the action of the subbilinear maximal operator \( \mathcal{M} \) (see Theorem 5.1 below). Some specific examples on the functions \( \varphi, \varphi_1 \) and \( \varphi_2 \) are also included (see Remarks 4.2, 4.4 and 5.2 below). In Section 6, as some applications of the above results, we obtain the boundedness on generalized Morrey spaces of the bilinear Hilbert transform, the bilinear singular integral, the first Calderón commutator and the bilinear oscillatory singular integral etc. We also indicate some typical examples of the main results of this paper (see Remark 6.8 below).

Finally, we list some conventions on notation. Throughout the whole paper, for any \( p \in (0, \infty) \), \( L^p_{\text{loc}}(\mathbb{R}^n) \) stands for the space of all locally \( p \)-integrable functions on \( \mathbb{R}^n \). Let \( C \) denote a positive constant which is independent of the main parameters, but it may change from line to line. Moreover, we let \( C_{(\alpha, \beta, \ldots)} \) and \( c_{(\alpha, \beta, \ldots)} \) denote a positive constant depending on the indicated parameters \( \alpha, \beta, \ldots \). For two real functions \( f \) and \( g \), we write \( f \lesssim g \) if \( f \leq C g \); \( f \sim g \) if \( f \leq g \sim f \). For any subset \( U \) of \( \mathbb{R}^n \), we let \( \chi_U \) denote its characteristic function. For any \( p \in [1, \infty) \), let \( p' \) denote the conjugate index of \( p \) defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \). Furthermore, let \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{Z}_+ := \{0\} \cup \mathbb{N} \).
2 Boundedness of $T$ from $L^{q,\varphi_1}(\mathbb{R}^n) \times L^{1,\varphi_2}(\mathbb{R}^n)$ to $L^p,\varphi(\mathbb{R}^n)$

In this section, we show the boundedness of $T$ from $L^{q,\varphi_1}(\mathbb{R}^n) \times L^{1,\varphi_2}(\mathbb{R}^n)$ to $L^p,\varphi(\mathbb{R}^n)$. To this end, we first recall the following condition on $\varphi$.

**Definition 2.1.** ([18, Definition 2.5.15]) We say that a function $\varphi : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ satisfies the weak doubling condition if there exists a positive constant $C(\varphi)$ such that, for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, $\varphi(x, 2t) \leq C(\varphi)\varphi(x, t)$.

**Remark 2.2.** The weak doubling condition is equivalent to the following statement: there exist positive constants $C$ and $\omega(\varphi) := \log_2 C(\varphi)$ such that, for any $x \in \mathbb{R}^n$ and $\lambda, t \in (1, \infty)$,

$$\varphi(x, \lambda t) \leq C \lambda^{\omega(\varphi)} \varphi(x, t).$$

The following theorem is an extension of [12, Theorem 1.1].

**Theorem 2.3.** Let $q, l \in (1, \infty)$ and $p \in [1, \infty)$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{l}$. Assume that $\omega(\varphi_i) \in (0, n)$ for any $i \in \{1, 2\}$, $\varphi$ and $\{\varphi_i\}_{i=1}^2$ are measurable functions from $\mathbb{R}^n \times (0, \infty) \to (0, \infty)$ satisfying $\varphi^{1/p} = \varphi_1^{1/q} \varphi_2^{1/l}$, and $T(f, g)$ be the bilinear operator as in (1.1). Let $\{\varphi_i\}_{i=1}^2$ satisfy (2.1). If there exists a positive constant $A_0$ such that, for any $f \in L^q(\mathbb{R}^n)$ and $g \in L^l(\mathbb{R}^n)$,

$$\|T(f, g)\|_{L^p(\mathbb{R}^n)} \leq A_0\|f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^l(\mathbb{R}^n)},$$

then there exists a positive constant $C$ such that, for any $t \in \mathbb{R}^n$ with $0 < |t| \leq 1$, $f \in L^{q,\varphi_1}(\mathbb{R}^n)$ and $g \in L^{1,\varphi_2}(\mathbb{R}^n)$,

$$\|T(f, g)\|_{L^p(\mathbb{R}^n)} \leq C \left( A_0 + |t|^{\frac{n-n(\varphi_1)}{q}} \right) \|f\|_{L^{q,\varphi_1}(\mathbb{R}^n)}\|g\|_{L^{1,\varphi_2}(\mathbb{R}^n)}.$$

**Proof.** We borrow some ideas from the proof of [12, Theorem 1.1]. Let $q$, $l$, $p$, $f$ and $g$ be as in this theorem. We write

$$f = \sum_{j=0}^{\infty} f_j \chi_{A_j} \quad \text{and} \quad g = \sum_{i=0}^{\infty} g_i \chi_{A_i},$$

where $\chi_E$ is the characteristic function of a set $E$. For any fixed $(x, r) \in \mathbb{R}^n \times (0, \infty)$, let $k_0 \in \mathbb{Z}$ satisfy $2^{k_0} \leq r < 2^{k_0+1}$, $A_0 := B(x, 2^{k_0+4}) =: B_0$ and, for any $i \in \mathbb{N}$,

$$A_i := \{y \in \mathbb{R}^n : 2^{k_0+3+i} < |y-x| \leq 2^{k_0+4+i}\} \quad \text{and} \quad B_i := \{y \in \mathbb{R}^n : |y-x| \leq 2^{k_0+4+i}\}.$$

Therefore,

$$\left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} |T(f, g)(z)|^p \, dz \right\}^{\frac{1}{p}} \leq \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| \sum_{(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n : j \geq i} T(f \chi_{A_j}, g \chi_{A_i})(z) \right|^p \, dz \right\}^{\frac{1}{p}}$$

$$\quad + \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| \sum_{(i,j) \in \mathbb{Z}^n \times \mathbb{Z}^n : j \geq l} T(f \chi_{A_j}, g \chi_{A_i})(z) \right|^p \, dz \right\}^{\frac{1}{p}}.$$
\[=: I + II,\]

where \( \mathbb{Z}_a := \{0\} \cup \mathbb{N}. \)

We first estimate \( I \). Indeed, we write

\[
I \leq \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| \sum_{(i,j) \in \mathbb{Z}^2 : |i| \leq 10} T(f\chi_{A_i}, g\chi_{A_j})(z) \right|^p \, dz \right\}^{\frac{1}{p}}
\]

\[
+ \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| \sum_{(i,j) \in \mathbb{Z}^2 : |j| > 10, |j| \leq 10} T(f\chi_{A_i}, g\chi_{A_j})(z) \right|^p \, dz \right\}^{\frac{1}{p}}
\]

\[=: I_1 + I_2.\]

To estimate \( I_1 \), by the Minkowski inequality, (2.1), (2.2) and \( r \geq 2^{k_0} \), we conclude that

\[
I_1 \leq \sum_{(i,j) \in \mathbb{Z}^2 : |i| \leq 10, |j| \leq 10} \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| T(f\chi_{A_i}, g\chi_{A_j})(z) \right|^p \, dz \right\}^{\frac{1}{p}}
\]

\[
\leq A_0 \left\{ \frac{1}{\varphi(x, r)} \right\}^{1/p} \sum_{(i,j) \in \mathbb{Z}^2 : |i| \leq 10, |j| \leq 10} \|f\chi_{A_i}\|_{L^p(\mathbb{R}^n)} \|g\chi_{A_j}\|_{L^p(\mathbb{R}^n)}
\]

\[
\leq A_0 \left\{ \frac{1}{\varphi(x, r)} \right\}^{1/p} \|f\|_{L^p(B_{10})} \|g\|_{L^p(B_{10})}
\]

\[
\leq A_0 \left[ \frac{\varphi_1(x, 2^{k_0+14})}{\varphi_1(x, r)} \right]^{1/2} \left[ \frac{\varphi_2(x, 2^{k_0+14})}{\varphi_2(x, r)} \right]^{1/2} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}
\]

\[
\leq A_0 \left[ \frac{2^{k_0+14}^{\varphi_2(x,r)}}{r^{\varphi_2(x,r)}} \right]^{1/2} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \leq A_0 \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)},
\]

where the second to the third inequality follows from the fact that \( A_j \subset B_j \) for any \( j \in \mathbb{Z} \), and there are only less than 11! terms in the summation.

