MULTIPLICATIVE BI-SKEW LIE TRIPLE DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra. For any $A, B \in \mathcal{A}$, a product $[A, B] = AB^* - BA^*$ is called a bi-skew Lie product. In this paper, it is shown that every bi-skew Lie triple derivation $\psi : \mathcal{A} \to \mathcal{A}$ is an additive $*$-derivation.

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1. Introduction

Let $\mathcal{A}$ be a $*$-algebra. Denote by $[A, B] = AB - BA$ and $[A, B]_* = AB - BA^*$, we mean the Lie product and the skew Lie product of $A, B \in \mathcal{A}$, respectively. Latter product is also called $*$-Lie product. Recall that a map $\psi : \mathcal{A} \to \mathcal{A}$ is called additive derivation if $\psi(A + B) = \psi(A) + \psi(B)$ and $\psi(A B) = \psi(A)B + A\psi(B)$ for all $A, B \in \mathcal{A}$. Additionally, we say $\psi$ is $*$-derivation if $\psi(A^*) = A^*$ for all $A \in \mathcal{A}$. Obviously, every $*$-derivation is a derivation. In particular, a derivation $\psi$ is called inner if there exists some $X \in \mathcal{A}$ such that $\psi(A) = AX - XA$ for all $A \in \mathcal{A}$. We say a map (not necessarily linear) $\psi$ from $\mathcal{A}$ to itself Lie derivation if

$$\psi([A, B]) = [\psi(A), B] + [A, \psi(B)]$$

for all $A, B \in \mathcal{A}$. Similarly, a map (not necessarily linear) $\psi : \mathcal{A} \to \mathcal{A}$ is called skew Lie derivation or $*$-Lie derivation if

$$\psi([A, B]_*) = [\psi(A), B]_* + [A, \psi(B)]_*,$$

for all $A, B \in \mathcal{A}$. In [4], Daif initially proved that each nonadditive derivation is additive on a 2-torsion free prime ring containing a nontrivial idempotent. Many authors have studied nonadditive derivations corresponding to these products on some operator algebras and have obtained several results. A growing interest in this field can be seen in [3, 5, 11, 12, 14, 16, 19, 20]. Furthermore, without linearity assumption, if a map $\psi : \mathcal{A} \to \mathcal{A}$ satisfies

$$\psi([[A, B], C]) = [[\psi(A), B], C] + [[A, \psi(B)], C] + [[A, B], \psi(C)]$$

for all $A, B, C \in \mathcal{A}$, then $\psi$ is called a multiplicative Lie triple derivation. The characterization of Lie triple derivations has attracted several authors attention, see for example [7, 8, 10, 17] and references therein. Similarly, a multiplicative
map \( \psi : A \rightarrow A \) is called multiplicative skew Lie triple derivation or \( * \)-Lie triple derivations if it satisfy

\[
\psi([[A, B], C]) = [\psi(A), B] + [[A, \psi(B)], C] + [[A, B], \psi(C)],
\]

for all \( A, B, C \in A \). Recently, Ashraf et al. [1] studied \( * \)-Lie triple derivations on standard operator algebras. Precisely, they established the following: Let \( A \) be a standard operator algebra on infinite dimensional complex Hilbert space \( H \) containing identity operator \( I \). If \( A \) is closed under adjoint operator and \( \delta : A \rightarrow B(H) \) is a multiplicative \( * \)-Lie triple derivation, then \( \delta \) is a linear \( * \)-derivation. Moreover, if there exists an operator \( S \in B(H) \) such that \( S + S^* = 0 \), then \( \delta(U) = US - SU \) for all \( U \in A \). Furthermore, they extended the case to a multiplicative \( * \)-Lie higher derivation on \( A \).

Inspired by Lie and skew Lie products, very recently Kong and Zhang [9] introduced the new product, namely bi-skew Lie product as \( [A, B] = AB - BA^* \) for all \( A, B \in A \). They proved that any multiplicative bi-skew Lie derivation i.e., a map \( \psi \) from \( A \) to itself satisfying

\[
\psi([A, B]) = [\psi(A), B] + [A, \psi(B)],
\]

for all \( A, B \in A \), is an additive \( * \)-derivation on \( A \) provided \( \text{dim}(A) \geq 2 \). Similarly, a mapping \( \psi : A \rightarrow A \) is called multiplicative bi-skew Lie triple derivation if it satisfy the condition

\[
\psi([[A, B], C]) = [\psi(A), B] + [[A, \psi(B)], C] + [[A, B], \psi(C)],
\]

for all \( A, B, C \in A \). Motivated by the above mentioned works, we will concentrate on giving a description of multiplicative bi-skew Lie triple derivation on a factor von Neumann algebra.

## 2. Preliminaries and Main Result

Before beginning detailed demonstration and stating our main result, we need to give some notation and preliminaries. Throughout the paper, unless otherwise mentioned, \( A \) represents a factor von Neumann algebra. As usual, \( \mathbb{R} \) and \( \mathbb{C} \) denote respectively the real field and complex field. Let \( H \) be a complex Hilbert space. We denote by \( B(H) \) the algebra of all bounded linear operators on \( H \). Let \( A \subseteq B(H) \) be a von Neumann algebra. Recall that \( A \) is a factor if its center is \( \mathbb{C}I \), where \( I \) is the identity of \( A \).

From ring theoretic perspective, standard operator algebras and factor von Neumann algebras are both prime. Recall that an algebra \( A \) is prime if \( AB = 0 \) implies either \( A = 0 \) or \( B = 0 \). Every standard operator algebra has the center \( \mathbb{C}I \), which is also the center of an arbitrary factor von Neumann algebra. An operator \( P \in B(H) \) is said to be a projection provided \( P^2 = P \) and \( P^2 = P \). Any operator \( A \in B(H) \) can be expressed as \( A = aI + i\xi I \), where \( i \) is the imaginary unit, \( \text{Re}A = \frac{a + a^*}{2} \) and \( \text{Im}A = \frac{a - a^*}{2i} \). Note that both \( \text{Re}A \) and \( \text{Im}A \) are self-adjoint.

