GRADIENT ESTIMATES FOR WEIGHTED LICHNEROWICZ EQUATION ON SMOOTH METRIC MEASURE SPACES

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ABSTRACT. In this paper, we consider the weighted Lichnerowicz equation
\[ \Delta_f u + cu^\sigma = 0 \]
on smooth metric measure spaces, where \( c \geq 0, \sigma \leq 1 \) are real constants. A local gradient estimate for positive solutions to this equation is derived and as an application, we give a corresponding Harnack inequality.

1. Introduction

In this paper, we will study the weighted Lichnerowicz equations
\[ \Delta_f u + cu^\sigma = 0 \tag{1.1} \]
here \( c \geq 0, \sigma \leq 1 \), on smooth metric measure space. Recall that smooth metric measure space is a triple \((M, g, d\mu)\), where \((M^n, g)\) is a complete \( n \)-dimensional Riemannian manifold and \( d\mu := e^{-f} dv \) with \( f \) a fixed smooth real-valued function on \( M \). The smooth metric measure space has a natural \( m \)-Bakry-Émery Ricci curvature, which is defined as
\[ \text{Ric}^m_f := \text{Ric} + \text{Hess} f - \frac{\nabla f \otimes \nabla f}{m - n}, (n < m \leq \infty). \]
In particular, when \( m = \infty \), \( \text{Ric}^\infty_f := \text{Ric}_f := \text{Ric} + \text{Hess} f \) is the classical Bakry-Émery Ricci curvature, which was introduced by Bakry-Émery [2] in the study of diffusion processes, and has been extensively investigated in the theory of Ricci flow. The case where \( m = n \) is only defined when \( f \) is a constant function.

Equation (1.1) can be seen as a simple version of the Lichnerowicz equation which arises from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity (see [4, 5] and the references therein). Li and Zhu [9][10] studied the simple Lichnerowicz equation and derived corresponding gradient estimates. The first author of this paper also studied the gradient estimates of Lichnerowicz equations and derived some Liouville theorems which can be referred to [15, 16, 17]. Moreover, Fang [6, 7, 8] investigated some Harnack inequalities under some geometric flows.

On the other hand, The motivations for studying the equation (1.1) come from weighted Yamabe problem. The static form of (1.1) for a special \( \sigma \) is related to the Euler-Lagrange equation for the weighted Yamabe quotient on compact smooth metric spaces. Case [2, 3] shows that Yamabe-type problem on \((M, g, d\mu)\) interpolates between Yamabe problem and the problem of finding minimizers for

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Perelmans’ entropy. Moreover, we known that Yamabe constant and Perelmans’ entropy are related to Sobolev and logarithmic Sobolev inequalities. Therefore, in this paper, we hope the gradient estimates of the positive solution to equation (1.1) can provide some ways to solving the weighted Yamabe problem.

However, the gradient estimates for the positive solutions to differential equations are usually obtained under Ricci curvature bounded below. Recently, many mathematicians consider similar problems on manifolds under upper curvature. Q.S. Zhang and M. Zhu [14] obtained the Li-Yau gradient bound for positive solutions to heat equation on collapsing manifolds under integral curvature condition. Later, Christian Rose [12] extended the above results to compact manifolds with negative parts of Ricci curvature in the Kato class. Then Xavier Ramos Olivé [11] generalized the estimate of [14] to compact Riemannian manifolds with Neumann boundary which satisfies the integral Ricci curvature condition. Motivated by the above work, in this paper, we will establish the local gradient estimate for positive boundary which satisfies the integral Ricci curvature condition. Motivated by the above work, in this paper, we will establish the local gradient estimate for positive solutions to equation (1.1) on the ball $B_x(R)$.