We now need to show that the following estimate holds true:

\[
I_2 \leq |t|^{-\frac{n-\varphi_2(x,r)}{2}} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.
\]

For any \( t \in \mathbb{R}^n \) with \( 0 < |t| \leq 1 \), there exists a nonnegative integer \( k_1 \) such that

\[
2^{k_1} \leq \frac{1}{|t|} < 2^{k_1+1}.
\]

Indeed, by [12, Lemma 2.2], we know that, for any \( z \in B(x, r) \),

\[
\sum_{(i,j) \in \mathbb{Z}^2 : |j| > 10, |j| \leq 10} (f\chi_{A_i})(z-t)(g\chi_{A_j})(z-y)
\]

\[= \sum_{(i,j) \in \mathbb{Z}^2 : |j| > 10, |j|, |j-k_1| \leq 4} (f\chi_{A_i})(z-t)(g\chi_{A_j})(z-y)
\]
Therefore, by this, the Minkowski inequality and (2.4), we conclude that, for any \( z \in B(x, r) \),

\[
\sum_{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+: j > 0, \ l \geq 0} T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \leq 10_{\mathbb{Z}_+}^{\mathbb{Z}_+} \leq 4}
\]

which implies that, for any \( z \in B(x, r) \),

\[
\sum_{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+: j > 0, \ l \geq 0} T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) = \sum_{j=11}^{\infty} \sum_{h=j-4}^{j+4} T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z).
\]

The relationship \( |h - j| \leq 4 \) means that \( h \sim j \) for all \( j > 10 \). Thus, to prove (2.3), it suffices to show that, for any \( z \in B(x, r) \),

\[
\bar{T}_2 := \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \sum_{j=1}^{\infty} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right|^p dz \right\}^{1/p} \leq 2 \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| dy.
\]

From (1.1), (2.4) and [12, Lemma 2.1], we deduce that, for any \( z \in B(x, r) \),

\[
\left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| \leq \int_{\mathbb{R}_+} \frac{|f(x, u)| |g(x, \cdot)|}{|y|^n} dy \leq 2 \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| dy.
\]

By the Hölder inequality, (2.4) and the fact that \( \sup(g(x, \cdot)) \subset A_{j+1} \) for any \( z \in B(x, r) \), and a change of variables, we further find that

\[
\left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| \leq \frac{1}{2^{(k_0 + j + 1)}} \left\| f \chi_{A_j} \cdot \chi_{A_{k+1}} \right\|_{L^p(\mathbb{R}_+)} \left\| g \chi_{A_j} \cdot \chi_{A_{k+1}} \right\|_{L^q(\mathbb{R}_+)} \leq 2 \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| dy.
\]

which, combined with (2.1) and \( 2^{k_0} \leq r < 2^{k_0+1} \), implies that, for any \( j \in \mathbb{Z}_+ \) with \( j > 10 \),

\[
\left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right|^p dz \right\}^{1/p} \leq 2 \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right| dy.
\]

Therefore, by this, the Minkowski inequality and (2.4), we conclude that

\[
\bar{T}_2 \leq \sum_{j=1}^{\infty} \left\{ \frac{1}{\varphi(x, r)} \int_{B(x, r)} \left| T(f \chi_{A_j} \cdot \chi_{A_{k+1}})(z) \right|^p dz \right\}^{1/p}
\]
Proof. Let \(\omega\) be as in this theorem. We write
\[
\|T(f, g)\|_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} 2^{-k_l} \left( \frac{n-\eta(q_2)}{\epsilon} \right)^{\frac{1}{q}} \left[ \|f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)} \right] \leq |t| \left[ \|f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)} \right],
\]
which, combined with the estimates of \(I_1\), implies that
\[
I \leq \left( A_0 + |t|^{\frac{n-\eta(q_2)}{\epsilon}} \right) \|f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)},
\]
From the proof of [12, Theorem 1.1] and some arguments used in the estimation of \(I\), and [12, Lemma 2.2], it follows that
\[
I \leq \left( A_0 + |t|^{\frac{n-\eta(q_2)}{\epsilon}} \right) \|f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)},
\]
which, together with the estimate of \(I\), further implies that
\[
\|T(f, g)\|_{L^p(\mathbb{R}^n)} \leq \sup_{(x, t) \in \mathbb{R}^n \times (0, \infty)} (I + II) \leq \left( A_0 + |t|^{\frac{n-\eta(q_2)}{\epsilon}} \right) \|f\|_{L^{q_1}(\mathbb{R}^n)}\|g\|_{L^{q_2}(\mathbb{R}^n)}.
\]
This completes the proof of Theorem 2.3. \(\square\)

3 Pointwise estimates of the modules \(\mathcal{M}_{p, \varphi}(T(f, g); x, r)\)

In this section, we obtain the following estimate which is important to the proofs of the boundedness of \(T\) on vanishing generalized Morrey spaces.

**Theorem 3.1.** Let \(p, q, l \in (1, \infty)\) with \(\frac{1}{p} = \frac{1}{q} + \frac{1}{l}\). Assume that \(\varphi, \varphi_1\) and \(\varphi_2\) are measurable functions from \(\mathbb{R}^n \times (0, \infty)\) to \((0, \infty)\) satisfying \(\varphi_1^{1/p} = \varphi_2^{1/q} \varphi_2^{1/l}\), and \(T\) is defined as in (1.1) satisfying (2.2). Then there exists a positive constant \(C\) such that, for any \((f, g) \in L^q_{\text{loc}}(\mathbb{R}^n) \times L^l_{\text{loc}}(\mathbb{R}^n), t \in \mathbb{R}\) with \(0 < |t| \leq 1, r \in (0, \infty)\) and \(x \in \mathbb{R}^n\),
\[
\mathcal{M}_{p, \varphi}(T(f, g); x, r) \leq \frac{C r^n |t|^{\frac{1}{q} \varphi_1(2s)}}{\varphi(x, r)} \left\{ \int_r^{\infty} \frac{\varphi_2(x, 2s)}{s^{\frac{n}{p} + 1}} \left[ \mathcal{M}_{q, \varphi_1}(f; x, 2s) \right]^{\frac{1}{2}} \left[ \mathcal{M}_{l, \varphi_2}(g; x, 2s) \right]^{\frac{1}{2}} ds \right\}^p.
\]

**Proof.** Let \(q, l, p, f, g, x\) and \(r\) be as in this theorem. We write
\[
f = f \chi_{B(x, 2r)} + f \chi_{\mathbb{R}^n \setminus B(x, 2r)} =: f_1 + f_2\quad \text{and}\quad g = g \chi_{B(x, 2r)} + g \chi_{\mathbb{R}^n \setminus B(x, 2r)} =: g_1 + g_2.
\]
Observe that
\[
\|T(f, g)\|_{L^p(B(x, r))} \leq \|T(f_1, g_1)\|_{L^p(B(x, r))} + \|T(f_1, g_2)\|_{L^p(B(x, r))} + \|T(f_2, g_1)\|_{L^p(B(x, r))} + \|T(f_2, g_2)\|_{L^p(B(x, r))}
=: I_T + II_T + III_T + IV_T.
\]
We first estimate \(I_T\). By (2.2) and \(0 < |t| \leq 1\), we obtain
\[
\|T(f_1, g_1)\|_{L^p(B(x, r))} \leq \|T(f_1, g_1)\|_{L^p(\mathbb{R}^n)} \leq \|f_1\|_{L^q(\mathbb{R}^n)} \|g_1\|_{L^l(\mathbb{R}^n)} \sim \|f\|_{L^q(B(x, 2r))} \|g\|_{L^l(B(x, 2r))}.
\]
From the H"older inequality, we deduce that

\[
\sim r^{\frac{\beta}{p}} \int_r^\infty s^{-\frac{\beta}{p}-1} \|f\|_{L^p(B(x,2r))} \|g\|_{L^q(B(x,2r))} ds \\
\leq r^{\frac{\beta}{p}} |t|^{-\frac{\beta}{p}} \int_r^\infty s^{-\frac{\beta}{p}-1} \|f\|_{L^p(B(x,2s))} \|g\|_{L^q(B(x,2s))} ds.
\]