The key task of this section is to prove our main theorem.

**Main Theorem.** Let \( A \) be a factor von Neumann algebra on a complex Hilbert space \( H \) with \( \text{dim}(A) \geq 2 \) and \( \psi : A \rightarrow A \) be a multiplicative bi-skew Lie triple
derivation. Then ψ is an additive *-derivation on A.

Let \( P_1 \in A \) be a projection. Write \( P_2 = I - P_1 \) and \( A_{ij} = P_i A P_j \). Then \( A = A_{11} + A_{12} + A_{21} + A_{22} \). Let \( \mathcal{S} = \{ \mathcal{A} \in A | \mathcal{A}^* = \mathcal{A} \} \) and \( \mathcal{G} = \{ \mathcal{A} \in A | \mathcal{A}^* = -\mathcal{A} \} \). \( \mathcal{S}_{12} = \{ P_1 S P_2 + P_2 S P_1 | S \in \mathcal{S} \} \) and \( \mathcal{G}_{ii} = P_i \mathcal{G} P_i \) \( (i = 1, 2) \). Thus, for every \( S \in \mathcal{G}, S = S_{11} + S_{12} + S_{22} \) for every \( S_{12} \in \mathcal{G}_{12} \) and \( S_{ii} \in \mathcal{G}_{ii} \) \( (i = 1, 2) \).

**Lemma 2.1.** \( \psi(0) = 0 \).

**Proof.** It follows that
\[
\psi(0) = \psi([0,0],0) = \psi([0,0],0) + \psi(0) = 0.
\]

**Lemma 2.2.** \( \psi(S)^* = -\psi(S) \) for every \( S \in \mathcal{G} \).

**Proof.** Observe, for any \( S \in \mathcal{G} \) that \( S = [-\frac{1}{2}I, S], \frac{1}{2}I \). Thus
\[
(2.1) \quad \psi(S) = \psi([-\frac{1}{2}I, S], \frac{1}{2}I)
\]
\[
= \psi(-\frac{1}{2}I, S), \frac{1}{2}I + \psi(-\frac{1}{2}I, \psi(S)), \frac{1}{2}I + \psi(-\frac{1}{2}I, S), \frac{1}{2}I
\]
\[
= \psi(-\frac{1}{2}I, S)^* - S\psi(-\frac{1}{2}I)^* + \frac{1}{2}\psi(S)^* + \frac{1}{2}\psi(S)^* - S\psi(-\frac{1}{2}I)^* - \frac{1}{2}\psi(S)^*.
\]
This implies
\[
(2.2) \quad \psi(S) = -\psi(S)^* + 2\psi(-\frac{1}{2}I)S^* - 2S\psi(-\frac{1}{2}I)^*
\]
\[
+ 2S\psi(-\frac{1}{2}I)^* - 2\psi(-\frac{1}{2}I)S^*.
\]
It follows that
\[
(2.3) \quad \psi(S)^* = -\psi(S) + 2S\psi(-\frac{1}{2}I)^* - 2\psi(-\frac{1}{2}I)S^*
\]
\[
+ 2\psi(-\frac{1}{2}I)S^* - 2S\psi(-\frac{1}{2}I)^*.
\]
Addition of (2.2) and (2.3) yields \( \psi(S)^* = -\psi(S) \). This completes the proof. □

**Lemma 2.3.** For any \( \mathcal{A}_{11} \in \mathcal{S}_{11}, \mathcal{B}_{12} \in \mathcal{S}_{12} \) and \( \mathcal{C}_{22} \in \mathcal{G}_{22} \), we have

(i) \( \psi(\mathcal{A}_{11} + \mathcal{B}_{12}) = \psi(\mathcal{A}_{11}) + \psi(\mathcal{B}_{12}) \);

(ii) \( \psi(\mathcal{B}_{12} + \mathcal{C}_{22}) = \psi(\mathcal{B}_{12}) + \psi(\mathcal{C}_{22}) \).

**Proof.** (i) Assume that \( \mathcal{T} = \psi(\mathcal{A}_{11} + \mathcal{B}_{12}) - \psi(\mathcal{A}_{11}) - \psi(\mathcal{B}_{12}) \). It is obvious that \( \mathcal{T} \in \mathcal{G} \), so it follows from Lemma 2.2 that \( \mathcal{T}^* = -\mathcal{T} \). Our aim is to show \( \mathcal{T} = 0 \).
We have
\[
\psi([P_2, A_{11} + B_{12}], P_1) = \psi([P_2, A_{11}], P_1) + \psi([P_2, B_{12}], P_1)
\]
\[
= [\psi(P_2), A_{11}, P_1] + [[P_2, \psi(A_{11})], P_1] + [[P_2, \psi(B_{12})], P_1] + [P_2, \psi(P_1)]
\]
\[
+ \psi([P_2, A_{11} + B_{12}], P_1) + [P_2, \psi(A_{11} + B_{12})], P_1]
\]

On the other hand,
\[
\psi([P_2, A_{11} + B_{12}], P_1) = [\psi(P_2), A_{11} + B_{12}, P_1] + [P_2, \psi(A_{11} + B_{12})], P_1]
\]
\[
+ [P_2, \psi(A_{11} + B_{12})], \psi(P_1)
\]

It follows from the last two relations that \([P_2, T], P_1] = 0\). Which gives \(T_{12} = \mathcal{T}_{21} = 0\). Next, since \([P_1, B_{12}], P_1] = 0\), so we have
\[
[[\psi(P_1), A_{11} + B_{12}], P_1] + [P_1, \psi(A_{11} + B_{12})], P_1]
\]
\[
+ \psi([P_1, A_{11} + B_{12}], P_1)
\]
\[
= [\psi(P_1), A_{11}, P_1] + [[P_1, \psi(A_{11})], P_1] + [[P_1, \psi(B_{12})], P_1] + [P_1, \psi(P_1)]
\]
\[
+ \psi([P_1, A_{11} + B_{12}], P_1) + [P_1, \psi(A_{11} + B_{12})], P_1]
\]
\[
+ [P_1, \psi(A_{11} + B_{12})], \psi(P_1)
\]

