Borel directions and the uniqueness of algebroid functions *

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Abstract: In this paper, by Nevanlinna theories we discuss the relations between the Borel directions and the uniqueness of algebroid functions. We get several uniqueness theorems of algebroid functions in angular domain which contains the Borel directions. These results extend the achievements of meromorphic functions obtained by some scholars.

Key words: algebroid function, order, uniqueness, angular domain

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1 Introduction and Preparation

Let $W(z)$ be an algebroid function in $\mathbb{D}_R = \{z||z| < R\}$, where $0 < R \leq \infty$. In this paper, we use $\mathbb{C}$ to denote the complex plane, $\overline{\mathbb{C}}$ to denote the extended complex plane. Suppose that $W(z)$ and $M(z)$ are two algebroid functions in $\mathbb{D}_R$, $a \in \mathbb{C}$ and $X \subseteq \mathbb{D}$, we say that $W(z)$ and $M(z)$ share a CM (counting multiplicities) in $X$ provided that $W(z) - a$ and $M(z) - a$ have the same zeros with the same multiplicities in $X$. Similarly, we say that $W(z)$ and $M(z)$ share a IM (ignoring multiplicities) in $X$ provided that $W(z) - a$ and $M(z) - a$ have the same zeros in $X$.

There are many achievements on the uniqueness of meromorphic functions (see[4]). As extention of meromorphic function, the uniqueness of algebroid functions is an important subject in the value distribution theory. But for multi-valued algebroid functions, owing to the complexity of its branch points, the uniqueness achievements of algebroid functions are fewer than those of meromorphic functions.

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The uniqueness problem of algebroid functions was firstly considered by G. Valiron. It is well know that G. Valiron (see [14]) obtained a famous $4v + 1$-valued theorem of the uniqueness of algebroid functions:

Let $W(z)$ and $M(z)$ be two $v$-valued algebroid functions, if $W(z)$ and $M(z)$ share $a_j \in \mathbb{C} (j = 1, 2, \cdots, 4v + 1)$ CM, then $W(z) \equiv M(z)$.

Afterwards some scholars have discussed the uniqueness problem of algebroid functions (see [1],[3],[13],[14]), they have got some achievements. In 2003, Jianhua Zheng firstly took into account uniqueness of meromorphic functions sharing values in an angular domain (see [5],[6]). Since then, the uniqueness of meromorphic functions in an angular domain attracted many investigations (see [2],[7],[11],[15]).

Jianhua Zheng proved the following theorems:

**Theorem A** (see [5]) Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions in $\mathbb{C}$ and for some $a \in \mathbb{C}$ and an integer $p \geq 0$, $\delta = \delta(a, f^{(p)}) > 0$. Assume that for $q$ radii $\arg z = \alpha_j (1 \leq j \leq q)$ satisfying

$$-\pi \leq \alpha_1 < \alpha_2 < \cdots < \alpha_q < \pi, \quad \alpha_{q+1} = \alpha_1 + 2\pi,$$

$f(z)$ and $g(z)$ share five distinct values IM in $X = \mathbb{C} \setminus \bigcup_{j=1}^{q} \{z|\arg z = \alpha_j\}$. If

$$\max \left\{ \frac{\pi}{\alpha_{j+1} - \alpha_j} : 1 \leq j \leq q \right\} < \lambda(f),$$

then $f(z) \equiv g(z)$, where $\lambda(f)$ is the order of $f(z)$.

**Theorem B** (see [6]) Let $f(z)$ and $g(z)$ be both transcendental meromorphic functions in $\mathbb{C}$ and let $f(z)$ be of the finite lower order $\mu$ and for some $a \in \mathbb{C}$, $\delta = \delta(a, f) > 0$. Given one angular domain $X = \{z|\alpha < \arg z < \beta\}$ with $0 \leq \alpha < \beta < 2\pi$ and

$$\beta - \alpha > \max \left\{ \frac{\pi}{\sigma}, 2\pi - \frac{4}{\sigma} \arcsin \left( \frac{\delta}{2} \right) \right\},$$

where $\mu \leq \sigma \leq \lambda$ and $\sigma < \infty$, we assume that $f(z)$ and $g(z)$ share four distinct values $a_j (j = 1, 2, \cdots, 4)$ IM in $X$ and $a_j \neq a(j = 1, 2, \cdots, 4)$, then $f(z) \equiv g(z)$, where $\lambda$ is the order of $f(z)$.