Various weighted $L^p$ norms of the function $h$ on a smooth metric measure space are

$$
\|h\|_{q,f,B_x(R)} = \left(\int_{B_x(R)} |h|^q e^{-f} d\text{vol}\right)^{\frac{1}{q}}
$$

$$
\|h\|_{q,f,a,R} = \sup_{x \in M} \left[ \int_{0}^{R} \int_{S^{n-1}} |h|^q e^{-at} \Lambda(f,\theta) d\theta dt \right]^{\frac{1}{q}}.
$$

Here $\Lambda(f,\theta)$ is the volume element of weighted measure $e^{-f} d\text{vol} = \Lambda(f,\theta) d\omega \wedge dt$ and $d\theta$ is the volume element of unit sphere. Let

$$
\varpi(q,f,a,R) = R^2 \sup_{x \in M} \left[ \frac{\int_{B_x(R)} \rho^q f - e^{-at} \Lambda(f,\theta) d\theta dt}{\int_{B_x(R)} e^{-f} d\text{vol}} \right]^{\frac{1}{q}},
$$

where $a$ is a positive constant.

We denote $\|h\|_q^*$ the normalized $q$-norm on the domain $\Omega$. Namely

$$
\|h\|_q^* = \|h\|^*_f,B_x(R) = \left( \frac{1}{\text{vol}_f(B_x(R))} \int_{B_x(R)} |h|^q e^{-f} d\text{vol} \right)^{\frac{1}{q}},
$$

where $\text{vol}_f(B_x(R)) = \int_{B_x(R)} e^{-f} d\text{vol}$, and

$$
\|h\|_{q,f,a,R}^* = \sup_{x \in M} \left[ \frac{\int_{B_x(R)} h e^{-at} \Lambda(f,\theta) d\theta dt}{\int_{B_x(R)} e^{-f} d\text{vol}} \right]^{\frac{1}{q}}.
$$

It is easy to observe that

$$
\|h\|_q^* \leq e^{\frac{aR}{2}} \|h\|_{q,f,a,R}^*.
$$

First, we give a new local gradient estimate for positive solutions to equation (1.1).

**Theorem 1.1** Let $(M,g,d\mu)$ be a smooth metric measure space. Suppose that $u$ is a positive solution to (1.1) on the ball $B_u(R) \subset M$. For $q > \frac{n}{2}$ and $R \leq 1$, if there exists a large enough positive constant $b$, such that $\overline{\kappa}(q,f,a,1) \leq \frac{1}{[q+1]^2}$, then

$$
\frac{|\nabla u|}{u} \leq C_s(n,q,V) \frac{R}{\text{vol}_u(B_u(R))} \text{ on } B_u(R/2),
$$

where $C_s(n,q,V)$ is a constant depending only on $n$, $q$ and $V$.
where $C_s(n, q, V)$ is a positive constant.

Remark 1.2 From Remark 2.4 in [13], we know there exists a constant $C(n, q, a, R)$ such that

$$k_m(q, f, a, R) \leq c(n, q, a, R)k_m(q, f, a, 1).$$

Since $k_m(q, f, a, R) = R^2 \| \text{Ric}^p_{f, a} \|_q^*$ and inequality (1.2), we get

$$\| \text{Ric}^p_{f, a} \|_q^* \leq C(n, q, a, R) \left( b + 1 \right)^2 R^2.$$  (1.3)

Remark 1.3 In the following sections of this paper, the constant $C(n, q, a, R)$ in Theorem 1.1 may change from line to line.

Second, as an application of the Theorem 1.1, we can obtain the following Corollary.

Corollary 1.2 Under the same conditions as in Theorem 1.1, given any $x, y \in B_o(R^2)$, and any minimal geodesic $\gamma(s) : [0, 1] \rightarrow B_o(R^2)$ with $\gamma(0) = y, \gamma(1) = x,$ The following Harnack inequality holds:

$$u(x) \leq u(y)e^{C_s(n, q, V)}.$$

2. PROOF OF THEOREM 1.1

Assume that $u$ is a positive solution to (1.1). The linearized operator of the weighted Laplacian at point $u \in C^2(M)$ is given by

$$L_f(v) = e^f \text{div}(e^{-f} \nabla f),$$

where

$$A_{ij} = g_{ij} + (p - 2) \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}.$$ 

Given a $C^3$ function $u$, if $|\nabla u| \neq 0$, then

$$L_f(|\nabla u|^2) = 2(|\text{Hess } u|^2 + \text{Ric}_f(\nabla u, \nabla u)) + 2|\nabla u|(|\nabla u, \nabla \Delta f u|).$$