From this, we get \([P_1, T], P_1] = 0\). In view of Lemma 2.2, we obtain from the last relation that \(T_{11} = 0\). In a similar manner, one can easily get \(T_{22} = 0\). Therefore \(T = 0\) i.e.,
\[
\psi(A_{11} + B_{12}) = \psi(A_{11}) + \psi(B_{12})
\]

We can establish \(ii\) in the similar manner. Hence the proof.

\[\square\]

**Lemma 2.4.** For any \(A_{11} \in S_{11}, B_{12} \in S_{12}\) and \(C_{22} \in S_{22}\), we have
\[
\psi(A_{11} + B_{12} + C_{22}) = \psi(A_{11}) + \psi(B_{12}) + \psi(C_{22})
\]

**Proof.** Let \(T = \psi(A_{11} + B_{12} + C_{22}) - \psi(A_{11}) - \psi(B_{12}) - \psi(C_{22})\). It follows from Lemma 2.3 and \([P_1, C_{22}], P_2] = 0\) that
\[
\psi([P_1, A_{11} + B_{12} + C_{22}], P_2] = 0
\]

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Proof. For any $i\in A$, Lemma 2.6.

Thus from the last two expressions, we have $[\{P_1, T\}, \{P_1, P_2\}] = 0$. This together with the fact $T^* = -T$ imply that $T_{12} = T_{21} = 0$. We now show that $T_{11} = T_{22} = 0$. Next, observe that $[\{P_2 - P_1, C_{22}\}, iF] = 0$. Reasoning as above, we obtain $T_{11} = T_{22} = 0$ i.e., $T = 0$. Hence the result.

Lemma 2.5. For any $A_{12}, B_{12} \in \mathcal{S}_{12}$, we have

$$\psi(A_{12} + B_{12}) = \psi(A_{12}) + \psi(B_{12}).$$

Proof. For any $A_{12}, B_{12} \in A_{12}$. Assume that $A_{12} = \chi_{12} - \chi_{12}^{*} \in \mathcal{S}_{12}$ and $B_{12} = \psi_{12} - \psi_{12}^{*} \in \mathcal{S}_{12}$. Thus,

$$[\{P_1 + i\chi_1 + i\chi_1^{*}, iP_2 + i\psi_1 + i\psi_1^{*}\}, \frac{1}{2}F] = (\chi_{12} - \chi_{12}^{*}) + (\psi_{12} - \psi_{12}^{*}) + (\chi_{12}\psi_{12}^{*} + \chi_{12}^{*}\psi_{12} - \psi_{12}^{*}\psi_{12} - \psi_{12}\psi_{12}^{*}) = A_{12} + B_{12} + A_{12}^{*}B_{12} - B_{12}^{*}A_{12}.

Note that $A_{12}B_{12}^{*} - B_{12}A_{12}^{*} = \chi_{12}\psi_{12}^{*} - \psi_{12}\chi_{12}^{*} + \chi_{12}\psi_{12} - \psi_{12}\chi_{12} = \psi_{11}^{*} + \psi_{22}$, where $\psi_{11} = \chi_{12}\psi_{12}^{*} - \psi_{12}\chi_{12}^{*} \in \mathcal{S}_{11}$ and $\psi_{22} = \psi_{12}\psi_{12}^{*} - \psi_{12}^{*}\psi_{12} \in \mathcal{S}_{22}$. Since $i\chi_{12} + i\chi_{12}^{*}, i\psi_{12} + i\psi_{12}^{*} \in \mathcal{S}_{12}$, so it follows from Lemma 2.3 and 2.4 that

$$\psi(A_{12} + B_{12}) + \psi(C_{11}) + \psi(C_{22}) = \psi(A_{12} + B_{12} + C_{11} + C_{22}) = \psi(A_{12} + B_{12} + A_{12}^{*}B_{12} - B_{12}^{*}A_{12}) = \psi([\{P_1 + i\chi_1 + i\chi_1^{*}, iP_2 + i\psi_1 + i\psi_1^{*}\}, \frac{1}{2}F]) + \psi([iP_1 + iP_2, \frac{1}{2}F]) + \psi([i\chi_1, \frac{1}{2}F] + \psi([i\psi_1, \frac{1}{2}F]) + \psi([i\psi_2, \frac{1}{2}F]), iP_2 + i\psi_1 + i\psi_1^{*}\frac{1}{2}F] = \psi(\{P_1 + i\chi_1 + i\chi_1^{*}, iP_2 + i\psi_1 + i\psi_1^{*}\}, \frac{1}{2}F]) = \psi([iP_1 + iP_2, \frac{1}{2}F]) + \psi([i\chi_1, \frac{1}{2}F] + \psi([i\psi_1, \frac{1}{2}F]) + \psi([i\psi_2, \frac{1}{2}F]) = \psi(A_{12}) + \psi(B_{12}) + \psi(A_{12}^{*}B_{12} - B_{12}^{*}A_{12}) = \psi(A_{12}) + \psi(B_{12} + \psi(C_{11}) + \psi(C_{22}).

Thus, we have $\psi(A_{12} + B_{12}) = \psi(A_{12}) + \psi(B_{12})$. Thereby the proof is completed.