However, there are relatively fewer results of the uniqueness of algebroid functions in angular domain. In this paper, we discuss the relations between the Borel directions and the uniqueness of algebroid functions. By using the conformal mapping and the properties of algebroid functions in the unit disc, we get several uniqueness theorems of algebroid functions in angular domain. These results extend the achievements of meromorphic functions.

Let $A_v(z), A_{v-1}(z), \cdots, A_0(z)$ be a group of holomorphic functions which have no common zeros in the unit disc, then the irreducible equation

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0, \quad (1.1)$$
defines a $v$-valued algebroid function in the unit disc.

The $v$-valued algebroid function $W(z)$ in the unit disc has two kinds of critical points:
(1) The root $z_0$ satisfies the equation $A_0(z) = 0$;
(2) The common root $z_0$ satisfies both the equation $\psi(z, W) = 0$ and $\psi_{W}(z, W) = 0$.
Namely, $z_0$ is a multiple root of the equation $\psi(z, W) = 0$.

We use the notations $S_z$ and $T_z = \mathbb{C}\setminus S_z$ to denote the set of critical points and the set of normal points of the $v$-valued algebroid function $W(z)$. Since each critical point $z_0$ of the $v$-valued algebroid function $W(z)$ is an isolated point and $|(z - z_0)^n W(z)|$ is bounded at the neighborhood of $z_0$, they are removable singularities or poles. So the $v$-valued algebroid function $W(z)$ is continuous according to the spherical metric on the sphere. Then we study the properties of the $v$-valued algebroid function $W(z)$ only on $T_z$.

The single valued domain of a $v$-valued irreducible algebroid function $W(z)$ is a connected Riemann surface $\overline{T}_z$. The point on the connected Riemann surface $\overline{T}_z$ is the regular function element $\tilde{b} = \{w_{b,j}(z), B(b, r)\}$, where $w_{b,j}(z)$ denotes an analytic function in the disc $B(b, r)$. There exists a path $\gamma \subset \overline{T}_z$ for any two regular function elements $(w_{b,j}(z), B(b, r))$ and $(w_{a,t}(z), B(a, r))$ to extend analytically to each other along it. We often write $W(z) = \{w_j(z), b\}_{j=1}^v$.

Let $W(z)$ be a $v$-valued algebroid function in the unit disc, we use the notations:

\[
m(r, W) = \frac{1}{v} \sum_{j=1}^{v} m(r, w_j) = \frac{1}{v} \sum_{j=1}^{v} \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |w_j(re^{i\theta})| d\theta.
\]

\[
N(r, W) = \frac{1}{v} \int_{0}^{r} \frac{n(t, W) - n(0, W)}{t} dt + \frac{n(0, W)}{v} \log r,
\]

where $w_j(z)(j = 1, 2, \ldots, v)$ is a one-valued branch of $W(z)$. In this paper, we use $E(a, W)$ to denote the set of zeros of $W(z) - a$, counting multiplicities; $\overline{E}(a, W)$ to denote the case of ignoring multiplicities. Then we say that $W(z)$ and $M(z)$ share a CM, resp. IM, provided that $E(a, W) = E(a, M)$, resp. $\overline{E}(a, W) = \overline{E}(a, M)$.

We use $\Omega(\alpha, \beta)$ to denote the set $\{z|\alpha < \arg z < \beta\}$. For any given complex number $a$, we use $n(r, \Omega(\alpha, \beta), W(z) = a)$ to denote the number of roots (counting multiplicities) of the equation $W(z) - a = 0$ in the domain $\Omega(\alpha, \beta) \cap \{|z| < r\}$.