Set $v = \log u$, from the above Lemma 2.1, we can obtain

$$L_f(|\nabla v|^2) = 2(|\text{Hess } v|^2 + \text{Ric}_f(\nabla v, \nabla v)) + (\nabla v, \nabla \Delta f v).$$

Moreover, in the terms of $v$, equation (1.1) has the following form

$$\Delta f v + cu = e^f \text{div}(e^{-f} \nabla v) + ce^v\sigma = e^v(|\nabla v|^2 + \Delta f v) + ce^v\sigma = 0.$$

That is to say,

$$\Delta f v = -ce^{(\sigma - 1)v} - |\nabla v|^2.$$  (2.1)

Let $w = |\nabla v|^2$, note that in terms of $w$,

$$\Delta f v = e^f \text{div}(e^{-f} \nabla v)$$

$$= \Delta f v + \frac{p - 2}{2}w^{-1}(\nabla w, \nabla v).$$

Therefore, equation (2.1) has the equivalent form

$$\Delta f v = -ce^{(\sigma - 1)v} - w.$$  (2.2)
Assume that $Q = |\nabla v|^2$, then

$$L_f(Q) = 2[[Hess v]^2 + Ric_f(\nabla v, \nabla v)] - 2(\nabla v, \nabla Q) - 2ch(\sigma - 1)Q,$$

where $h = e^{(\sigma - 1)v}$.

Next, we will estimate $|\nabla \nabla v|^2$. We only need to estimate it over the points where $w > 0$. Choose a local orthonormal frame $\{e_i\}_{i=1}^n$ near any such a given point so that $\nabla v = |\nabla v|e_1$. Then $w = v_1^2$, $w_1 = 2v_{11}v_1 = 2v_{11}v_1$, and for $j \geq 2$, $w_j = 2v_{j1}v_1$.

Hence $2v_{j1} = \frac{w_j}{w_{11}}$.

From (2.2), we obtain that

$$\sum_{j=2}^n v_{jj} = -ch - v_{11} + f_1v_1 - w$$

$$= -ch - v_{11} + f_1v_1 - w.$$ 

It is easy to see that

$$|Hess v|^2 = \sum_{i,j=1}^n v_{ij}^2$$

$$= v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2 + \sum_{i,j=2}^n v_{ij}^2$$

$$\geq v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2 + \frac{1}{n-1}(\sum_{j=2}^n v_{jj})^2.$$ 

Therefore, we get

$$|Hess v|^2 \geq v_{11}^2 + 2\sum_{j=2}^n v_{j1}^2$$

$$+ \frac{1}{n-1}(-chw - v_{11} + f_1v_1 - w)^2$$

$$\geq \alpha \sum_{j=1}^n v_{j1}^2 + \frac{1}{m-1}(ch + w)^2$$

$$+ \frac{2v_{11}}{m-1}(ch + w) - \frac{(f_1v_1)^2}{m-n},$$

where $\alpha = \min\{2, \frac{m}{m-1}\}$, and we applied the inequality $(a - b)^2 \geq \frac{a^2}{1+\delta} - \frac{b^2}{\delta}$ with $\delta = \frac{m-n}{m-1} > 0$. Substituting the identities,

$$2wv_{11} = \langle \nabla v, \nabla w \rangle, \sum_{j=1}^n v_{j1}^2 = \frac{1}{4}\frac{|
abla w|^2}{w},$$

we can obtain

$$|Hess v|^2 \geq \frac{\alpha}{4}\frac{|
abla w|^2}{w} + \frac{w^2}{m-1}(chw^{-1} + 1)^2$$