Lemma 2.6. For every $A_{ii}, B_{ii} \in \mathcal{S}_{ii}$ ($i = 1, 2$), we have

(i) $\psi(A_{11} + B_{11}) = \psi(A_{11}) + \psi(B_{11})$;

(ii) $\psi(A_{22} + B_{22}) = \psi(A_{22}) + \psi(B_{22})$. 
Proof. Let $\mathcal{T} = \psi(\mathcal{A}_1 + \mathcal{B}_1) - \psi(\mathcal{A}_1) - \psi(\mathcal{B}_1)$. We shall prove $\mathcal{T} = 0$. To show this, see that

$$\psi([[P_2, \mathcal{A}_1 + \mathcal{B}_1], \mathcal{P}_1]) = \psi([[P_2, \mathcal{A}_1], \mathcal{P}_1]) + \psi([[P_2, \mathcal{B}_1], \mathcal{P}_1])$$

$$+ [[P_2, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \mathcal{P}_1] + [[P_2, \psi(\mathcal{A}_1)], \mathcal{P}_1]$$

$$+ [[P_2, \psi(\mathcal{B}_1)], \mathcal{P}_1] + [[P_2, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \psi(\mathcal{P}_1)]$$

$$+ [[P_2, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \psi(\mathcal{P}_1)].$$

Alternatively,

$$\psi([[P_2, \mathcal{A}_1 + \mathcal{B}_1], \mathcal{P}_1]) = [[\psi(P_2), \mathcal{A}_1 + \mathcal{B}_1], \mathcal{P}_1]$$

$$+ [[P_2, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \mathcal{P}_1] + [[P_2, \psi(\mathcal{A}_1)], \mathcal{P}_1]$$

Thus, we have $[[P_2, \mathcal{T}, \mathcal{P}_1], \mathcal{T}] = 0$, and hence $\mathcal{T}_{12} = \mathcal{T}_{21} = 0$. Observe next that, for any $\mathcal{X}, \mathcal{Y} \subset \mathcal{A}, \mathcal{Z} = \mathcal{X}_{12} - \mathcal{Y}_{12} \in \mathcal{S}_{12}$. Then $[[\mathcal{C}, \mathcal{A}_1], \mathcal{X}], [[\mathcal{C}, \mathcal{B}_1], \mathcal{X}] \in \mathcal{S}_{12}$. Therefore, it follows from Lemma 2.5 that

$$[[\psi(\mathcal{C}), \mathcal{A}_1 + \mathcal{B}_1], \mathcal{X}] + \frac{1}{2} \mathcal{Y} + [[\mathcal{C}, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \mathcal{X}] + \frac{1}{2} \mathcal{Y}$$

$$+ [[\mathcal{C}, \mathcal{A}_1 + \mathcal{B}_1], \psi(\frac{1}{2} \mathcal{Y})]$$

$$= \psi([[\mathcal{C}, \mathcal{A}_1 + \mathcal{B}_1], \mathcal{X}] + \frac{1}{2} \mathcal{Y})$$

$$= \psi([[\mathcal{C}, \mathcal{A}_1 + \mathcal{B}_1], \mathcal{X}] + \frac{1}{2} \mathcal{Y}) + \psi([[\mathcal{C}, \mathcal{B}_1], \mathcal{X}] + \frac{1}{2} \mathcal{Y})$$

$$= [[\psi(\mathcal{C}), \mathcal{A}_1 + \mathcal{B}_1], \mathcal{X}] + \frac{1}{2} \mathcal{Y} + [[\mathcal{C}, \psi(\mathcal{A}_1 + \mathcal{B}_1)], \mathcal{X}] + \frac{1}{2} \mathcal{Y}$$

Using the similar arguments as used above, we get $[[\mathcal{C}, \mathcal{T}], \mathcal{X}] + \frac{1}{2} \mathcal{Y} = 0$. This leads to $\mathcal{T}_{11} = \mathcal{T}_{22} = 0$, which completes the proof. \hfill \square

Remark 2.7. It follows from Lemma 2.3–2.6 that $\psi$ is additive on $\mathcal{S}$.

Lemma 2.8. $\psi(\mathcal{I}) = 0$.

Proof. (i) In view of Remark 2.7 that

$$\psi(4i\mathcal{I}) = \psi([[i, \mathcal{I}], \mathcal{I}]) = \psi(4i\mathcal{I}) + 4i(\psi(\mathcal{I}) + \psi(\mathcal{I}^*)).$$

This implies $\psi(\mathcal{I}^*) = -\psi(\mathcal{I})$. Also, for any $\mathcal{S} \in \mathcal{S}$, we have

$$\psi(4S) = \psi([[\mathcal{S}, \mathcal{I}], \mathcal{I}]) = \psi(4S) + 2(\psi(\mathcal{I})S + S\psi(\mathcal{I}^*)).$$

Since $\psi(\mathcal{I}^*) = -\psi(\mathcal{I})$, so we have $\mathcal{S}\psi(\mathcal{I}) = \psi(\mathcal{I})S$ for all $\mathcal{S} \in \mathcal{S}$. This implies $\psi(\mathcal{I}) \in i\mathcal{I}$. Without loss of generality, we may assume $\psi(\mathcal{I}) = i\gamma\mathcal{I}$ for some $\gamma \in \mathbb{R}$. We claim that $\gamma = 0$. To prove this, let us assume $\psi(\mathcal{P}_1) = \mathcal{H} + i\mathcal{H}'$, where $\mathcal{H}, \mathcal{H}' \in \mathcal{I}$. Thus, we have

$$0 = \psi([[\mathcal{I}, \mathcal{P}_1], \mathcal{I}^*]),$$

$$= \psi(\mathcal{P}_1^* - \psi(\mathcal{I}))\mathcal{P}_1 - \mathcal{P}_1\psi(\mathcal{I}^*),$$

$$= 2i\gamma\mathcal{P}_1 - 2i\mathcal{H}'.
This gives $\mathcal{H}^\prime = \gamma \mathcal{P}_1$, and hence $\psi(\mathcal{P}_1) = \mathcal{H} + i\gamma \mathcal{P}_1$. Next, for any $\mathcal{A} \in \mathcal{G}$ observe that $[[\mathcal{A}, i\mathcal{I}]*, \mathcal{I}]_* = 0$. Thus $\psi([[\mathcal{A}, i\mathcal{I}]*, \mathcal{I}]_*) = 0$ gives $\psi(i\mathcal{I}) \in i\mathbb{R}$. Assume, for some $\eta \in \mathbb{R}$ that $\psi(i\mathcal{I}) = i\eta \mathcal{I}$. Now, in view of Remark 2.7, we have

$$4\psi(i\mathcal{P}_1) = \psi([[i\mathcal{I}, \mathcal{P}_1]*, \mathcal{I}]*)$$

$$= [[\psi(i\mathcal{I}), \mathcal{P}_1]*, \mathcal{I}]* + [[i\mathcal{I}, \psi(\mathcal{P}_1)]*, \mathcal{I}]*$$