Now we give the following definitions of this paper.

**Definition 1** The order of $W(z)$ on the complex plane $\mathbb{C}$ is defined by $\lambda(W)$, where

\[
\lambda(W) = \lim_{r \to \infty} \frac{\log^+ T(r, W)}{\log r}.
\]

**Definition 2** The order of $W(z)$ in the unit disc is defined by $\sigma(W)$, where

\[
\sigma(W) = \lim_{r \to 1} \frac{\log^+ T(r, W)}{\log \frac{1}{1-r}}.
\]
Definition 3 Let $W(z)$ be a $v$-valued algebroid function of finite order $\lambda(0 < \lambda < \infty)$ on the complex plane $\mathbb{C}$, the half line $L$: $\text{arg}z = \theta_0(0 \leq \theta_0 < 2\pi)$ is called its Borel direction with finite order $\lambda$ if for any small $\varepsilon > 0(0 < \varepsilon < \frac{\pi}{2})$

$$\lim_{r \to +\infty} \frac{\log n(r, \Omega(\theta_0 - \varepsilon, \theta_0 + \varepsilon), W(z) = b)}{\log r} = \lambda$$

for all $b \in \mathbb{C}$ with at most $2v$ exceptional values.

Definition 4 ([8]) For a nonnegative; continuous and increasing function $T(r)$ with infinite order, that is $\lim_{r \to +\infty} \frac{\log T(r)}{\log r} = +\infty$, there exist a precise infinite order $\rho(r)$ such that

(i) $\rho(r)$ is continuous and increasing function for $r \geq r_0$ and $\rho(r) \to +\infty$ as $r \to +\infty$,

(ii) $\lim_{r \to +\infty} \frac{\log U(R)}{\log T(r)} = 1$ where $U(r) = r^{\rho(r)}$, $R = r(1 + \frac{1}{\log T(r)})$,

(iii) $\lim_{r \to +\infty} \frac{\log T(r)}{\rho(T)\log r} = 1$.

When an algebroid function $W(z)$ has infinite order, then the precise order $\rho(r)$ of $T(r, W)$ is also called the precise order of $W(z)$.

Definition 5 Let $W(z)$ be a $v$-valued algebroid function of infinite order on the complex plane $\mathbb{C}$, the half line $L$: $\text{arg}z = \theta_0(0 \leq \theta_0 < 2\pi)$ is called its Borel direction with the precise order $\rho(r)$ if for any small $\varepsilon > 0(0 < \varepsilon < \frac{\pi}{2})$

$$\lim_{r \to +\infty} \frac{\log n(r, \Omega(\theta_0 - \varepsilon, \theta_0 + \varepsilon), W(z) = b)}{\rho(r)\log r} = 1$$

for all $b \in \mathbb{C}$ with at most $2v$ exceptional values.

In [7], Jianren Long and Pengcheng Wu obtained the following theorem:

Theorem C Let $f(z)$ and $g(z)$ be two meromorphic functions on the complex plane $\mathbb{C}$. $\rho(r)$ is the precise order of $f(z)$, $M(\rho(r))$ is the set of meromorphic functions which satisfy $\lim_{r \to +\infty} \frac{\log T(r,q)}{\rho(T)\log r} \leq 1$, $g(z) \in M(\rho(r))$, the half line $L$: $\text{arg}z = \theta(0 \leq \theta < 2\pi)$ is the Borel direction with the precise order $\rho(r)$ of $f(z)$. If there exist five distinct complex numbers $a_j \in \mathbb{C}(j = 1,2,\cdots,5)$, for any small $\varepsilon(\leq \pi) > 0$ and $\mathbb{E}(a_j, f(z)) = \mathbb{E}(a_j, g(z))(j = 1,2,\cdots,5)$ in angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, then $f(z) \equiv g(z)$.

The following Theorem 2 extends Theorem C.