To estimate $I_1$, for any $z \in B(x,r)$, $z - y \in \mathbb{R}^n \setminus B(x,2r)$ and a given $\beta \in \left(\frac{2}{p}, n\right)$, we know that $|x - z| + |y| \geq |x - (z - y)| \geq 2r$ which implies that $|y| \geq r$, and hence

\[
|T(f_1, g_2)(z)| \leq \int_{\mathbb{R}^n} \frac{|f_1(z - ty)g_2(z - y)|}{|y|^\beta} dy \leq \int_{\mathbb{R}^n \setminus B(0,r)} \frac{|f_1(z - ty)g(z - y)|}{|y|^\beta} dy \\
\sim \int_{\mathbb{R}^n \setminus B(0,r)} \frac{|f_1(z - ty)g(z - y)|}{|y|^\beta} \left(\int_r^\infty ds \right) dy \\
\leq \int_r^\infty \frac{1}{s^{\beta+1}} \left(\int_{\{y \in \mathbb{R}^n : |y| \leq s\}} \frac{|f_1(z - ty)g(z - y)|}{|y|^\beta} dy \right) ds.
\]

From the H"older inequality, we deduce that

\[
\int_{\{y \in \mathbb{R}^n : |y| \leq s\}} \frac{|f_1(z - ty)g(z - y)|}{|y|^\beta} dy \\
\leq \|f_1(z - t \cdot)g(z - \cdot)\|_{L^p(B(0,s), B(0,r))} \|y|^{-(\alpha - \beta)}\|_{L^q(B(0,s), B(0,r))} \\
= I \cdot II,
\]

where $I := \|f_1(z - t \cdot)g(z - \cdot)\|_{L^p(B(0,s), B(0,r))}$ and $II := \|y|^{-(\alpha - \beta)}\|_{L^q(B(0,s), B(0,r))}$.

To estimate $I$, by the H"older inequality, $0 < |t| \leq 1$, the fact that if $s \geq r$ and $z \in B(x,r)$, then $B(z, |t|s) \subset B(z,s) \subset B(x,2s)$, and the change of variables, we conclude that

\[
I \leq \left(\int_{\{y \in \mathbb{R}^n : |y| \leq s\}} |f_1(z - ty)|^q dy \right)^{\frac{1}{q}} \left(\int_{\{y \in \mathbb{R}^n : |y| \leq s\}} |g(z - y)|^q dy \right)^{\frac{1}{q}} \\
\leq \left[\int_{B(z,|t|s)} |f(y)|^q \frac{1}{|t|^q} dy \right]^{\frac{1}{q}} \left[\int_{B(z,s)} |g(y)|^q \frac{1}{|y|^q} dy \right]^{\frac{1}{q}} \\
\leq |t|^{-\frac{n}{2}} \|f\|_{L^q(B(x,2s))} \|g\|_{L^q(B(x,2s))}.
\]

On the other hand, by $\beta \in \left(\frac{2}{p}, n\right)$, it is easy to verify that

\[
II \leq \left[\int_{|y| \leq s} |y|^{-(\alpha - \beta)p^r} dy \right]^{\frac{1}{p}} \sim \left[\int_{|t| \leq 1} \int_0^s r^{-(\alpha - \beta)p^r} r^{p-1} dr d\theta \right]^{\frac{1}{p}} \sim s^{-(\alpha + \beta + \beta')} \sim s^{\frac{n}{p}}
\]

with the notation $\mathbb{S}^{n-1}$ being the unit sphere in $\mathbb{R}^n$, which, combined with the estimate of $I$, (3.3) and (3.4), implies that

\[
|T(f_1, g_2)(z)| \leq \int_r^\infty s^{-\frac{\beta}{p}-1} s^{\frac{\beta}{p}-\frac{n}{p}} |t|^{-\frac{n}{2}} \|f\|_{L^q(B(x,2s))} \|g\|_{L^q(B(x,2s))} ds \\
\sim |t|^{-\frac{n}{2}} \int_r^\infty s^{-\frac{\beta}{p}-1} \|f\|_{L^q(B(x,2s))} \|g\|_{L^q(B(x,2s))} ds.
\]
By this, we easily conclude that

(3.5) \[ \|T(f_1, g_2)\|_{L^p(B(x, r), r)} \leq r^{\frac{\sigma}{2}}|t|^{-\frac{\sigma}{2}} \int_r^\infty s^{-\frac{\sigma}{p} - 1}\|f\|_{L^p(B(x, 2s), r)}\|g\|_{L^p(B(x, 2s), r)} ds. \]

For any \( z \in B(x, r) \) and \( z - ty \in \mathbb{R}^n \setminus B(x, 2r) \), we have \( |x - z| + |ty| \geq |x - (z - ty)| \geq 2r \), which implies that \(|y| \geq |ty| \geq r\). From this and some arguments similar to those used in the proof of (3.5), it follows that

(3.6) \[ \|T(f_2, g_2)\|_{L^p(B(x, r), r)} \leq r^{\frac{\sigma}{2}}|t|^{-\frac{\sigma}{2}} \int_r^\infty s^{-\frac{\sigma}{p} - 1}\|f\|_{L^p(B(x, 2s), r)}\|g\|_{L^p(B(x, 2s), r)} ds. \]

Finally, we turn to estimate \( IV_T \). For any \( z \in B(x, r) \), \( z - y \in \mathbb{R}^n \setminus B(x, 2r) \) and \( z - ty \in \mathbb{R}^n \setminus B(x, 2r) \), we know that \(|x - z| + |ty| \geq |x - (z - ty)| \geq 2r \) and hence \(|y| \geq |ty| > r\). If \( 0 \in \text{supp}(f_2(z - t)) \cap \text{supp}(g_2(z - \cdot)) \), then \(|z - x| \geq 2r \) which is a contradiction to \(|z - x| < r\). Thus, \( 0 \notin \text{supp}(f_2(z - t)) \cap \text{supp}(g_2(z - \cdot)) \) and hence

\[ |T(f_2, g_2)(z)| \leq \int_{\mathbb{R}^n} \frac{|f_2(z - ty)g_2(z - y)|}{|y|^n} dy. \]

From this and some arguments similar to those used in the proof of (3.5), we deduce that

(3.7) \[ \|T(f_2, g_2)\|_{L^p(B(x, r), r)} \leq r^{\frac{\sigma}{2}}|t|^{-\frac{\sigma}{2}} \int_r^\infty s^{-\frac{\sigma}{p} - 1}\|f\|_{L^p(B(x, 2s), r)}\|g\|_{L^p(B(x, 2s), r)} ds. \]

By (3.2), (3.5), (3.6), (3.7) and \( \varphi^{1/p} = \varphi_1^{1/p} \varphi_2^{1/p} \), we conclude that

\[ \mathcal{M}_{p, \varphi}(T(f, g); x, r) = \frac{1}{\varphi(x, r)} \int_{B(x, r)} |T(f, g)(z)|^p dz \]
\[ \leq \frac{1}{\varphi(x, r)} \left[ \|T(f_1, g_1)\|_{L^p(B(x, r))}^p + \|T(f_1, g_2)\|_{L^p(B(x, r))}^p + \|T(f_2, g_1)\|_{L^p(B(x, r))}^p + \|T(f_2, g_2)\|_{L^p(B(x, r))}^p \right] \]
\[ \leq \frac{1}{\varphi(x, r)} \left( r^{\frac{\sigma}{2}}|t|^{-\frac{\sigma}{2}} \int_r^\infty s^{-\frac{\sigma}{p} - 1}\|f\|_{L^p(B(x, 2s), r)}\|g\|_{L^p(B(x, 2s), r)} ds \right)^p \]
\[ \sim r^{p} |t|^{-\frac{p}{\sigma}} \frac{1}{\varphi(x, r)} \left\{ \int_r^\infty \frac{1}{s^{\frac{p}{\sigma} + 1}} \left[ \mathcal{M}_{q, \varphi_1}(f; x, 2s) \right]^\frac{1}{q} \mathcal{M}_{l, \varphi_2}(g; x, 2s) \right\}^p ds \]

which completes the proof of this theorem.

\[ \square \]