$$+ \frac{1}{m-1}(1 + chw^{-1})(\nabla v, \nabla w) - \frac{(f_1v_1)^2}{m-n}.$$ 

Therefore, we have
$L_f(Q) = L_f(w)$
\[\geq \left[ \frac{\alpha}{2} \frac{|\nabla w|^2}{w} + \frac{2}{m-1} (w + ch)^2 \right.\]
\[+ \frac{2}{m-1} (1 + chw^{-1}) \langle \nabla v, \nabla w \rangle \]
\[+ 2 Ric^m_j (\nabla v, \nabla v) - 2 \langle \nabla v, \nabla w \rangle - 2 ch(\sigma - 1) w \]
\[\leq \frac{\alpha}{2} w^{-1} |\nabla w|^2 + \frac{2}{m-1} w^2 (1 + chw^{-1})^2 \]
\[+ \frac{2}{m-1} (1 + chw^{-1}) \langle \nabla v, \nabla w \rangle \]
\[+ 2 Ric^m_j (\nabla v, \nabla v) - 2 \langle \nabla v, \nabla w \rangle - 2 ch(\sigma - 1) w \]
\[\geq \frac{\alpha}{2} w^{-1} |\nabla w|^2 + \frac{2}{m-1} w^2 (1 + chw^{-1})^2 \]
\[+ \frac{2}{m-1} (1 + chw^{-1}) \langle \nabla v, \nabla w \rangle \]
\[+ 2 Ric^m_j (\nabla v, \nabla v) - 2 \langle \nabla v, \nabla w \rangle - 2 ch(\sigma - 1) w \]
\[= \frac{\alpha}{2} w^{-1} |\nabla w|^2 + \frac{2}{m-1} w^2 (1 + chw^{-1})^2 \]
\[+ \left[ \frac{2}{m-1} (1 + chw^{-1}) - 2 \langle \nabla v, \nabla w \rangle + 2 Ric^m_j (\nabla v, \nabla v) \right. \quad (2.3) \]

In order to prove the Theorem 1.1, we also need the following Sobolev inequality.

**Lemma 2.1 (see [13])** Given $q > \frac{n}{2}$ and $K > 0$, there exists $\epsilon = \epsilon(n, q, K)$ such that if $M$ is a complete manifold with $\pi(q, f, a, 1) \leq \epsilon$, then for any $R \leq 1$ and $u \in C^\infty_0(B_n(R))$,
\[
\left( \int_{B_n(R)} u^{\frac{2m}{n-1}} \right)^{\frac{n-1}{2m}} \leq C_s(n, a) V^{\frac{1}{2m}} \left( \int_{B_n(R)} u^2 + R^2 \int_{B_n(R)} |\nabla u|^2 \right),
\]

here $V = vol_f(B_n(R))$.

**Proof.** From Corollary 3.7 in [13], they get the following Sobolev inequality
\[
\|u\|_{L^\infty, f, B_n(R)} \leq C_s R \|\nabla u\|_{L^2, f, B_n(R)}^n,
\]

where $C_s = C_s(n, a)$ denote the Sobolev constant, in the following of this paper, it may change from line to line. We can get
\[
\left( \int_{B_n(R)} u^{\frac{2m}{n-1}} \right)^{\frac{n-1}{2m}} \leq C_s V^{\frac{1}{2m}} R \int_{B_n(R)} 2|u| |\nabla u| \]
\[
\leq C_s V^{\frac{1}{2m}} \left( \int_{B_n(R)} u^2 + R^2 \int_{B_n(R)} |\nabla u|^2 \right). \]

The inequality (2.3) is always right wherever $w$ is strictly positive. Let $K = \{x \in \Omega : w(x) = 0\}$, here $\Omega \subset M$ is an open set. Then for any nonnegative function
\psi with compact support in \( \Omega \setminus K \), we have
\[
- \int_{\Omega} \left( \frac{1}{2} \nabla w, \nabla \psi \right)
\geq \int_{\Omega} \left( \frac{\alpha}{4} w^{-1} |\nabla w|^2 + \frac{1}{m-1} (1 + chw^{-1}) w^2 \right.
+ \left[ \frac{1}{m-1} (1 + chw^{-1}) - 1 \right] |\nabla v, \nabla w| + \text{Ric}^m_{\nu}(\nabla v, \nabla v) \psi.
\tag{2.4}
\]

In particular, let \( \psi = w^b \eta^2 \), where \( \eta \in C_0^\infty(B_R) \) is nonnegative, \( b > 1 \) is to be determined later. Then direct computation yields
\[
\nabla \psi = bw^{b-1} \eta^2 \nabla w + 2w^b \eta \nabla \eta.
\]