2 Lemmas

Lemma 1 ([10]) (The second fundamental theorem) Let $W(z)$ be a $v$-valued algebroid function in the unit disc defined by (1.1), and let $a_j \in \mathbb{C}(j = 1,2,\cdots,q)$ be $q$ distinct complex numbers, then

$$(q - 2v)T(r,W) \leq \sum_{j=1}^{q} N\left(r, \frac{1}{W - a_j}\right) + S(r,W),$$
where

\[ S(r, W) = O(\log T(r, W)) + O \left( \log \frac{1}{1 - r} \right), \]

possibly outside an exceptional set \( E \subset [0, 1) \) such that \( \int_E \frac{dr}{1 - r} < +\infty. \)

**Lemma 2** Let \( W(z) \) and \( M(z) \) be \( v \)-valued and \( \mu \)-valued algebroid functions in the unit disc respectively and \( \mu \leq v \). \( \lim_{r \to 1} \frac{T(r, W)}{\log \frac{1}{1 - r}} = \infty \) or \( \lim_{r \to 1} \frac{T(r, M)}{\log \frac{1}{1 - r}} = \infty \), there are \( 4v + 1 \) distinct complex numbers \( a_j \in \mathbb{C} (j = 1, 2, \cdots, 4v + 1) \), if \( E(a_j, W(z)) = E(a_j, M(z)) (j = 1, 2, \cdots, 4v + 1) \), then \( W(z) \equiv M(z) \).

**Proof** Now let \( W(z) \) be determined by (1.1) and \( M(z) \) be determined by the following equation

\[ \varphi(z, M) = B_{\mu}(z)M^\mu + B_{\mu-1}(z)M^{\mu-1} + \cdots + B_1(z)M + B_0(z) = 0. \]

\( \mu_0(r, a) \) denotes the common values of \( W(z) = a \) and \( M(z) = a \) in \( |z| < r \) ignoring multiplicities. Let

\[ N_0(r, a) = \mu + v \int_0^r \frac{\mu_0(t, a) - \mu_0(0, a)}{t} dt + \frac{\mu + v}{2\mu v} \mu_0(0, a) \log r \]

\[ N_{12}(r, a) = N \left( r, \frac{1}{W - a} \right) + N \left( r, \frac{1}{M - a} \right) - 2N_0(r, a). \]

Using the second fundamental theorem of algebroid function in the unit disc, let \( p = 4v + 1 \) we have

\[ (p - 2v)T(r, W) \leq \sum_{k=1}^p N \left( r, \frac{1}{W - a_k} \right) + S(r, W). \quad (2.1) \]

\[ (p - 2\mu)T(r, M) \leq \sum_{k=1}^p N \left( r, \frac{1}{M - a_k} \right) + S(r, M). \quad (2.2) \]

By (2.1), (2.2) and \( \mu \leq v \), we have

\[ (p - 2v)[T(r, W) + T(r, M)] \leq \sum_{k=1}^p N_{12}(r, a_j) + 2 \sum_{k=1}^p N_0(r, a_j) \]

\[ + O \left( \log \left[ \frac{1}{1 - r} T(r, W)T(r, M) \right] \right). \quad (2.3) \]

If \( W(z) \neq M(z) \), then we have

\[ \sum \mu_0(r, a) \leq n \left( r, \frac{1}{R(\psi, \varphi)} \right). \]

\( R(\psi, \varphi) \) denotes the resultant of \( \psi(z, W) \) and \( \varphi(z, M) \), it can be written as the following

\[ R(\psi, \varphi) = [A_v(z)]^\mu [B_\mu(z)]^v \prod_{1 \leq k \leq \mu} [w_j(z) - m_k(z)]. \]
It can be written in another form (see [14])

\[
R(\psi, \varphi) = \begin{vmatrix}
A_v(z) & A_{v-1}(z) & \cdots & \cdots & A_0(z) & 0 & \cdots & 0 \\
0 & A_v(z) & A_{v-1}(z) & \cdots & A_1(z) & A_0(z) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & A_v(z) & A_{v-1}(z) & \cdots & \cdots & A_0(z) \\
B_\mu(z) & B_{\mu-1}(z) & \cdots & \cdots & B_0(z) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & B_\mu(z) & B_{\mu-1}(z) & \cdots & \cdots & B_0(z)
\end{vmatrix}
\]