4  Boundedness of \( T \) on \( V_0 L^{p, \varphi}(\mathbb{R}^n) \) and \( V_\alpha L^{p, \varphi}(\mathbb{R}^n) \)

In this section, we show that the bilinear operator \( T \) as in (1.1) are bounded on both spaces \( V_0 L^{p, \varphi}(\mathbb{R}^n) \) and \( V_\alpha L^{p, \varphi}(\mathbb{R}^n) \). Since the boundedness of \( T \) on \( L^{p, \varphi}(\mathbb{R}^n) \) is already known, we only need to show that the \( V_0 \) and \( V_\alpha \) properties are preserved under the action of \( T \).
Theorem 4.1. Let \( p, q, l, \varphi, \varphi_1 \) and \( \varphi_2 \) as in Theorem 3.1. Then the bilinear operator \( T \) as in (1.1) is bounded from \( V_0L^{l,p}(\mathbb{R}^n) \times V_0L^{l,p}(\mathbb{R}^n) \) to \( V_0L^q(\mathbb{R}^n) \), if \( T \) satisfies (2.2), \( \{\varphi_j\}_{j=1}^2 \) satisfy (1.2), (1.3) and (2.1). \( \varphi \) satisfies (1.2), (1.3) and, for any \( \delta \in (0, \infty), \)

\[
(4.1) \quad c(\delta) := \int_0^\infty \sup_{x \in \mathbb{R}^n} \frac{\varphi^{\frac{1}{p}}(x, 2s)}{s^{\frac{q}{p} + 1}} ds < \infty,
\]

and there exists a positive constant \( C_0 \) such that, for any \( r \in (0, \infty) \) and \( x \in \mathbb{R}^n, \)

\[
(4.2) \quad \int_r^\infty \frac{\varphi^{\frac{1}{p}}(x, 2s)}{s^{\frac{q}{p} + 1}} ds \leq C_0 \frac{\varphi^{\frac{1}{p}}(x, r)}{r^{\frac{q}{p}}}. \]

Proof. Let \( p, q, l, \varphi, \varphi_1 \) and \( \varphi_2 \) as in this theorem, \( f \in V_0L^{l,p}(\mathbb{R}^n) \) and \( g \in V_0L^{l,p}(\mathbb{R}^n) \). It follows from Theorem 2.3 that \( T \) is bounded from \( L^{l,p}(\mathbb{R}^n) \times L^{l,p}(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \). Thus, to prove this theorem, it suffices to show that if

\[
(4.3) \quad \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{M}_{q,\varphi_1}(f; x, 2r) = 0 = \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{M}_{l,\varphi_2}(g; x, 2r),
\]

then

\[
(4.4) \quad \limsup_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathcal{M}_{p,\varphi}(T(f, g); x, r) = 0.
\]

To show (4.4), we split the right-hand side of (3.1) as follows:

\[
(4.5) \quad \mathcal{M}_{p,\varphi}(T(f, g); x, r) \leq C_3 \left[ I_{\delta_0}(x, r) + J_{\delta_0}(x, r) \right],
\]

where we assume that \( r \in (0, \delta_0) \) and \( C_3 \) is a positive constant independent of \( f, g, \delta_0, r \) and \( x, \)

\[
I_{\delta_0}(x, r) := \frac{r^q}{\varphi(x, r)} \left\{ \int_{\delta_0}^{\delta_0} \frac{\varphi^{\frac{1}{p}}(x, 2s)}{s^{\frac{q}{p} + 1}} \left[ \mathcal{M}_{q,\varphi_1}(f; x, 2s) \right]^{\frac{1}{q}} \left[ \mathcal{M}_{l,\varphi_2}(g; x, 2s) \right]^{\frac{1}{l}} ds \right\}^p
\]

and

\[
J_{\delta_0}(x, r) := \frac{r^q}{\varphi(x, r)} \left\{ \int_{\delta_0}^{\infty} \frac{\varphi^{\frac{1}{p}}(x, 2s)}{s^{\frac{q}{p} + 1}} \left[ \mathcal{M}_{q,\varphi_1}(f; x, 2s) \right]^{\frac{1}{q}} \left[ \mathcal{M}_{l,\varphi_2}(g; x, 2s) \right]^{\frac{1}{l}} ds \right\}^p.
\]

By (4.3), we choose a positive constant \( \delta_0 \) such that, for any \( s \in (0, \delta_0), \)

\[
(4.6) \quad \sup_{x \in \mathbb{R}^n} \mathcal{M}_{q,\varphi_1}(f; x, 2s) < \frac{E}{2C_3C_0^p} \quad \text{and} \quad \sup_{x \in \mathbb{R}^n} \mathcal{M}_{l,\varphi_2}(g; x, 2s) < \frac{E}{2C_3C_0^p},
\]

where we assume that \( C_0 \) and \( C_3 \) are constants in (4.2) and (4.5), respectively.

From (4.6) and (4.2), it follows that, for any \( r \in (0, \delta_0), \)

\[
(4.7) \quad \sup_{x \in \mathbb{R}^n} I_{\delta_0}(x, r)C_3 < \frac{E}{2}.
\]
For the estimate of $J_{\delta_0}(x, r)$, by (4.1) and (4.6), we have

$$J_{\delta_0}(x, r) \leq \left( c_{(\delta_0)} \| f \|_{L^{p\phi_1}(\mathbb{R}^n)} \| g \|_{L^{p\phi_2}(\mathbb{R}^n)} \right)^p \frac{r^n |r|^{-\frac{m}{q}}}{\varphi(x, r)},$$

where $c_{(\delta_0)}$ is the constant from (4.1). Then, by this, (1.2) and choosing $r \in (0, \infty)$ small enough such that

$$\sup_{x \in \mathbb{R}^n} \frac{r^n}{\varphi(x, r)} \left( c_{(\delta_0)} \| f \|_{L^{p\phi_1}(\mathbb{R}^n)} \| g \|_{L^{p\phi_2}(\mathbb{R}^n)} \right)^p < \frac{\varepsilon^m}{2},$$

then we obtain

$$\sup_{x \in \mathbb{R}^n} J_{\delta_0}(x, r) \leq \frac{\varepsilon}{2},$$

Therefore, from (4.5), (4.7) and (4.8), we deduce that, if we choose $r \in (0, \infty)$ small enough, then

$$\sup_{x \in \mathbb{R}^n} \mathfrak{R}_{p, q}(T(f, g); x, r) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which completes the proof of (4.4) and hence of this theorem.

\[\square\]

**Remark 4.2.** We list some specific examples for Theorem 4.1, which are borrowed from [27, Remark 3.5]. Let $p$, $q$ and $l$ be as in Theorem 4.1.

1. Then $\varphi_i(r) := a_ir^{\lambda_i}$, where $r \in (0, \infty)$, $i \in \{1, 2\}$, $a_i \in (0, \infty)$ and $\lambda_i \in (0, n)$, satisfies (1.2), (1.3) and (2.1), and $\varphi := \varphi_1^{\frac{p}{q_1}} \varphi_2^{\frac{p}{q_2}}$ satisfies (1.2), (1.3), (4.1) and (4.2), and hence Theorems 3.1 and 4.1 hold true in this case.

2. Let $i \in \{1, 2\}$, $\varphi_i$ satisfy (1.2), (1.3) and (2.1), and $\varphi := \varphi_1^{\frac{p}{q_1}} \varphi_2^{\frac{p}{q_2}}$ satisfy (1.2), (1.3), (4.1) and (4.2). For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $i \in \{1, 2\}$, let $\tilde{\varphi}(x, r) := \left[ \varphi_1(x, r) \right]^{\frac{p}{q_1}} \left[ \varphi_2(x, r) \right]^{\frac{p}{q_2}}$, where $\omega_i$ satisfies that there exist two positive constants $c$ and $C$ such that, for any $x \in \mathbb{R}^n$, $c \leq \omega_i(x) \leq C$, and $\tilde{\varphi}(x, r) := \omega_i(x) \varphi_i(x)$. Then $\tilde{\varphi}_i$ satisfies (1.2), (1.3) and (2.1), and $\tilde{\varphi}$ satisfies (1.2), (1.3), (4.1) and (4.2), and hence Theorems 3.1 and 4.1 hold true in this case.

**Theorem 4.3.** Let $p$, $q$, $l$, $\varphi_1$ and $\varphi_2$ as in Theorem 3.1. Assume that $\varphi$ satisfies (1.3), (1.4) and (4.2). Then the bilinear operator $T$ as in (1.1) is bounded from $V_\infty L^{p\phi_1}(\mathbb{R}^n)$ to $V_\infty L^{p\phi_2}(\mathbb{R}^n)$ if $T$ satisfies (2.2) and $\{\varphi_i\}_{i=1}^2$ satisfy (1.3), (1.4) and (2.1).

**Proof.** Let $p$, $q$, $l$, $\varphi_1$ and $\varphi_2$ be as in this theorem, $f \in V_\infty L^{p\phi_1}(\mathbb{R}^n)$ and $g \in V_\infty L^{p\phi_2}(\mathbb{R}^n)$. By Theorem 2.3, to prove this theorem, it suffices to show that

$$\lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \mathfrak{R}_{p, q}(T(f, g); x, r) = 0.$$
From these, (3.1) and (4.2), it follows that
\[ \mathfrak{M}_{p,\varphi}(T(f, g); x, r) \leq \varepsilon, \]
which completes the proof of (4.9) and hence of Theorem 4.3.