For the above Ricci term, we can estimate it as follows
\[
\int_{B_o(R)} \text{Ric}^m_{\nu}(\nabla v, \nabla v) w^b \eta^2 d\mu
\geq -\int_{B_o(R)} \| \text{Ric}^m_{\nu} \| w^{1+b} \eta^2
\geq -\| \text{Ric}^m_{\nu} \| \eta \left( \int_{B_o(R)} (w^{1+b} \eta^2)^{\frac{2}{1+b}} \right)^{\frac{1+b}{2}}.
\tag{2.5}
\]

By the Lemma 2.1, apply the Sobolev inequality to \( w^{1+b} \eta^2 \), we have
\[
(\int_{B_o(R)} (w^{1+b} \eta^2)^{\frac{2}{1+b}})^{\frac{1+b}{2}}
\leq V^{\frac{1}{2}} \left( \frac{C_s}{4} R^2 \int_{B_o(R)} |\nabla (w^{1+b} \eta^2)|^2 w + \int_{B_o(R)} w^{1+b} \eta^2 \right)
= V^{\frac{1}{2}} \frac{C_s}{4} R^2 \int_{B_o(R)} [(b+1)^2 w^{b-1} |\nabla w|^2 \eta^2 + 4w^{b+1} |\nabla \eta|^2
+ 2(b+1) \eta w^b (\nabla w, \nabla \eta)] + V^{\frac{1}{2}} \int_{B_o(R)} w^{1+b} \eta^2.
\tag{2.6}
\]

Set \( \delta = \| \text{Ric}^m_{\nu} \| \eta \), by the inequality (1.3) and the assumption of the Theorem 1.1, we get the estimation of curvature item
\[
\int_{B_o(R)} \text{Ric}^m_{\nu}(\nabla v, \nabla v) w^b \eta^2 d\mu
\geq (-V^{\frac{1}{2}} \delta) \int_{B_o(R)} w^{1+b} \eta^2 - \left( \frac{C_s}{4} V^{\frac{1}{2}} R^2 \delta \right) \int_{B_o(R)} [(b+1)^2 w^{b-1} |\nabla w|^2 \eta^2
+ 4w^{b+1} |\nabla \eta|^2 + 2(b+1) \eta w^b (\nabla w, \nabla \eta)]
\geq (-V^{\frac{1}{2}} \delta) \int_{B_o(R)} w^{1+b} \eta^2 - \frac{C}{(b+1)^2} \int_{B_o(R)} [(b+1)^2 w^{b-1} |\nabla w|^2 \eta^2
+ 4w^{b+1} |\nabla \eta|^2 + 2(b+1) \eta w^b (\nabla w, \nabla \eta)].
\tag{2.7}
\]

Where \( C \) is a positive constant, it may change from line to line throughout the paper.