$$+ [[i\mathcal{I}, \mathcal{P}_1]*, \psi(\mathcal{I})]*$$

$$= [[i\eta \mathcal{I}, \mathcal{P}_1]*, \mathcal{I}]* + [[i\mathcal{I}, \mathcal{H} + i\gamma \mathcal{P}_1]*, \mathcal{I}]*$$

$$+ [[i\mathcal{I}, \mathcal{P}_1]*, i\gamma \mathcal{I}]*$$

$$= 4i\eta \mathcal{P}_1 + 4i\mathcal{H}.$$

So, we have $\psi(i\mathcal{P}_1) = i\eta \mathcal{P}_1 + i\mathcal{H}$. Also, we have

$$4\psi(i\mathcal{P}_1) = \psi([[i\mathcal{P}_1, \mathcal{P}_1]*, \mathcal{I}]*)$$

$$= [[\psi(i\mathcal{P}_1), \mathcal{P}_1]*, \mathcal{I}]* + [[i\mathcal{P}_1, \psi(\mathcal{P}_1)]*, \mathcal{I}]*$$

$$+ [[i\mathcal{P}_1, \mathcal{P}_1]*, \psi(\mathcal{I})]*$$

$$= [[i\eta \mathcal{P}_1 + i\mathcal{H}, \mathcal{P}_1]*, \mathcal{I}]* + [[i\mathcal{P}_1, \mathcal{H} + i\gamma \mathcal{P}_1]*, \mathcal{I}]*$$

$$+ [[i\mathcal{P}_1, \mathcal{P}_1]*, i\gamma \mathcal{I}]*$$

$$= 4i\eta \mathcal{P}_1 + 4i\mathcal{H} \mathcal{P}_1 + 4i\mathcal{P}_1 \mathcal{H}.$$

We obtain from (2.6) and (2.7) that

$$\mathcal{H} = \mathcal{H} \mathcal{P}_1 + \mathcal{P}_1 \mathcal{H}.$$

Multiply (2.8) by $\mathcal{P}_1$ from left and $\mathcal{P}_2$ from right, alternatively. This yields

$$\mathcal{P}_1 \mathcal{H} \mathcal{P}_1 = \mathcal{P}_2 \mathcal{H} \mathcal{P}_2 = 0.$$

Let $\mathcal{A}_1 \in \mathcal{A}$ and taking $\mathcal{X} = \mathcal{A}_1 - \mathcal{A}_2 \in \mathcal{G}$. Then, it follows from Remark 2.7 and Lemma 2.2 that

$$2\psi(\mathcal{X}) = -\psi([[\mathcal{P}_1, \mathcal{X}]*, \mathcal{I}]*$$

$$= -[[\psi(\mathcal{P}_1), \mathcal{X}]*, \mathcal{I}]* - [[\mathcal{P}_1, \psi(\mathcal{X})]*, \mathcal{I}]*$$

$$= 2\psi(\mathcal{P}_1)\mathcal{X} + \mathcal{X}\psi(\mathcal{P}_1)^* + \mathcal{P}_1\psi(\mathcal{X}) + \psi(\mathcal{X})\mathcal{P}_1.$$

Multiply above equation by $\mathcal{P}_1$ from left and $\mathcal{P}_2$ from right. So, we get

$$\mathcal{P}_1\psi(\mathcal{P}_1)\mathcal{X}\mathcal{P}_2 + \mathcal{P}_1\mathcal{X}\psi(\mathcal{P}_1)^*\mathcal{P}_2 = 0.$$

Since $\mathcal{X} = \mathcal{A}_1 - \mathcal{A}_2^*$, so we obtain

$$\mathcal{P}_1\psi(\mathcal{P}_1)\mathcal{A}_1 + \mathcal{A}_2\psi(\mathcal{P}_1)^*\mathcal{P}_2 = 0.$$

Also, as $\psi(\mathcal{P}_1) = \mathcal{H} + i\gamma \mathcal{P}_1$, it follows that

$$i\gamma \mathcal{A}_1 + \mathcal{P}_1 \mathcal{H} \mathcal{A}_1 + \mathcal{A}_1 \mathcal{H} \mathcal{P}_2 = 0.$$

Thus,

$$i\gamma \mathcal{A}_1 + (\mathcal{P}_1 \mathcal{H} \mathcal{P}_1) \mathcal{A}_2 + \mathcal{P}_1 \mathcal{A} (\mathcal{P}_2 \mathcal{H} \mathcal{P}_2) = 0.$$

In view of (2.9), we have $\gamma \mathcal{A}_1 = 0$. Therefore $\gamma = 0$, and hence $\psi(\mathcal{I}) = 0$. □

**Lemma 2.9.** For any $\mathcal{H} \in \mathcal{H}$, $\psi(\mathcal{H})^* = \psi(\mathcal{H})$.  

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Proof. Since for any $H \in \mathfrak{S}$, $[[\mathcal{I}, H], \mathcal{I}] = 0$, so from Lemma 2.8 and the hypothesis, we have
\begin{equation}
0 = \psi([[\mathcal{I}, H], \mathcal{I}]) = [[\mathcal{I}, \psi(H)], \mathcal{I}] = \psi(H) - \psi(H)^*.
\end{equation}
Therefore, we have $\psi(H)^* = \psi(H)$ for all $H \in \mathfrak{S}$. \hfill \Box

Lemma 2.10. For any $H \in \mathfrak{S}$, $\psi(iH) = i\psi(H) + i\eta H$.

Proof. In view of Lemma 2.2, 2.8 and 2.9, we have
\begin{equation}
\psi([i, \mathcal{I}], H) = [[\psi(i, \mathcal{I}), \mathcal{I}], H] + [[i, \mathcal{I}], \psi(H)] = 4\psi(i, \mathcal{I})H + 4\psi(H).
\end{equation}
Also, since $\psi([i, \mathcal{I}], H) = 4\psi(iH)$ and $\psi(i, \mathcal{I}) = i\eta$, so $\psi(iH) = i\psi(H) + i\psi(H)$. \hfill \Box

Lemma 2.11. $\psi$ is additive on $\mathfrak{S}$.