**Remark 4.4.** We list some specific examples for Theorem 4.3, which are borrowed from [27, Remark 3.6]. Let \( p, q, l \) be as in Theorem 4.3.

1. Then \( \varphi_i(r) := a_i r^{\lambda_i} \), where \( r \in (0, \infty) \), \( i \in \{1, 2\} \), \( a_i \in (0, \infty) \) and \( \lambda_i \in (0, n) \), satisfy (1.3), (1.4) and (2.1), and \( \varphi := \varphi_1^{\frac{p}{q}} \varphi_2^{\frac{q}{p}} \) satisfy (1.3), (1.4) and (4.2), and hence Theorem 4.3 holds true in this case.

2. Let \( i \in \{1, 2\} \), \( \varphi_i \) satisfy (1.3), (1.4) and (2.1), and \( \varphi := \varphi_1^{\frac{p}{q}} \varphi_2^{\frac{q}{p}} \) satisfy (1.3), (1.4) and (4.2). For any \( x \in \mathbb{R}^n \), \( r \in (0, \infty) \) and \( i \in \{1, 2\} \), let \( \overline{\varphi}(x, r) := [\varphi_1(x, r)]^{\frac{p}{q}} [\varphi_2(x, r)]^{\frac{q}{p}} \), where \( \omega_i \) satisfies that there exist two positive constants \( c \) and \( C \) such that, for any \( x \in \mathbb{R}^n \), \( c \leq \omega_i(x) \leq C \), and \( \overline{\varphi}(x, r) := \omega_i(x) \varphi_i(x) \). Then \( \overline{\varphi} \) satisfies (1.3), (1.4) and (2.1), and \( \overline{\varphi} \) satisfies (1.3), (1.4) and (4.2), and hence Theorem 4.3 holds true in this case.

## 5 Subbilinear maximal operator \( M \)

This section is devoted to the proof of the boundedness of the subbilinear maximal operator \( M \) from \( V^{(s)} L^{q, \varphi_1}(\mathbb{R}^n) \times V^{(s)} L^{q, \varphi_2}(\mathbb{R}^n) \) to \( V^{(s)} L^{p, \varphi}(\mathbb{R}^n) \).

**Theorem 5.1.** Let \( p, q, l, \varphi, \varphi_1 \) and \( \varphi_2 \) as in Theorem 3.1. Assume that, for any \( i \in \{1, 2\} \), \( \varphi_i \) satisfies
\[
\frac{\varphi_i(x, r)}{r^l} < \infty 
\]
and \( \varphi \) satisfies
\[
\lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \frac{\varphi(x, r)}{r^l} = 0.
\]

Then the subbilinear maximal operator \( M \) defined by setting, for any \( (f, g) \in V^{(s)} L^{q, \varphi_1}(\mathbb{R}^n) \times V^{(s)} L^{q, \varphi_2}(\mathbb{R}^n) \), \( t \in \mathbb{R}^n \) with \( 0 < |t| \leq 1 \), and \( x \in \mathbb{R}^n \),
\[
M(f, g) = \sup_{r > 0} \frac{1}{v_n r^l} \int_{B(0, r)} |f(x - ty)| dy \cdot g(x - y),
\]
is bounded from \( V^{(s)} L^{q, \varphi_1}(\mathbb{R}^n) \times V^{(s)} L^{q, \varphi_2}(\mathbb{R}^n) \) to \( V^{(s)} L^{p, \varphi}(\mathbb{R}^n) \), where \( v_n \) denotes the Lebesgue measure of the unit ball in \( \mathbb{R}^n \).

**Proof.** Let \( p, q, l, \varphi, \varphi_1, \varphi_2, f \) and \( g \) be as in this theorem. We first show that \( M(f, g) \) is bounded on \( L^{p, \varphi}(\mathbb{R}^n) \). By the Hölder inequality, \( y \in B(0, r) \) and a change of variables, we conclude that, for any \( t \in \mathbb{R}^n \) with \( 0 < |t| \leq 1 \), and \( x \in \mathbb{R}^n \),
\[
M(f, g)(x)
\]
which implies that
\[
\begin{align*}
\leq & \sup_{r \in (0, \infty)} \frac{1}{V_{n+r^2}} \left( \int_{B(0,r)} |f(x-ty)|^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \left( \int_{B(0,r)} |g(x-y)|^{\frac{t}{p}} dy \right)^{\frac{t}{p}} \\
\leq & \left( \sup_{r \in (0, \infty)} \frac{1}{V_{n+r^2}} \int_{B(x,r)} |f(y)|^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \left( \sup_{r \in (0, \infty)} \frac{1}{V_{n+r^2}} \int_{B(x,r)} |g(y)|^{\frac{t}{p}} dy \right)^{\frac{t}{p}} |t|^{-\frac{m}{q}} \\
= & \left[ M \left( |f|^{\frac{q}{p}} \right)(x) \right]^{\frac{p}{q}} \left[ M \left( |g|^{\frac{t}{p}} \right)(x) \right]^{\frac{t}{p}} |t|^{-\frac{m}{q}},
\end{align*}
\]
which, combined with the boundedness of $M$ on $L^p(\mathbb{R}^n)$ for any $p \in (1, \infty)$ and the Hölder inequality, implies that

\[
\|M(f, g)\|_{L^p(\mathbb{R}^n)} \leq |t|^{-\frac{m}{q}} \left\{ \int_{\mathbb{R}^n} \left[ M \left( |f|^{\frac{q}{p}} \right)(x) \right]^{\frac{p}{q}} \left[ M \left( |g|^{\frac{t}{p}} \right)(x) \right]^{\frac{t}{p}} dx \right\}^{\frac{1}{p}}
\]

\[
\leq |t|^{-\frac{m}{q}} \left\{ \int_{\mathbb{R}^n} \left[ M \left( |f|^{\frac{q}{p}} \right)(x) \right] dx \right\}^{\frac{1}{q}} \left\{ \int_{\mathbb{R}^n} \left[ M \left( |g|^{\frac{t}{p}} \right)(x) \right] dx \right\}^{\frac{1}{p}}
\]

\[
\leq |t|^{-\frac{m}{q}} \left\| |f|^{\frac{q}{p}} \right\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}} \left\| |g|^{\frac{t}{p}} \right\|_{L^p(\mathbb{R}^n)}^{\frac{t}{p}} \sim |t|^{-\frac{m}{q}} \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.
\]

From (5.4), $\varphi^{1/p} = \varphi_1^{1/q} \varphi_2^{1/l}$ and [37, Corollary 4.2], we deduce that

\[
\|M(f, g)\|_{L^{p,q}(\mathbb{R}^n)}
\]

\[
\leq |t|^{-\frac{m}{q}} \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} \left\{ \frac{1}{\varphi(x, r)} \int_{B(x,r)} \left[ M \left( |f|^{\frac{q}{p}} \right)(y) \right]^{\frac{p}{q}} \left[ M \left( |g|^{\frac{t}{p}} \right)(y) \right]^{\frac{t}{p}} dy \right\}^{\frac{1}{p}}
\]

\[
\leq |t|^{-\frac{m}{q}} \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} \frac{1}{\varphi_1(x, r)} \left\{ \int_{B(x,r)} \left[ M \left( |f|^{\frac{q}{p}} \right)(y) \right]^{p} dy \right\}^{\frac{1}{p}} \left\{ \int_{B(x,r)} \left[ M \left( |g|^{\frac{t}{p}} \right)(y) \right]^{p} dy \right\}^{\frac{1}{p}}
\]

\[
\leq |t|^{-\frac{m}{q}} \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} \left\{ \frac{1}{\varphi_1(x, r)} \int_{B(x,r)} \left[ M \left( |f|^{\frac{q}{p}} \right)(y) \right]^{p} dy \right\}^{\frac{1}{p}}
\]

\[
\times \left\{ \varphi_2(x, r) \int_{B(x,r)} \left[ M \left( |g|^{\frac{t}{p}} \right)(y) \right]^{p} dy \right\}^{\frac{1}{q}}
\]

\[
\leq |t|^{-\frac{m}{q}} \left\| M \left( |f|^{\frac{q}{p}} \right) \right\|_{L^{p,q_1}(\mathbb{R}^n)} \left\| M \left( |g|^{\frac{t}{p}} \right) \right\|_{L^{p,q_2}(\mathbb{R}^n)} \leq |t|^{-\frac{m}{q}} \left\| f \right\|_{L^{p,q_1}(\mathbb{R}^n)} \left\| g \right\|_{L^{p,q_2}(\mathbb{R}^n)}
\]

\[
\sim |t|^{-\frac{m}{q}} \|f\|_{L^{p,q_1}(\mathbb{R}^n)} \|g\|_{L^{p,q_2}(\mathbb{R}^n)}
\]

which implies that $\mathcal{M}$ is bounded from $L^{p,q_1}(\mathbb{R}^n) \times L^{p,q_2}(\mathbb{R}^n)$ to $L^{p,\varphi}(\mathbb{R}^n)$. Thus, to finish the proof of Theorem 5.1, it suffices to show that, if $\lim_{N \to \infty} \mathcal{A}_{N,q}(f) = 0 = \lim_{N \to \infty} \mathcal{A}_{N,l}(g)$, then

\[
\lim_{N \to \infty} \mathcal{A}_{N,p}(M(f, g)) = 0.
\]