Inserting (2.7) into (2.4), we derive
\[
- \int_{B_o(R)} \left( \frac{1}{2} \nabla w, bw^{b-1} \eta^2 \nabla w + 2w^b \eta \nabla \eta \right)
\]
\[
\begin{align*}
&\geq - (V - \frac{1}{4} \delta) \int_{B_a(R)} w^{1+b} \eta^2 + \left( \frac{\alpha}{4} - \frac{C}{4} \right) \int_{B_a(R)} w^{b-1} |\nabla w|^2 \eta^2 \\
&\quad + \frac{1}{m-1} (1 + chw^{-1}) \int_{B_a(R)} w^{2+b} \eta^2 + \left( \frac{1}{m-1} (1 + chw^{-1}) - 1 \right) \int_{B_a(R)} w^b \eta^2 \langle \nabla w, \nabla \eta \rangle \\
&\quad - \frac{C}{(b+1)^2} \int_{B_a(R)} w^{b+1} |\nabla \eta|^2 + \frac{C}{2(b+1)} \int_{B_a(R)} \eta w^b \langle \nabla w, \nabla \eta \rangle,
\end{align*}
\]
which implies
\[
\begin{align*}
\int_{B_a(R)} \frac{b}{2} w^{b-1} \eta^2 |\nabla w|^2 - \int_{B_a(R)} w^b \eta \langle \nabla w, \nabla \eta \rangle \\
\geq - (V - \frac{1}{4} \delta) \int_{B_a(R)} w^{1+b} \eta^2 + \left( \frac{\alpha}{4} - \frac{C}{4} \right) \int_{B_a(R)} w^{b-1} |\nabla w|^2 \eta^2 \\
&\quad + \frac{1}{m-1} (1 + chw^{-1}) \int_{B_a(R)} w^{2+b} \eta^2 + \left( \frac{1}{m-1} (1 + chw^{-1}) - 1 \right) \int_{B_a(R)} w^b \eta^2 \langle \nabla w, \nabla \eta \rangle \\
&\quad - \frac{C}{(b+1)^2} \int_{B_a(R)} w^{b+1} |\nabla \eta|^2 + \frac{C}{2(b+1)} \int_{B_a(R)} \eta w^b \langle \nabla w, \nabla \eta \rangle.
\end{align*}
\]
Since
\[
\begin{align*}
\int_{B_a(R)} w^b \langle \nabla w, \nabla \eta \rangle \\
\leq \int_{B_a(R)} w^b |\nabla w||\nabla \eta||\eta|
\end{align*}
\]
let \( \beta = 1 + chw^{-1} \), with these inequalities we can get
\[
\begin{align*}
C \int_{B_a(R)} w^b |\nabla w||\nabla \eta||\eta| + \int_{B_a(R)} w^b \eta^2 \langle \nabla w, \nabla v \rangle \\
\geq \int_{B_a(R)} \left( \frac{\alpha}{4} + \frac{b}{2} - \frac{C}{4} \right) w^{b-1} \eta^2 |\nabla w|^2 + \int_{B_a(R)} \frac{1}{m-1} \beta^2 w^{2+b} \eta^2 \\
&\quad + \int_{B_a(R)} \frac{1}{m-1} (1 + chw^{-1}) w^b \eta^2 \langle \nabla w, \nabla v \rangle + \int_{B_a(R)} [-V - \frac{1}{4} \delta] w^{b+1} \eta^2 \\
&\quad + \frac{C}{(b+1)^2} \int_{B_a(R)} w^{1+b} |\nabla \eta|^2.
\end{align*}
\]
From now on we use \( a_1, a_2, \ldots \) to denote some constants. The constant \( b > 1 \) is to be determined later. It is easy to see that
\[
\begin{align*}
\int_{B_a(R)} w^b |\nabla w||\nabla \eta||\eta| \\
\leq \frac{b}{12C} \int_{B_a(R)} w^{b-1} |\nabla w|^2 \eta^2 + \frac{a_1}{b} \int_{B_a(R)} w^{1+b} \nabla \eta|^2,
\end{align*}
\]
we also have
\[
\begin{align*}
\int_{B_a(R)} w^b \eta^2 \langle \nabla w, \nabla v \rangle \\
\leq \frac{b}{12} \int_{B_a(R)} w^{b+\frac{1}{2}} \eta^2 |\nabla w| \\
\leq \frac{b}{12} \int_{B_a(R)} w^{b-1} |\nabla w|^2 \eta^2 + \frac{a_2}{b} \int_{B_a(R)} w^{2+b} \eta^2,
\end{align*}
\]
and the following inequality holds
\[
\int_{B_a(R)} \frac{1}{m-1} \beta w^{b} \eta^2 \langle \nabla w, \nabla v \rangle \\
\geq - \int_{B_a(R)} \frac{1}{m-1} \beta w^{b+\frac{3}{2}} \eta^2 |\nabla w|
\geq - \frac{b}{12} \int_{B_a(R)} w^{b-1} |\nabla w|^2 \eta^2 - \frac{a_3 \beta^2}{b} \int_{B_a(R)} w^{2+b} \eta^2.
\]
By requiring \[
\frac{a_3}{b} - \frac{1}{m-1} \leq 0.
\tag{2.8}
\]
Since \( \beta = 1 + chw^{-1} \geq 1 \), we get
\[
\left( \frac{a_3}{b} - \frac{1}{m-1} \right) \int_{B_a(R)} \beta^2 w^{2+b} \eta^2 \leq \left( \frac{a_3}{b} - \frac{1}{m-1} \right) \int_{B_a(R)} w^{2+b} \eta^2.
\]
Combining these inequalities, we derive
\[
- \int_{B_a(R)} \left( \frac{\alpha}{4} + \frac{b}{4} - \frac{C}{4} \right) w^{b-1} \eta^2 |\nabla w|^2 + \frac{a_2}{b} \int_{B_a(R)} w^{2+b} \eta^2
\]
\[
+ \left( \frac{a_3}{b} - \frac{1}{m-1} \right) \int_{B_a(R)} w^{2+b} \eta^2 + \int_{B_a(R)} \left( \frac{a_1}{b} - \frac{C}{b+1} \right) w^{1+b} |\nabla \eta|^2
\geq \int_{B_a(R)} (- V^{\frac{\alpha}{b+1}} \delta) w^{1+b} \eta^2,
\]
For the first term on the LHS of the above inequality, we use
\[
|\nabla (w^{\frac{1+b}{2}} \eta)|^2 \leq \frac{(b+1)^2}{2} w^{b-1} |\nabla w|^2 \eta^2 + 2w^{b+1} |\nabla \eta|^2.
\]
Substituting it into the above inequality, we can get
\[
\frac{2}{(b+1)^2} \left( \frac{\alpha}{4} + \frac{b}{4} - \frac{C}{4} \right) \int_{B_a(R)} |\nabla (w^{\frac{1+b}{2}} \eta)|^2 - 2 \int_{B_a(R)} w^{b+1} |\nabla \eta|^2
\]
\[
+ \left( \frac{a_3}{b} - \frac{1}{m-1} - \frac{a_2}{b} - \frac{a_3}{b} \right) \int_{B_a(R)} w^{2+b} \eta^2 + \int_{B_a(R)} \left( \frac{a_1}{b} - \frac{C}{b+1} \right) w^{1+b} |\nabla \eta|^2
\leq \int_{B_a(R)} V^{-\frac{\alpha}{b+1}} \delta w^{1+b} \eta^2.
\tag{2.9}
\]
We can control the constant \( C \) to make the coefficient of the first term of LHS of inequality (2.9) satisfy
\[
\frac{\alpha}{4} + \frac{b}{4} - \frac{C}{4} \geq \frac{b}{8}.
\]
Therefore, we can get the following inequality of a simple form.
\[
\int_{B_a(R)} |\nabla (w^{\frac{1+b}{2}} \eta)|^2 + b \int_{B_a(R)} w^{2+b} \eta^2
\leq a_4 \int_{B_a(R)} w^{1+b} |\nabla \eta|^2 + b \int_{B_a(R)} w^{1+b} \eta^2,
\]
where \( d_1, d_2 \) are some constants.