So we know that \( R(\psi, \varphi) \) is a holomorphic function, then we have

\[
N\left(r, \frac{1}{R(\psi, \varphi)} \right) = \frac{1}{2\pi} \int_0^{2\pi} \log |R(\psi, \varphi)| d\theta + \log \left| \frac{1}{R(\psi, \varphi)} \right|_{z=0} = \frac{\mu}{2\pi} \int_0^{2\pi} \log \left| A_v(re^{i\theta}) \right| d\theta + \frac{v}{2\pi} \int_0^{2\pi} \log \left| B_\mu(re^{i\theta}) \right| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \left| \prod_{1 \leq j \leq v} \left| w_j(re^{i\theta}) - m_k(re^{i\theta}) \right| \right| d\theta + O(1)
\]

\[
\leq \mu v [T(r, W) + T(r, M)] + O(1).
\]

Then we have

\[
\sum N_0(r, a) \leq \frac{2\mu v}{\mu + v} \{T(r, W) + T(r, M)\} + O(1)
\]

\[
\leq v \{T(r, W) + T(r, M)\} + O(1). \tag{2.4}
\]

By (2.3) and (2.4) we have

\[
(p - 4v) [T(r, W) + T(r, M)] \leq \sum_{k=1}^{p} N_{12}(r, a_j) + O \left( \log \left[ \frac{1}{1 - r} T(r, W) T(r, M) \right] \right).
\]

By the conditions of Lemma 2, we know \( \overline{N}_{12}(r, a_j) = 0 \). So we get

\[
T(r, W) + T(r, M) < O \left( \log \left[ \frac{1}{1 - r} T(r, W) T(r, M) \right] \right).
\]

So it must be \( W(z) \equiv M(z) \). \( \square \)
3 Main Results

**Theorem 1** Let \( W(z) \) and \( M(z) \) be \( v \)-valued and \( \mu \)-valued algebroid functions on the complex plane \( \mathbb{C} \) respectively and \( \mu \leq v \). If the half line \( L: \arg z = \theta_0(0 \leq \theta_0 < 2\pi) \) is the Borel direction with the finite order \( \lambda(\frac{1}{2} < \lambda < \infty) \) of algebroid functions \( W(z) \) or \( M(z) \), there exist \( 4v + 1 \) distinct complex numbers \( a_j \in \mathbb{C}(j = 1, 2, \ldots, 4v + 1) \) and

\[
E(a_j, W(z)) = E(a_j, M(z)) \quad (j = 1, 2, \ldots, 4v + 1)
\]

in angular domain \( \Omega(\theta_0 - \delta, \theta_0 + \delta) \), where \( \delta(\frac{\pi}{2^3} < \delta < \pi) \), then we have \( W(z) \equiv M(z) \).

**Corollary 1** Let \( W(z) \) and \( M(z) \) be two \( v \)-valued algebroid functions on the complex plane \( \mathbb{C} \). If the half line \( L: \arg z = \theta_0(0 \leq \theta_0 < 2\pi) \) is the Borel direction with the finite order \( \lambda(\frac{1}{2} < \lambda < \infty) \) of algebroid functions \( W(z) \) or \( M(z) \), there exist \( 4v + 1 \) distinct complex numbers \( a_j \in \mathbb{C}(j = 1, 2, \ldots, 4v + 1) \) and

\[
E(a_j, W(z)) = E(a_j, M(z)) \quad (j = 1, 2, \ldots, 4v + 1)
\]

in angular domain \( \Omega(\theta_0 - \delta, \theta_0 + \delta) \), where \( \delta(\frac{\pi}{2^3} < \delta < \pi) \), then we have \( W(z) \equiv M(z) \).