We decompose $f$ and $g$, respectively, into

\[
f = f \chi_{\Omega_{n/2}} + f \chi_{\mathbb{R}^n \setminus \Omega_{n/2}} =: f_1 + f_2 \quad \text{and} \quad g = g \chi_{\Omega_{n/2}} + g \chi_{\mathbb{R}^n \setminus \Omega_{n/2}} =: g_1 + g_2,
\]
where \( \Omega_{x,N/2} := B(x,2) \cap [\mathbb{R}^n \setminus B(0,N/2)] \) for any \( N \in \mathbb{N} \). Since \( M \) is sublinear, we have

(5.7) \[
\mathcal{A}_{N,p}(M(f,g)) \leq \mathcal{A}_{N,p}(M(f_1,g_1)) + \mathcal{A}_{N,p}(M(f_1,g_2)) \\
+ \mathcal{A}_{N,p}(M(f_2,g_1)) + \mathcal{A}_{N,p}(M(f_2,g_2)) \\
=: I_N + II_N + III_N + IV_N.
\]

We show next that all terms in the right-hand side of (5.7) tend to zero as \( N \to \infty \).

We first estimate \( I_N \). By (5.5), we obtain

(5.8) \[
\int_{B(x,1)} [M(f_1,g_1)(y)]^p \chi_N(y) dy \leq \int_{\mathbb{R}^n} [M(f_1,g_1)(y)]^p dy \\
\leq \left( \int_{\mathbb{R}^n} |f_1(y)|^q dy \right)^{\frac{p}{q}} \left( \int_{\mathbb{R}^n} |g_1(y)|^l dy \right)^{\frac{p}{l}} |r|^{-\frac{pl}{q}} \\
\leq \left( \int_{\Omega_{x,N/2}} |f(y)|^q dy \right)^{\frac{p}{q}} \left( \int_{\Omega_{x,N/2}} |g(y)|^l dy \right)^{\frac{p}{l}} |r|^{-\frac{pl}{q}}
\]

with the implicit positive constants independent of \( f, g, N \) and \( x \). Since \( f \in V^{(s)}L^{q,\varphi_1}(\mathbb{R}^n) \) and \( g \in V^{(s)}L^{l,\varphi_2}(\mathbb{R}^n) \), the right-hand side in (5.8) tends to zero uniformly in \( x \) as \( N \to \infty \). It was shown by [5, Lemma 3.4] that the property \( V^{(s)} \) does not depend on the particular value of the radii taken in the balls centered at \( x \) by changing variables. Therefore,

\[
\lim_{N \to \infty} \mathcal{A}_{N,p}(M(f_1,g_1)) = 0.
\]

Then we estimate \( II_N \). From (5.5), we deduce that

(5.9) \[
\int_{B(x,1)} [M(f_1,g_2)(y)]^p \chi_N(y) dy \leq \int_{B(x,1)} [M(f_1,g_2)(y)]^p dy \\
\leq \left( \int_{\mathbb{R}^n} |f_1(y)|^q dy \right)^{\frac{p}{q}} \left( \int_{B(x,1)} |g_2(y)|^l dy \right)^{\frac{p}{l}} |r|^{-\frac{pl}{q}} \\
\leq \left\| g \right\|_{L^{l,\varphi_2}(\mathbb{R}^n)} \left( \int_{\Omega_{x,N/2}} |f(y)|^q dy \right)^{\frac{p}{q}} \left[ \varphi_2(x,1) \right]^p
\]

with the implicit constants independent of \( f, g, N \) and \( x \). Since \( f \in V^{(s)}L^{q,\varphi_1}(\mathbb{R}^n) \) and \( g \in V^{(s)}L^{l,\varphi_2}(\mathbb{R}^n) \), by (5.1), we know that the right-hand side in the equation tends to zero uniformly in \( x \) as \( N \to \infty \). Therefore, \( \lim_{N \to \infty} II_N = 0 \). From some arguments similar to those used in the estimation of \( II_N \), it follows that \( \lim_{N \to \infty} III_N = 0 \).

Now, we deal with \( IV_N \). Let \( \varepsilon \in (0,\infty) \) be arbitrary. Then, by (5.2), we find that there exists \( r_1 \in (1,\infty) \) such that \( r^{-n}\varphi(y,r) < \varepsilon \) for any \( r \in [r_1,\infty) \). For such fixed \( r_1 \), we have

\[
\int_{B(x,1)} [M(f_2,g_2)(y)]^p \chi_N(y) dy \leq I_1(x,N) + I_2(x,N),
\]

where

\[
I_1(x,N) := \int_{B(x,1)} \chi_N(y) \sup_{r \in (0,r_1)} \left[ \frac{1}{V_n r^n} \int_{B(0,r)} |f_2(y-tz)g_2(y-z)| dz \right]^p dy
\]

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and

\[ I_2(x, N) := \int_{B(x, 1)} \chi_N(y) \sup_{r \in [r_1, \infty)} \left[ \frac{1}{v_n r^n} \int_{B(0, r)} |f(y - tz)g(y - z)|dz \right]^p dy. \]

First, we estimate \( I_2(x, N) \). By the Hölder inequality and a change of variables, we conclude that

\[
\frac{1}{v_n r^n} \int_{B(0, r)} |f_2(y - tz)g_2(y - z)|dz \\
\leq \frac{1}{v_n r^n} \left[ \int_{B(0, r)} |f(y - tz)|^{\frac{q}{r}} dz \right]^\frac{r}{q} \left[ \int_{B(0, r)} |g(y - z)|^{\frac{q}{r}} dz \right]^\frac{r}{q} \\
\leq \frac{1}{v_n r^n} \left[ \int_{B(y, r)} |f(z)|^{\frac{q}{r}} dz \right]^{\frac{r}{q}} \left[ \int_{B(y, r)} |g(z)|^{\frac{q}{r}} dz \right]^{\frac{r}{q}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))} \\
\leq r^{-\frac{n}{r}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))} \\
\sim r^{-\frac{n}{r}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))} \\
\Rightarrow I_2(x, N) \leq \int_{B(y, r)} \sup_{r \in [r_1, \infty)} r^{-\frac{n}{r}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))} dy \\
\leq \sup_{r \in [r_1, \infty)} r^{-\frac{n}{r}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))} \leq \epsilon r^{-\frac{n}{r}} \|f\|_{L^q(B(y, r))} \|g\|_{L^q(B(y, r))},
\]

and hence

\[ (5.9) \quad \lim_{N \to \infty} \sup_{x \in \mathbb{R}^n} I_2(x, N) = 0. \]

Next, we deal with \( I_1(x, N) \). Observe that \( y - z \notin \Omega_{x\cdot N^2} \) if and only if \( y - z \notin B(x, 2) \) or \( y - z \notin B(0, \frac{N}{2}) \), and \( y - tz \notin \Omega_{x\cdot N^2} \) if and only if \( y - tz \notin B(x, 2) \) or \( y - tz \notin B(0, \frac{N}{2}) \). Thus, for any \( y \in B(x, 1) \setminus B(0, N) \) and \( z \in B(0, r) \), we further consider the following four cases:

**Case 1:** If \( y - z \in B(0, \frac{N}{2}) \) and \( y - tz \in B(0, \frac{N}{2}) \), then \( r \geq |z| = |y - (y - z)| \geq |y| - |y - z| > \frac{N}{2} \) and \( r \geq |z| = |y - (y - tz)| \geq |y| - |y - tz| > \frac{N}{2} \). Therefore, if \( r \in (0, r_1) \) and \( N \geq 2r_1 \), then \( I_1(x, N) = 0 \).