In order to prove the main theorem, we need the following Lemma.
Lemma 2.2 For $b_0 > 0$ large enough and $R > 0$, there exists $d_3 > 0$ such that

$$
\|u\|_{L^{(b_0+1)\frac{m}{m-1}}(B_o(\frac{3}{4}R))} \leq d_3 b_0^{b_0+3} V^{\frac{m-1}{m(b_0+1)}} R^{-2}.
$$

Proof. From the Lemma 2.1, we have

$$
(\int_{B_o(R)} w^{(1+b)\frac{2m}{m-1}} \eta^{\frac{2m}{m-1}})^{\frac{m-1}{m}} \leq C_s(n, q) V^{-\frac{n}{m}} \left( \int_{B_o(R)} (|\nabla (w^{\frac{b_0+1}{b}} \eta)|^2 + \int_{B_o(R)} w^{b+1} \eta^2) \right) \leq b_0 V^{-\frac{n}{m}} \left( a_4 R^2 \int_{B_o(R)} w^{1+b} |\nabla \eta|^2 + b_0 R^2 \int_{B_o(R)} w^{1+b} \eta^2 \right) - b_1 \int_{B_o(R)} w^{2+b} \eta^2 + \int_{B_o(R)} w^{b+1} \eta^2),
$$

where $b_0 = C_s(n, q)$ large enough to make $b_0$ satisfy (2.8), then we have

$$
(\int_{B_o(R)} w^{(1+b)\frac{2m}{m-1}} \eta^{\frac{2m}{m-1}})^{\frac{m-1}{m}} \leq a_4 R^2 b_0 V^{-\frac{n}{m}} \left( \int_{B_o(R)} w^{1+b} |\nabla \eta|^2 + b_0 R^2 \int_{B_o(R)} w^{b+1} \eta^2 \right) \leq a_4 R^2 b_0 V^{-\frac{n}{m}} \left( \int_{B_o(R)} w^{1+b} |\nabla \eta|^2 + a_5 b_0^2 b V^{-\frac{1}{m}} \right) \int_{B_o(R)} w^{b+1} \eta^2. (2.10)
$$