**Theorem 2** Let \( W(z) \) and \( M(z) \) be \( v \)-valued and \( \mu \)-valued algebroid functions on the complex plane \( \mathbb{C} \) respectively and \( \mu \leq v \). If the half line \( L: \arg z = \theta_0(0 \leq \theta_0 < 2\pi) \) is the Borel direction with the precise infinite order \( \rho(r) \) of algebroid functions \( W(z) \) or \( M(z) \), there exist \( 4v + 1 \) distinct complex numbers \( a_j \in \mathbb{C}(j = 1, 2, \ldots, 4v + 1) \) and

\[
E(a_j, W(z)) = E(a_j, M(z)) \quad (j = 1, 2, \ldots, 4v + 1)
\]

in angular domain \( \Omega(\theta_0 - \delta, \theta_0 + \delta) \), where \( \delta(0 < \delta < \pi) \), then we have \( W(z) \equiv M(z) \).

**Corollary 2** Let \( W(z) \) and \( M(z) \) be two \( v \)-valued algebroid functions on the complex plane \( \mathbb{C} \). If the half line \( L: \arg z = \theta_0(0 \leq \theta_0 < 2\pi) \) is the Borel direction with the precise infinite order \( \rho(r) \) of algebroid functions \( W(z) \) or \( M(z) \), there exist \( 4v + 1 \) distinct complex numbers \( a_j \in \mathbb{C}(j = 1, 2, \ldots, 4v + 1) \) and

\[
E(a_j, W(z)) = E(a_j, M(z)) \quad (j = 1, 2, \ldots, 4v + 1)
\]

in angular domain \( \Omega(\theta_0 - \delta, \theta_0 + \delta) \), where \( \delta(0 < \delta < \pi) \) is a sufficiently small positive number, then we have \( W(z) \equiv M(z) \).

**Corollary 3** Under the conditions of Corollary 2, if \( v = 1 \) then we get Theorem C.

4 Proofs of Theorems

**Proof of Theorem 1** Firstly, without loss of generality, we may assume that the half line \( L: \arg z = \theta_0 = 0 \) is the Borel direction with the finite order \( \lambda \) of \( W(z) \). We set

\[
u(z) = \frac{z^\frac{\pi}{2^3} - 1}{z^\frac{\pi}{2^3} + 1}, \quad z(u) = \frac{1 + u}{1 - u}^{\frac{\pi}{2^3}}.\]
Then \( u(z) \) maps the angular domain \( \Omega(-\delta, \delta) \) conformally on \( |u| < 1 \). We use the notation \( z(u) \) to denote the inverse mapping of \( u(z) \). Set \( z = r e^{i\varphi} \), for any \( \psi : 0 < \psi < \delta \), when \( |\varphi| \leq \psi \), then we have

\[
|u(z)| = \left| \frac{\rho^{\pi} \cos \frac{\pi \varphi}{2\delta} - 1 + i \rho^{\pi} \sin \frac{\pi \varphi}{2\delta}}{\rho^{\pi} \cos \frac{\pi \varphi}{2\delta} + 1 + i \rho^{\pi} \sin \frac{\pi \varphi}{2\delta}} \right|
\]

\[
= \sqrt{\frac{\rho^{\pi} + 1 - 2 \rho^{\pi} \cos \frac{\pi \varphi}{2\delta}}{\rho^{\pi} + 1 + 2 \rho^{\pi} \cos \frac{\pi \varphi}{2\delta}}} 
\leq \sqrt{\frac{\rho^{\pi} + 1 - 2 \rho^{\pi} \cos \pi \psi}{\rho^{\pi} + 1 + 2 \rho^{\pi} \cos \pi \psi}} 
= 1 - 2 \cos \frac{\pi \psi}{2\delta} \rho^{-\frac{\pi}{2\delta}} + O(\rho^{-\frac{\pi}{2\delta}}).
\]

So there is \( r_0 > 1 \), where \( r_0 \) only depends on \( \psi, \delta \). When \( \rho \geq r_0 \),

\[
|u(z)| < 1 - \cos \frac{\pi \psi}{2\delta} \rho^{-\frac{\pi}{2\delta}}.
\]