Proof. We know that $iH, iH' \in \mathfrak{S}$, where $H, H' \in \mathfrak{S}$. Then, in view of Remark 2.7 and Lemma 2.10, we have
\begin{equation}
\psi(iH + iH') = \psi(iH) + \psi(iH') = i\psi(H) + i\psi(H') + i\eta(H + H').
\end{equation}
Also
\begin{equation}
\psi(i(H + H')) = i\psi(H + H') + i\eta(H + H').
\end{equation}
From (2.17) and (2.18), we have the desired result. \hfill \Box

Lemma 2.12. $\psi(\alpha^*) = \psi(\alpha)^*$ for all $\alpha \in \mathcal{A}$.

Proof. It follows, for any $H, H' \in \mathfrak{S}$, Remark 2.7, Lemma 2.8, 2.10 and $[[H, \mathcal{I}], \mathcal{I}] = 0$ that
\begin{equation}
\psi([iH, \mathcal{I}], \mathcal{I}) = \psi([[H, \mathcal{I}], \mathcal{I}], \mathcal{I}) + \psi([iH', \mathcal{I}], \mathcal{I}) = 4i\psi(H') + i\eta H' + 4i\eta H.
\end{equation}
On the other hand,
\begin{equation}
\psi([iH, \mathcal{I}], \mathcal{I}) = 4i\psi(H') + i\eta H'.
\end{equation}
So, we have from (2.19) and (2.20) that
\begin{equation}
4i\psi(H') + i\eta H' = 2\psi(H + iH') - \psi(H + iH'^*).
\end{equation}
Also, note that $[i, \mathcal{I}, H, \mathcal{I}] = 0$, so we have
\begin{equation}
4i\psi(H) + i\eta H = 2i\psi(H + iH') + \psi(H + iH'^*) + 4i\eta H.
\end{equation}
In view of (2.21) and (2.22), we get
\begin{equation}
\psi(H + iH') = \psi(H) + i\psi(H') + i\eta H'.
\end{equation}
Next, since we know that any element $\alpha \in \mathcal{A}$ can be written as $\alpha = H + iH'$ for all $H, H' \in \mathfrak{S}$, so it follows from (2.23), Lemma 2.9 and 2.11 that
\begin{equation}
\psi(\alpha)^* = \psi(H + iH')^* = (\psi(H) + i\psi(H') + i\eta H')^* = \psi(H) - i\psi(H') - i\eta H' = \psi(H - iH') = \psi(\alpha^*).
This gives the assertion.

\textbf{Lemma 2.13.} \( \psi \) is additive on \( \mathcal{A} \).

\textit{Proof.} Let \( \mathcal{A}, \mathcal{B} \in \mathcal{A} \) such that \( \mathcal{A} = \mathcal{H}_1 + i\mathcal{H}_2 \) and \( \mathcal{B} = \mathcal{X}_1 + i\mathcal{X}_2 \) for all \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{X}_1, \mathcal{X}_2 \in \mathcal{H} \). Observe from (2.23) and Lemma 2.11

\begin{equation}
\psi(\mathcal{A} + \mathcal{B}) = \psi((\mathcal{H}_1 + \mathcal{X}_1) + i(\mathcal{H}_2 + \mathcal{X}_2)) \nonumber
\end{equation}

\begin{equation}
= \psi(\mathcal{H}_1 + \mathcal{X}_1) + i\psi(\mathcal{H}_2 + \mathcal{X}_2) + i\eta(\mathcal{H}_2 + \mathcal{X}_2) \nonumber
\end{equation}

\begin{equation}
= (\psi(\mathcal{H}_1) + i\psi(\mathcal{H}_2) + i\eta\mathcal{H}_2) + (\psi(\mathcal{X}_1) + i\psi(\mathcal{X}_2) + i\eta\mathcal{X}_2) \nonumber
\end{equation}

\begin{equation}
= \psi(\mathcal{H}_1 + i\mathcal{H}_2) + \psi(\mathcal{X}_1 + i\mathcal{X}_2) \nonumber
\end{equation}

\begin{equation}
= \psi(\mathcal{A}) + \psi(\mathcal{B}). \nonumber
\end{equation}

Hence the result.

\textbf{Lemma 2.14.} \( \psi(i\mathcal{A}) = 0 \).

\textit{Proof.} Since we know from Lemma 2.8 that \( \psi(i\mathcal{A}) = i\eta \mathcal{A} \). Our aim is to show \( \eta = 0 \). Also, we know that \( \psi(i\mathcal{P}_1) = i\eta\mathcal{P}_1 + i\mathcal{H} \). In view of (2.8) and (2.9), we have

\begin{equation}
\psi(i\mathcal{P}_1) = i\eta\mathcal{P}_1 + i\mathcal{P}_1\mathcal{H}\mathcal{P}_2 + i\mathcal{P}_2\mathcal{H}\mathcal{P}_1. \nonumber
\end{equation}

Observe, for any \( \mathcal{A}_{12} \in \mathcal{A} \) that

\begin{equation}
\psi([[i\mathcal{P}_1, \mathcal{A}_{12} - \mathcal{A}_{12}^*] \mathcal{A}]_\bullet) = -2i\psi(i(\mathcal{A}_{12} + \mathcal{A}_{12}^*)). \nonumber
\end{equation}

In view of Lemma 2.10, we have

\begin{equation}
2i\psi(i(\mathcal{A}_{12} + \mathcal{A}_{12}^*)) = -2i(\psi(\mathcal{A}_{12}) + \psi(\mathcal{A}_{12})^* + \eta(\mathcal{A}_{12} + \mathcal{A}_{12}^*)). \nonumber
\end{equation}

Thus,

\begin{equation}
\psi([[i\mathcal{P}_1, \mathcal{A}_{12} - \mathcal{A}_{12}^*] \mathcal{A}]_\bullet) = -2i(\psi(\mathcal{A}_{12}) + \psi(\mathcal{A}_{12})^* + \eta(\mathcal{A}_{12} + \mathcal{A}_{12}^*)). \nonumber
\end{equation}