**Case 2:** If \( y - z \in B(0, \frac{N}{2}) \) and \( y - tz \notin B(x, 2) \), then \( r \geq |z| = |y - (y - z)| \geq |y| - |y - z| > \frac{N}{2} \). Thus, if \( r \in (0, r_1) \) and \( N \geq 2r_1 \), then \( I_1(x, N) = 0 \).

**Case 3:** If \( y - z \notin B(x, 2) \) and \( y - tz \in B(0, \frac{N}{2}) \), then \( r \geq |z| = |y - (y - tz)| \geq |y| - |y - tz| > \frac{N}{2} \). Thus, if \( r \in (0, r_1) \) and \( N \geq 2r_1 \), then \( I_1(x, N) = 0 \).

**Case 4:** If \( y - z \notin B(x, 2) \) and \( y - tz \notin B(x, 2) \), then \( r \geq |z| = |y - z - x - (y - x)| \geq 1 \). Hence, it remains to estimate \( I_1(x, N) \) when the supremum is taken over all \( r \in (1, r_1) \). For such values of \( r \), we observe that

\[
\frac{1}{r^n} \int_{B(0, r)} |f_2(y - tz)g_2(y - z)|dz \leq \int_{B(0, r_1)} |f_2(y - tz)g_2(y - z)|dz,
\]
which, combined with the Minkowski inequality, the Hölder inequality and a change of variables, implies that, for any $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$,

\[
I_1(x, N) \leq \int_{B(x, 1)} \chi_N(y) \left[ \int_{\mathbb{R}^n} \left| f_2(y - tz)g_2(y - z) \right|^p \right]^{\frac{1}{p}} \, dy
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \chi_N(y) \left[ \int_{B(x, 1)} \left| f_2(y - tz)g_2(y - z) \right|^{p} \, dz \right]^{\frac{1}{p}} \right\}^{p}
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \chi_N(y) \left[ \int_{B(x, 1)} \left| f_2(y - tz)g_2(y - z) \right|^{q} \, dz \right]^{\frac{1}{q}} \right\}^{p}
\]

\[
\leq \left\{ \int_{\mathbb{R}^n} \chi_N(y) \left[ \sup_{v \in \mathbb{R}^n} \int_{B(v, 1)} \left| f(u) \right|^q \chi_N(y - |z|) \, du \right]^{\frac{1}{q}} \right\}^{p}
\]

\[
= \left( \int_{\mathbb{R}^n} F_N(z) \, dz \right)^p
\]

with a modified notation of $\chi_a$ in (1.5): $\chi_a := 1$ if $a \in (-\infty, 0]$, and $\chi_a := \chi_{\mathbb{R}^n \setminus B(0, a)}$ if $a \in (0, \infty)$. By $f \in V^{(r)} L^{q, \varphi_1}(\mathbb{R}^n)$, $g \in V^{(s)} L^{q, \varphi_2}(\mathbb{R}^n)$ and the Lebesgue convergence theorem, we conclude that $\int_{\mathbb{R}^n} F_N(z) \, dz \to 0$ as $N \to \infty$, which implies that $\lim_{N \to \infty} \sup_{x \in \mathbb{R}^n} I_1(x, N) = 0$. This, together with (5.9), implies that $\lim_{N \to \infty} IV_N = 0$, which, combined with the estimates of $I_N$, $II_N$ and $III_N$, further completes the proof of (5.6) and hence of this theorem. \qed

**Remark 5.2.** We list some specific examples for Theorem 5.1, which are borrowed from [27, Remark 4.4]. Let $p$, $q$, $l$ be as in Theorem 5.1. Then

1. $\varphi_i(r) := a_i r^{\lambda_i}$, where $r \in (0, \infty)$, $i \in \{1, 2\}$, $a_i \in (0, \infty)$ and $\lambda_i \in (0, n);$ 

2. $\varphi_i(r) := a_i e^{-\lambda_i r} + b_i$, where $r \in (0, \infty)$, $i \in \{1, 2\}$ and $a_i, b_i, \lambda_i \in (0, \infty);$ 

3. $\varphi_i(r) := \begin{cases} a_i - \ln(\lambda_i r), & r \in (0, \frac{1}{\lambda_i}), \\ a_i, & r \in \left[ \frac{1}{\lambda_i}, \infty \right), \end{cases}$ where $i \in \{1, 2\}$ and $a_i, \lambda_i \in (0, \infty)$ 

satisfy (5.1), and $\varphi := \varphi_1^{\frac{p}{l}} \varphi_2^{\frac{q}{l}}$ satisfy (4.2) and (5.2), and hence Theorem 5.1 holds true in these cases.

4. Let $i \in \{1, 2\}$, $\varphi_i$ satisfy (5.1), and $\varphi := \varphi_1^{\frac{p}{l}} \varphi_2^{\frac{q}{l}}$ satisfy (4.2) and (5.2). For any $x \in \mathbb{R}^n$, $r \in (0, \infty)$ and $i \in \{1, 2\}$, let $\widetilde{\varphi}(x, r) := [\varphi_1(x, r)]^{\frac{p}{l}} [\varphi_2(x, r)]^{\frac{q}{l}}$, where $\omega_i$ satisfies that there exist two positive constants $c$ and $C$ such that, for any $x \in \mathbb{R}^n$, $c \leq \omega_i(x) \leq C$, and $\overline{\varphi}(x, r) := \omega_i(x) \varphi_i(x)$. Then $\varphi_i$ satisfies (5.1), and $\varphi$ satisfies (4.2) and (5.2), and hence Theorem 5.1 holds true in this case.
6 Some applications

As applications, we can easily obtain the boundedness on generalized Morrey spaces for several well known bilinear operators in harmonic analysis. These operators include the bilinear Hilbert transform, the bilinear oscillatory Hilbert transform, the bilinear (maximal) singular integral with rough kernels and the first Calderón commutator.

Our first result is about the bilinear Hilbert transform, which is an easy consequence of Theorem 2.3 and [24, Theorem 1.1], and we omit the details.

**Corollary 6.1.** Let $p, q, l, \varphi_1$ and $\varphi_2$ be as in Theorem 2.3 with $\varphi_1^{1/p} = \varphi_1^{1/q} \varphi_2^{1/l}$. Then there exists a positive constant $C$ such that, for any $f \in L^{\varphi_1}(\mathbb{R}^n)$ and $g \in L^{\varphi_2}(\mathbb{R}^n)$,

$$\|H(f, g)\|_{L^{\varphi_0}(\mathbb{R}^n)} \leq \|f\|_{L^{\varphi_1}(\mathbb{R}^n)} \|g\|_{L^{\varphi_2}(\mathbb{R}^n)},$$

where $H$ is the bilinear Hilbert transform defined by setting, for any suitable functions $f$ and $g$, and $x \in \mathbb{R}^n$,

$$H(f, g)(x) := \text{p.v.} \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{y} dy.$$ 

We remark that p.v. in Corollary 6.1 represents the principle value of the integral, which is omitted in the following context for the notational simplicity.

Next, we consider the bilinear oscillatory Hilbert transform defined by setting, for any suitable functions $f$ and $g$, and $x \in \mathbb{R}^n$,

$$T_P(f, g)(x) := \int_{\mathbb{R}^n} \frac{e^{iP(x, y)}f(x-y)g(x+y)}{y} dy,$$

where $P$ is a binary real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$. In [9, Theorem 2.6], the authors proved that the operator $T_P$ is bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $1/p + 1/q = 1/l$ and $p, q, l \in (0, \infty)$. Also they showed that the operator norm of $T_P$, $\|T_P\|_{L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)}$, is independent of the coefficients of $P$. By this result and Theorem 2.3, it is easy to obtain the following result, and we omit the details here.

**Corollary 6.2.** Let $p, q, l, \varphi_1$ and $\varphi_2$ be as in Theorem 2.3 with $\varphi_1^{1/p} = \varphi_1^{1/q} \varphi_2^{1/l}$. Then there exists a positive constant $C$ such that, for any $f \in L^{\varphi_1}(\mathbb{R}^n)$ and $g \in L^{\varphi_2}(\mathbb{R}^n)$,

$$\|T_P(f, g)\|_{L^{\varphi_0}(\mathbb{R}^n)} \leq C \|f\|_{L^{\varphi_1}(\mathbb{R}^n)} \|g\|_{L^{\varphi_2}(\mathbb{R}^n)}.$$

Moreover, the operator norm of $T_P$, $\|T_P\|_{L^{\varphi_1}(\mathbb{R}^n) \times L^{\varphi_2}(\mathbb{R}^n) \rightarrow L^{\varphi_0}(\mathbb{R}^n)}$, is independent of the coefficients of the binary polynomial $P$.