We note that $a_5 b_0 b w^{1+b} < \frac{1}{2} b_1 \int_{B_o(R)} w^{2+b}$, when $w > a_5 b_0 R^{-2}$. Thus in the evaluation of the second term on the right hand side of inequality (2.10), we decompose $\omega$ into subregions, one over the places $w > a_5 b_0 R^{-2}$ and the second region is the complement of the first region. With this decomposition we have

$$
a_5 b_0^2 b V^{-\frac{1}{m}} \int_{B_o(R)} w^{b+1} \eta^2 \leq \frac{1}{2} b_1 \int_{B_o(R)} w^{2+b} \eta^2 + a_5 b_0^2 b V^{-\frac{1}{m}} \left( \frac{b_0^2}{b} \right) \int_{B_o(R)} w^{b+1} \eta^2.
$$

Let $\eta \in C^\infty_0(\Omega)$ satisfy $0 \leq \eta \leq 1, \eta \equiv 1$ in $B_o(\frac{3}{4}R), |\nabla \eta| \leq \frac{C}{R}$, and let $\eta = \eta^{\frac{b+1}{b}}$. The first term on the right hand side of (2.6) satisfies

$$
a_4 R^2 b_0 \int_{B_o(R)} w^{1+b} |\nabla \eta|^2 \leq a_5 b^2 b_0 \int_{B_o(R)} w^{1+b} \eta^{\frac{2(b+1)}{b+2}} \leq a_5 b^2 b_0 \left( \int_{B_o(R)} w^{b+2} \eta^2 \right)^{\frac{2(b+1)}{b+2}} V^{\frac{1}{b+2}} \leq \frac{1}{2} b_1 \int_{B_o(R)} w^{2+b} \eta^2 + b_0 b^{b+2} d_1 R^{-2(1+b)} V.
$$

Hence

$$
(\int_{B_o(R)} w^{(1+b)\frac{2m}{m-1}} \eta^{\frac{2m}{m-1}})^{\frac{m-1}{m}} \leq d_2 b^{b+3} d_1 R^{-2(1+b)} V^{1-\frac{m}{m}}.
$$
Assume that $b = b_0$ and $b_1 = (1 + b_0) \frac{m}{m-1}$, we can get
\[
\|w\|_{L^{b_0+1}} \leq d_3 b_0^{b_0+3} V^{\frac{m-1}{m(b_0+1)}} R^{-2}.
\]

Finally, we can prove the Theorem 1.1,

Proof. Since
\[
\lim_{b \to \infty} \|w\|_{L^b(B_o(\frac{R}{2}))} = \|w\|_{L^\infty(B_o(\frac{R}{2}))},
\]
from the definition of limit, we know for any $\epsilon > 0$, $\|w\|_{L^\infty(B_o(\frac{R}{2}))} \leq \|w\|_{L^b(B_o(\frac{R}{2}))} + \epsilon$, therefore, by the Lemma 2.2, we have
\[
\frac{|\nabla u|}{u} \leq C_s(n, q, V) R \text{ on } B_o(\frac{R}{2}).
\]

Proof of Corollary 1.3 Let minimal geodesic $\gamma(s) : [0, 1] \to M^n$, so that $\gamma(0) = y, \gamma(1) = x$, then
\[
\ln \frac{u(x)}{u(y)} = \int_0^1 \frac{d\ln(u(\gamma(s)))}{ds} ds
\]
\[
= \int_0^1 \frac{|\nabla u(\gamma(s))|}{u(\gamma(s))} ds
\]
\[
\leq \int_0^1 \frac{|\nabla u(\gamma(s))|}{u(\gamma(s))} ds = r(x, y) \int_0^1 \frac{|\nabla u|}{u(\gamma(s))} ds
\]
\[
\leq r(x, y) \int_0^1 \frac{C_s}{R} ds
\]
\[
\leq \int_0^1 C_s(n, q). 
\]

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