Then we have

\[
\{ u(re^{i\varphi}) : r_0 \leq r, |\varphi| \leq \psi \} \subset \{ |u(z)| < 1 - \cos \frac{\pi \psi}{2\delta} r^{-\frac{\pi}{2\delta}} \}.
\]

We use \( d \) to denote the distance from \( \{ u(re^{i\varphi}) | 1 \leq r \leq r_0, |\varphi| \leq \psi \} \) to \( |u| = 1 \), where \( d \) only depends on \( r_0, \psi \), that is to say, \( d \) only depends on \( \psi, \delta \). Let \( \eta : 0 < \eta < d r_0^{-\frac{\pi}{2\delta}} \), then we have

\[
\{ u(re^{i\varphi}) | r_0 \leq r, |\varphi| \leq \psi \} \subset \{ |u| \leq 1 - d \}
\subset \{ |u| < 1 - \eta r^{-\frac{\pi}{2\delta}} \} \subset \{ |u| < 1 - \eta r^{-\frac{\pi}{2\delta}} \} \quad (r > r_0).
\]

At the same time, let \( \eta < \cos \frac{\pi \psi}{2\delta} \), then we get

\[
\{ u(re^{i\varphi}) | r_0 \leq r, |\varphi| \leq \psi \} \subset \{ |u| < 1 - \cos \frac{\pi \psi}{2\delta} r^{-\frac{\pi}{2\delta}} \} \subset \{ |u| < 1 - \eta r^{-\frac{\pi}{2\delta}} \}.
\]

So for any \( \psi \in (0, \delta) \), let \( \{ \eta : 0 < \eta < \min\{ \cos \frac{\pi \psi}{2\delta}, d r_0^{-\frac{\pi}{2\delta}} \} \} \) where \( \eta \) only depends on \( \psi, \delta \). When \( r \) is large enough, then we have

\[
\{ u(re^{i\varphi}) : 1 \leq r \leq r_0, |\varphi| \leq \psi \} \subset \{ |u| < 1 - \eta r^{-\frac{\pi}{2\delta}} \}.
\]

Since the half line \( L: \arg z = 0 \) is the Borel direction with the finite order \( \lambda \) of \( W(z) \). Then there is a sequence of points \( r_n \), such that for any given small \( \varepsilon(>0) \) we have

\[
n(r_n, \Omega(-\psi, \psi), W(z) = b) > r_n^{\lambda - \varepsilon}.
\]

Set

\[
y_n = 1 - \eta r_n^{-\frac{\pi}{2\delta}}, \quad r_n = \left( \frac{\eta}{1 - y_n} \right)^{\frac{2\delta}{\pi}}.
\]

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So we get
\[ n(y_n, W(z(u)) = b) \geq n(r_n, \Omega(-\psi, \psi), W(z) = b) > r_n^{\lambda - \varepsilon}. \]

Set
\[ r'_n = 2r_n, \quad y'_n = 1 - \eta(r'_n)^{-\frac{\pi}{2\varepsilon}}. \]

Hence
\[ y'_n = 1 - \frac{1 - y_n}{2\varepsilon} = y_n + (1 - y_n)(1 - 2^{-\frac{\pi}{2\varepsilon}}). \]

Then we have
\[ T(y'_n, W(z(u))) \geq N(y'_n, W(z(u)) = b) - B \geq \frac{1}{v} \int_{y_n}^{y'_n} n(y, W(z(u)) = b) \frac{dy}{y} - B \geq \frac{1}{v} n(y_n, W(z(u)) = b) \log \frac{y'_n}{y_n} - B \geq An(y_n, W(z(u)) = b)(1 - y_n) \geq Ar_n^{\lambda - \varepsilon}(1 - y_n) = A \left( \frac{1}{1 - y_n} \right)^{\frac{2\delta}{\pi} \lambda - 1 - \frac{2\delta}{\pi} \varepsilon} = A \left( \frac{1}{1 - y_n} \right)^{\frac{2\delta}{\pi} \lambda - 1 - \frac{2\delta}{\pi} \varepsilon}. \]