Alternatively, from (2.26) and Lemma 2.8, we have

\begin{equation}
\psi([[i\mathcal{P}_1, \mathcal{A}_{12} - \mathcal{A}_{12}^*] \mathcal{A}]_\bullet) \nonumber
\end{equation}

\begin{equation}
= [[\psi(i\mathcal{P}_1), \mathcal{A}_{12} - \mathcal{A}_{12}^*] \mathcal{A}]_\bullet + [[i\mathcal{P}_1, \psi(\mathcal{A}_{12} - \mathcal{A}_{12}^*)] \mathcal{A}]_\bullet \nonumber
\end{equation}

\begin{equation}
= [[i\eta\mathcal{P}_1 + i\mathcal{P}_1\mathcal{H}\mathcal{P}_2 + i\mathcal{P}_2\mathcal{H}\mathcal{P}_1, \mathcal{A}_{12} - \mathcal{A}_{12}^*] \mathcal{A}]_\bullet \nonumber
\end{equation}

\begin{equation}
+ [[i\mathcal{P}_1, \psi(\mathcal{A}_{12}) - \psi(\mathcal{A}_{12}^*)] \mathcal{A}]_\bullet \nonumber
\end{equation}

\begin{equation}
= 2(i\eta\mathcal{P}_1 + i\mathcal{P}_1\mathcal{H}\mathcal{P}_2 + i\mathcal{P}_2\mathcal{H}\mathcal{P}_1)(\mathcal{A}_{12}^* - \mathcal{A}_{12}) \nonumber
\end{equation}

\begin{equation}
+ 2(\mathcal{A}_{12} - \mathcal{A}_{12}^*)(i\eta\mathcal{P}_1 + i\mathcal{P}_1\mathcal{H}\mathcal{P}_2 + i\mathcal{P}_2\mathcal{H}\mathcal{P}_1) \nonumber
\end{equation}

\begin{equation}
+ 2i\mathcal{P}_1(\psi(\mathcal{A}_{12})^* - \psi(\mathcal{A}_{12})) + 2i(\psi(\mathcal{A}_{12}) - \psi(\mathcal{A}_{12}^*))\mathcal{P}_1. \nonumber
\end{equation}

Now from (2.27) and (2.28), we obtain

\begin{equation}
\psi(\mathcal{A}_{12}) - \psi(\mathcal{A}_{12})^* - \eta(\mathcal{A}_{12} - \mathcal{A}_{12}^*) \nonumber
\end{equation}

\begin{equation}
= (\eta\mathcal{P}_1 + \mathcal{P}_1\mathcal{H}\mathcal{P}_2 + \mathcal{P}_2\mathcal{H}\mathcal{P}_1)(\mathcal{A}_{12}^* - \mathcal{A}_{12}) \nonumber
\end{equation}

\begin{equation}
+ (\mathcal{A}_{12} - \mathcal{A}_{12}^*)(\eta\mathcal{P}_1 + \mathcal{P}_1\mathcal{H}\mathcal{P}_2 + \mathcal{P}_2\mathcal{H}\mathcal{P}_1) \nonumber
\end{equation}

\begin{equation}
+ \mathcal{P}_1(\psi(\mathcal{A}_{12})^* - \psi(\mathcal{A}_{12})) + (\psi(\mathcal{A}_{12}) - \psi(\mathcal{A}_{12}^*))\mathcal{P}_1. \nonumber
\end{equation}
Multiply (2.29) by \( P_1 \) from left and \( P_2 \) from right, we get \( P_1 \psi(\mathcal{A}_{12})^* P_2 = 0 \). Next, consider

\[
(2.30) \quad 2(\psi(\mathcal{A}_{12}) - \psi(\mathcal{A}_{12})^*) = \psi([iP_1, i(\mathcal{A}_{12} + \mathcal{A}_{12}^*)] \cdot \mathcal{I}, \cdot) = \psi([i\psi(P_1), i(\mathcal{A}_{12} + \mathcal{A}_{12}^*)] \cdot \mathcal{I}, \cdot) + [[iP_1, \psi(i(\mathcal{A}_{12} + \mathcal{A}_{12}^*))] \cdot \mathcal{I}, \cdot].
\]

It follows from (2.31) and (2.32) that \( \psi = 0 \).

\[\eta \mathcal{A}_{12} = 0. \]

This complete the proof. \( \square \)

**Lemma 2.15.** \( \psi(i \mathcal{A}) = i \psi(\mathcal{H}) \) for all \( \mathcal{A} \in \mathcal{A} \).

**Proof.** It follows from Lemma 2.10 and 2.14 that \( \psi(i \mathcal{H}) = i \psi(\mathcal{H}) \) for all \( \mathcal{H} \in \mathcal{Y} \). Thus, for any \( \mathcal{A} \in \mathcal{A} \) and \( \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{Y} \) and using the fact that \( \psi \) is additive on \( \mathcal{A} \), we have

\[\psi(i \mathcal{A}) = \psi(i \mathcal{H}_1 - \mathcal{H}_2) = i \psi(\mathcal{H}_1) - i \psi(\mathcal{H}_2) = i(\psi(\mathcal{H}_1) + i \psi(\mathcal{H}_2)) = i \psi(\mathcal{A}).\]

This gives the result. \( \square \)

**Lemma 2.16.** \( \psi \) is a derivation on \( \mathcal{A} \).

**Proof.** First we prove that \( \psi \) is a derivation on \( \mathcal{Y} \), and then on \( \mathcal{A} \). Let \( \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{Y} \). Then

\[
(2.31) \quad 2\psi(\mathcal{H}_1 \mathcal{H}_2 - \mathcal{H}_2 \mathcal{H}_1) = \psi([\mathcal{H}_1, \mathcal{H}_2] \cdot \mathcal{I}, \cdot) = [\psi(\mathcal{H}_1), \mathcal{H}_2] \cdot \mathcal{I}, \cdot + [[\mathcal{H}_1, \psi(\mathcal{H}_2)] \cdot \mathcal{I}, \cdot] = 2(\psi(\mathcal{H}_1) \mathcal{H}_2 - \mathcal{H}_1 \psi(\mathcal{H}_2) + \mathcal{H}_1 \psi(\mathcal{H}_2) - \psi(\mathcal{H}_1) \mathcal{H}_2).
\]