In the high dimensional case, we consider the bilinear singular integral with rough kernel $T_\Omega$ and its related subbilinear maximal operator $T_\Omega^*$ defined, respectively, by setting, for any suitable functions $f$ and $g$, and $x \in \mathbb{R}^n$,

$$T_\Omega(f, g)(x) := \int_{\mathbb{R}^n} \frac{f(x-y)g(x+y)}{|y|^n} \Omega \left( \frac{y}{|y|} \right) dy.$$
and

\[ T_\Omega^*(f, g)(x) := \sup_{\varepsilon \in (0, \infty)} \left| T_{\Omega, \varepsilon}(f, g)(x) \right|, \]

where \( T_{\Omega, \varepsilon} \) is the truncated bilinear operator for any \( \varepsilon \in (0, \infty) \), namely,

\[ T_{\Omega, \varepsilon}(f, g)(x) := \int_{|y| > \varepsilon} \frac{f(x - y)g(x + y)}{|y|^n} \Omega(y') dy \]

and \( \Omega(y') \) is a function defined on the unit sphere \( S^{n-1} \) in the Euclidean space \( \mathbb{R}^n \), and \( y' := y/|y| \) for any \( y \in \mathbb{R}^n \setminus \{0\} \).

The following result is an easy consequence of Corollary 6.3 and Lemma 6.4 (see the proof of [12, Corollary 6.3]).

**Corollary 6.3.** Let \( p, q, l, \varphi_1 \) and \( \varphi_2 \) be as in Theorem 2.3 with \( \varphi_1/p = \varphi_1/q = \varphi_1/1/l \). Assume that \( \Omega \in L^\infty(S^{n-1}) \) is an odd function and \( T_\Omega \) is bounded from \( L^p(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \). Then there exists a positive constant \( C \) such that, for any \( f \in L^{\varphi_1'}(\mathbb{R}^n) \) and \( g \in L^{\varphi_2'}(\mathbb{R}^n) \),

\[ \|T_\Omega(f, g)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{\varphi_1}(\mathbb{R}^n)} \|g\|_{L^{\varphi_2}(\mathbb{R}^n)}. \]

Recall that the maximal bilinear Hilbert transform \( H^* \) is defined by setting, for any suitable functions \( f \) and \( g \), and \( x \in \mathbb{R}^n \),

\[ H^*(f, g)(x) := \sup_{\varepsilon \in (0, \infty)} \left| \int_{|t| > \varepsilon} \frac{f(x - t)g(x + t)}{t} \right| dt. \]

Lacey [25, Theorem 1.5] obtained the following remarkable result.

**Lemma 6.4.** Let \( q, l \in (1, \infty) \) and \( 1/p = 1/q + 1/l \). If \( p \in (2/3, \infty) \), then there exists a positive constant \( C \) such that, for any \( f \in L^{\varphi_1}(\mathbb{R}^n) \) and \( g \in L^{\varphi_2}(\mathbb{R}^n) \),

\[ \|H^*(f, g)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^{\varphi_1}(\mathbb{R}^n)} \|g\|_{L^{\varphi_2}(\mathbb{R}^n)}. \]

The following result is an easy consequence of Corollary 6.3 and Lemma 6.4 (see the proof of [12, Corollary 6.5]), and we omit the details here.

**Corollary 6.5.** Let \( p, q, l, \varphi_1 \) and \( \varphi_2 \) as in Theorem 2.3 with \( \varphi_1/p = \varphi_1/q = \varphi_1/l \). Assume that \( \Omega \in L^\infty(S^{n-1}) \) is an odd function, then there exists a positive constant \( C \) such that, for any \( f \in L^{\varphi_1}(\mathbb{R}^n) \) and \( g \in L^{\varphi_2}(\mathbb{R}^n) \),

\[ \|T_{\Omega}^*(f, g)\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^{\varphi_1}(\mathbb{R}^n)} \|g\|_{L^{\varphi_2}(\mathbb{R}^n)}. \]

The first Calderón commutator is defined by setting, for any suitable functions \( f \) and \( g \), and Lipschitz function \( \psi \) and \( x \in \mathbb{R}^n \),

\[ C_\psi f(x) := \int_{\mathbb{R}^n} \frac{\psi(x) - \psi(y)}{(x - y)^2} f(y) dy, \]

where, for any Lipschitz function \( \psi \), we may write

\[ \frac{\psi(x) - \psi(y)}{x - y} = \frac{1}{x - y} \int_x^y \psi'(t) dt = \int_0^1 \psi'((1 - t)x + ty) dt. \]
Thus, from [12, pp. 155-156], (6.1) and the Fubini theorem, it follows that

\begin{equation}
C_{\psi}f(x) = \int_{\mathbb{R}^n} \psi(x) - \psi(y) (x-y)^2 f(y) dy = \int_{\mathbb{R}^n} f(y) \left( \int_0^1 \psi'((1-t)y + tx) dt \right) dy
= \int_0^1 \int_{\mathbb{R}^n} f(x-y)\psi'(x-(1-t)y) \frac{dy}{y} dt =: \int_0^1 H_t(f,\psi')(x) dt.
\end{equation}

A celebrated result due to Grafakos and Li [13, p. 891] is stated as follows.

**Lemma 6.6.** Let \( q, l \in (2, \infty) \) and \( p := q/(q+l) \in (1,2) \). Then there is a constant \( C := C_{(q,l)} \) such that, for any \( f \in L^q(\mathbb{R}^n) \) and \( g \in L^l(\mathbb{R}^n) \),

\[
\sup_{t \in \mathbb{R}^n} \| H_t(f,g) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^q(\mathbb{R}^n)} \| g \|_{L^l(\mathbb{R}^n)}.
\]

Consequently, we have the following result for the operator \( C_{\psi} \).

**Corollary 6.7.** Let \( p, q, l, \varphi, \varphi_1 \) and \( \varphi_2 \) as in Theorem 2.3 with \( \varphi^{1/p} = \varphi_1^{1/q} \varphi_2^{1/l} \). Assume that \( \psi \) is a Lipschitz function. Then there exists a positive constant \( C \) such that, for any \( f \in L^{\varphi_{\psi}}(\mathbb{R}^n) \),

\[
\| C_{\psi}f(t) \|_{L^{\varphi_{\psi}}(\mathbb{R}^n)} \leq \| f \|_{L^{\varphi_{\psi_1}}(\mathbb{R}^n)} \| \psi' \|_{L^{\varphi_{\psi_2}}(\mathbb{R}^n)}.
\]

**Proof.** By [22, Proposition 1.2.2], (6.2) and Theorem 2.3, we conclude that

\[
\| C_{\psi}(f) \|_{L^{\varphi_{\psi}}(\mathbb{R}^n)} \leq \int_0^1 \| H_t(f,\psi') \|_{L^{\varphi_{\psi}}(\mathbb{R}^n)} dt \leq \int_0^1 \left( A_0 + |t|^\frac{\varphi_1}{2} \right) \| f \|_{L^{\varphi_{\psi_1}}(\mathbb{R}^n)} \| \psi' \|_{L^{\varphi_{\psi_2}}(\mathbb{R}^n)} dt
\]

\[
\leq \left(1 + \int_0^1 |t|^\frac{\varphi_1}{2} dt \right) \| f \|_{L^{\varphi_{\psi_1}}(\mathbb{R}^n)} \| \psi' \|_{L^{\varphi_{\psi_2}}(\mathbb{R}^n)} \leq \| f \|_{L^{\varphi_{\psi_1}}(\mathbb{R}^n)} \| \psi' \|_{L^{\varphi_{\psi_2}}(\mathbb{R}^n)},
\]

which completes the proof of this corollary.

**Remark 6.8.** We point out that the boundedness of \( H, T_P, T_{\Omega}, T_{\Omega_1}^* \) and \( C_{\psi} \) on vanishing generalized Morrey spaces \( V_0 L^{p,\varphi}(\mathbb{R}^n) \) and \( V_\infty L^{p,\varphi}(\mathbb{R}^n) \) can be shown immediately, and we omit the details here.

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