Where \( A \) and \( B \) are positive numbers, which may be different at different places. So we have for sufficiently small \( \varepsilon > 0 \)

\[ \lim_{y'_n \to 1} \frac{T(y'_n, W(z(u)))}{\log \frac{1}{1 - y'_n}} \geq \lim_{y'_n \to 1} \frac{A \left( \frac{1}{1 - y_n} \right)^{\frac{2\delta}{\pi} \lambda - 1 - \frac{2\delta}{\pi} \varepsilon}}{\log \frac{1}{1 - y'_n}} = \infty. \]

Then, since \( E(a_j, W(z)) = E(a_j, M(z)) \) in angular domain \( \Omega(-\delta, \delta) \), it follows that \( E(a_j, W(z(u))) = E(a_j, M(z(u))) \) holds in \( |u| < 1 \). So by Lemma 2, we obtain \( W(z(u)) \equiv M(z(u)) \) holds in \( |u| < 1 \). Therefore, we can obtain that \( W(z) \equiv M(z) \) holds in the domain \( \Omega(-\delta, \delta) \). Then by the uniqueness theorem of analytic functions, we have \( W(z) \equiv M(z) \) in \( \mathbb{C} \). \( \square \)

**Proof of Theorem 2** Similar to the proof of Theorem 1, without loss of generality, we may assume that the half line \( L: \arg z = \theta_0 = 0 \) is the Borel direction with the precise infinite order \( \rho(r) \) of \( W(z) \). We set
\[ u(z) = \frac{z^{\frac{\pi}{2\varepsilon}} - 1}{z^{\frac{\pi}{2\varepsilon}} + 1}, \quad z(u) = \left( \frac{1 + u}{1 - u} \right)^{\frac{\pi}{2\varepsilon}}. \]
Since the half line \( L: \arg z = 0 \) is the Borel direction with the precise infinite order \( \rho(r) \) of \( W(z) \). Then there is a sequence of points \( r_n \), such that for any given \( \varepsilon > 0 \) we have

\[ n(r_n, \Omega(-\psi, \psi), W(z) = b) > U^{1-\varepsilon}(r_n). \]

Set

\[ y_n = 1 - \eta r_n^{-\frac{\pi}{2\delta}}, \quad r_n = \left( \frac{\eta}{1 - y_n} \right)^{\frac{2\delta}{\pi}}. \]

Then we have

\[ n(y_n, W(z(u)) = b) \geq n(r_n, \Omega(-\psi, \psi), W(z) = b) > U^{1-\varepsilon}(r_n). \]

Set

\[ r'_n = 2r_n, \quad y'_n = 1 - \eta (r'_n)^{-\frac{\pi}{2\delta}}. \]

Thus

\[ y'_n = 1 - \frac{1 - y_n}{2\pi} = y_n + (1 - y_n)(1 - 2^{-\frac{\pi}{2\delta}}). \]

So we get

\[
T(y'_n, W(z(u))) \geq N(y'_n, W(z(u)) = b) - B \\
\geq \frac{1}{v} \int_{y_n}^{y'_n} \frac{n(y, W(z(u)) = b)}{y} dy - B \\
\geq \frac{1}{v} n(y_n, W(z(u)) = b) \log \frac{y'_n}{y_n} - B \\
\geq A n(y_n, W(z(u)) = b)(1 - y_n) \\
\geq AU^{1-\varepsilon}(r_n)(1 - y_n) \\
= U^{-2\varepsilon}(r_n).
\]

Where \( A \) and \( B \) are positive numbers, which may be different at different places. So we have

\[
\lim_{y'_n \to 1} \frac{T(y'_n, W(z(u)))}{\log \frac{1}{1-y'_n}} \geq \lim_{r_n \to \infty} \frac{U^{1-2\varepsilon}(r_n)}{\frac{\pi}{2\delta} \log r_n + \log \left( \frac{\pi}{2\delta\eta} \right)} = \infty
\]

Similarly as above, we also have \( W(z) \equiv M(z) \) in \( \mathbb{C} \). \( \square \)

References