Also,

\[
(2.32) \quad 2i\psi(\mathcal{H}_1 \mathcal{H}_2 + \mathcal{H}_2 \mathcal{H}_1) = \psi([i\mathcal{H}_1, \mathcal{H}_2] \cdot \mathcal{I}, \cdot) = [\psi(i\mathcal{H}_1), \mathcal{H}_2] \cdot \mathcal{I}, \cdot + [[i\mathcal{H}_1, \psi(\mathcal{H}_2)] \cdot \mathcal{I}, \cdot] = 2i(\psi(\mathcal{H}_1) \mathcal{H}_2 + \mathcal{H}_1 \psi(\mathcal{H}_2) + \mathcal{H}_1 \psi(\mathcal{H}_2) - \psi(\mathcal{H}_1) \mathcal{H}_2).
\]

It follows from (2.31) and (2.32) that \( \psi(\mathcal{H}_1 \mathcal{H}_2) = \psi(\mathcal{H}_1) \mathcal{H}_2 + \mathcal{H}_1 \psi(\mathcal{H}_2) \) for all \( \mathcal{H}_1, \mathcal{H}_2 \in \mathcal{Y} \). Next, for any \( \mathcal{A}, \mathcal{B} \in \mathcal{A} \) assume that \( \mathcal{A} = \mathcal{H}_1 + i \mathcal{H}_2 \) and \( \mathcal{B} = \mathcal{H}_3 + i \mathcal{H}_4 \).
Given the consideration of bi-skew Lie derivations and bi-skew Lie triple derivations, we can further develop them in a natural way. Suppose that \( n \geq 2 \) is a fixed positive integer. Let us see a sequence of polynomials with involution

\[
\begin{align*}
 p_1(X_1) & = X_1, \\
p_2(X_1, X_2) & = [p_1(X_1), X_2]_\bullet = [X_1, X_2]_\bullet, \\
p_3(X_1, X_2, X_3) & = [p_2(X_1, X_2), X_3]_\bullet = [[X_1, X_2]_\bullet, X_3]_\bullet, \\
p_4(X_1, X_2, X_3, X_4) & = [p_3(X_1, X_2, X_3), X_4]_\bullet = [[[X_1, X_2]_\bullet, X_3]_\bullet, X_4]_\bullet, \\
& \quad \vdots \\
p_n(X_1, X_2, \ldots, X_n) & = [p_{n-1}(X_1, X_2, \ldots, X_{n-1}), X_n]_\bullet \\
& = \ldots[[X_1, X_2]_\bullet, X_3]_\bullet, \ldots, X_{n-1}]_\bullet, X_n]_\bullet.
\end{align*}
\]

Accordingly, a multiplicative/nonlinear bi-skew Jordan \( n \)-derivation is a mapping \( \delta : \mathcal{A} \rightarrow \mathcal{A} \) satisfying the condition

\[
(3.1) \quad \delta(p_n(X_1, X_2, \ldots, X_n)) = \sum_{k=1}^{n} p_n(X_1, \ldots, X_{k-1}, \delta(X_k), X_{k+1}, \ldots, X_n),
\]

for all \( X_1, X_2, \ldots, X_n \in \mathcal{A} \). This notion makes the best use of the definition of Lie \( n \)-derivations. By the definition, it is clear that every bi-skew Lie derivation is a bi-skew \( 2 \)-derivation and every bi-skew Lie triple derivation is a bi-skew Lie \( 3 \)-derivation. One can easily check that every multiplicative bi-skew Lie derivation on any \( * \)-algebra is a multiplicative bi-skew Lie triple derivation. But we do not know whether the converse statement is still valid. bi-skew \( 2 \)-derivations, bi-skew \( 3 \)-derivations and bi-skew Lie \( n \)-derivations are collectively referred to as bi-skew Lie-type derivations. This leads us to write the following open problem, which in fact more interesting, as:
**Question 1.** Let $\mathcal{A}$ be a factor von Neumann algebra/a von Neumann algebra on a complex Hilbert space $\mathcal{H}$ and $\psi : \mathcal{A} \to \mathcal{A}$ be a multiplicative bi-skew Lie-type derivation. Then what can we say about the structure of $\psi$?

Let $p_n(X_1, X_2, ..., X_n)$ be the polynomial defined by $n$ indeterminates $X_1, ..., X_n$ and their bi-skew Lie $n$-derivations. Let $\mathbb{N}$ be the set of non-negative integers and $\mathcal{D} = \{\delta_n\}_{n \in \mathbb{N}}$ be a family of nonlinear mappings $\delta_n : \mathcal{A} \to \mathcal{A}$ such that $\delta_0 = \text{id}_\mathcal{A}$, the identity mapping on $\mathcal{A}$. Then $\mathcal{D}$ is called a multiplicative/nonlinear bi-skew Lie $n$-higher derivation if $\mathcal{D}$ satisfies the condition

$$\delta_m(p_n(X_1, X_2, ..., X_n)) = \sum_{i_1+\cdots+i_n=m} p_n(\delta_{i_1}(X_1), \delta_{i_2}(X_2), ..., \delta_{i_n}(X_n))$$

for all $X_1, X_2, ..., X_n \in \mathcal{A}$. In the case of $n = 2$, $\delta_m$ is called a bi-skew Lie higher derivation, and is called a bi-skew Lie triple higher derivation whenever $n = 3$. In the spirit of above literature, we write the following:

**Question 2.** Let $\mathcal{A}$ be a factor von Neumann algebra/a von Neumann algebra on a complex Hilbert space $\mathcal{H}$ and $\psi : \mathcal{A} \to \mathcal{A}$ be a multiplicative bi-skew Lie triple higher derivation. Then what can we say about the structure of $\psi$?

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**References**


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