Measure pseudo Bloch periodic solutions for some difference and differential equations with piecewise constant argument

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Abstract In this work, we show the existence and uniqueness of a measure pseudo (w, k) Bloch periodic solution for differential equation model with piecewise constant argument, contains a generator of an evolution process. We define the new notion of pseudo (w, k) Bloch periodic sequences. After this, some properties of these functions are mentioned. By using the contraction mapping, we prove the main result.

Keywords : Differential equation; difference equation; Measure pseudo Bloch periodic solutions; piecewise constant argument; periodic evolution process.
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1 Introduction

The existence and uniqueness of almost periodic solutions represents one of the most popular questions in the field of differential equations. The diversity of application domains for this type of function has motivated several researchers to study the properties of these functions and their generalisations to solve these problems (See [10, 17, 18, 19, 20]).
Among the first works dealing the study of differential equations with piecewise constant argument, we can quote the article of Shah and Wiener[23]. This type of differential equations has recently found great interest. In fact, this class of equations can describe some hybrid dynamical systems and biomedical models. These equations combines both differential and difference equations.
Yuan and Hong [22] investigated the existence of almost periodic solutions of the differential equation:

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + g(t, x(t), x([t])).$$

Xia et al. [25] discussed the existence of almost periodic solution for the forced perturbed equations with piecewise constant argument of the form:

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + f(t) + \varepsilon g(t, x(t), x([t]), \varepsilon), \quad t \in \mathbb{R}$$

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Piao [21] considered the following differential equation:

$$x'(t) = Mx(t) + M_b x([t]) + g(t), \tag{1}$$

where M and M_b are constant matrices, g is pseudo almost periodic function. He introduced the notion of pseudo almost periodic sequences to prove the existence of pseudo almost periodic solution and he gave sufficient hypothesis to obtain the uniqueness.
In [16], the authors have addressed a more general model

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + M_0 x(t-[t]) + M_1 x'(t-[t]) + g(t), \quad t \in \mathbb{R}.$$

They gave a sufficient conditions to obtain a pseudo almost periodic solution of the equation in different cases.
Zhang and Li [28] presented some existence and uniqueness results concerning weighted pseudo almost periodic solution by applying the theory of exponential dichotomy and the contraction mapping.
In [3], the authors used the composition theorems to find the uniqueness of measure pseudo almost periodic solution for generalized differential equation with piecewise constant argument.
Dimbour and Valmorin in [14], investigated the existence and uniqueness of asymptotically antiperiodic solution for the following nonlinear differential equation:

$$x'(t) = Mx(t) + M_b x([t]) + g(t, x([t]))$$

Later, Dimbour [13] studied the differential equation (1), where M generates a semi group. He presented sufficient hypothesis that allows to obtain a pseudo asymptotic periodic solution and to ensure its uniqueness.
More recently, Ait Dads et al. [2] presented some properties of exponential dichotomiy and proved the existence and uniqueness of pseudo-S-asymptotically w periodic solution of equation with piecewise constant argument.
After its definition by the physicist Bloch, the type of periodic functions bearing his name found increasing interest. Indeed, applications of this type of periodicity can be found in many fields such as solid state physics, condensed matter and quantum mechanics.
Bloch periodic functions are wave functions describing the quantum states of nearly free electrons subjected to the periodic potential of the infinite perfect crystal lattice. It should be mentioned that periodicity and anti-periodicity are two special cases of this kind.
In practice, we may encounter some functions that are not exactly Bloch periodic but asymptotically Bloch periodic which contains two components [15]: the principal (Bloch periodic) component and the corrective part, or pseudo asymptotically Bloch ([4, 5, 6, 9]) or semi-bloch k periodic or Stepanov semi Bloch periodic function and Stepanov semi Bloch anti-periodic function ([8]) or pseudo (w, k) Bloch type periodic function which is uniquely written as the sum of a periodic part and an ergodic perturbation.
Chang and Wei [24] defined the space of weighted pseudo Bloch periodic functions. They established some important properties such as composition and convolution theorems of such functions (see also [7]).
Motivated by this work, we treat the following differential equations with piecewise constant argument:

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + g(t), \tag{2}$$

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + f(t, x(t), x([t])), \tag{3}$$

where $M(t)$ generates an exponentially stable evolutionary process, $M_b(t)$ a bounded linear operator, $[.]$ is the greatest integer function, $g : \mathbb{R} \to \mathbb{X}$ is a measure pseudo Bloch (w, k)-periodic function and $f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is bounded continuous.
We prove the existence and uniqueness of a measure pseudo Bloch periodic solution for the equation model without using the exponential dichotomy outcomes.
To organise our work, we divide the content into four parts :
In the second part, the corresponding difference equation is treated and its explicit solution is given. After this, we recall some definitions and preliminary lemmas, we define the new notion of pseudo (w, k) Bloch periodic sequences and we give some important properties. The main results concerning the existence and uniqueness of solution are mentioned in the third part.
Finally, we finish by giving an illustrative example.

2 Preliminaries

2.1 Measure pseudo Bloch periodic functions

Let U be the set of positives measures λ such that $\lambda([r, s]) < \infty$, $\forall r, s \in \mathbb{R}$, and $\lambda(\mathbb{R}) = +\infty$.
We assume the hypothesis
(H) $\forall \tau \in \mathbb{R}$, there $\exists \alpha > 0$ and a bounded interval I such that

$$\lambda(a + \tau : a \in A) \leq \alpha \lambda(A), \tag{4}$$

when $A \in \mathbb{L}$ and $A \cap I = \emptyset$ with $\lambda([-s, s]) = \int_{-s}^{s} d\lambda(t)$; for $s > 0$ and $\lambda(j) = \lambda(j, j + 1) = \int_{j}^{j+1} d\lambda(t)$ for $j \in \mathbb{Z}$.

Lemma 1 *If λ verify **(H)**, then :*
i) For $\tau \in \mathbb{R}$ and $s \in \mathbb{R}$, there exists $\alpha > 0$ such that :

$$\lambda([s + \tau, s + \tau + 1]) \leq \alpha \lambda([s, s + 1]). \tag{5}$$

***ii)** For $\tau > 0$ and $s > \tau$, there exists $\beta > 0$ such that :*

$$\lambda([-s - \tau, s + \tau]) \leq \beta \lambda([-s, s]). \tag{6}$$

Proof. i) When we take $A = [s, s + 1]$, we find the result.

ii) For $A = [-s, s]$, we have

$$\lambda([-s + \tau, s + \tau]) \leq \alpha_\tau \lambda([-s, s])$$

and

$$\lambda([-s-\tau, s-\tau]) \leq \alpha_{-\tau}\lambda([-s,s]).$$

Then

$$\lambda([-s-\tau, s+\tau]) \leq \lambda([-s+\tau, s+\tau]) + \lambda([-s-\tau, s-\tau]) \leq \beta\lambda([-s,s]).$$

Remark 1 *Let $\rho : \mathbb{R} \to]0,\infty)$ a function locally integrable over $\mathbb{R}$ such that :*

i) *For $s > 0$, $\lim\limits_{s\to\infty} \int_{-s}^{s} \rho(t)dt = \infty$.*

ii) *For all $\tau \in \mathbb{R}$ and $s \in \mathbb{R}$, there exists $v > 0$ such that :*

$$\rho(s+\tau) \leq v\rho(s).$$

Hence,

if $d\lambda(t) = \rho(t)dt$ then (5) and (6) are satisfied.

Let $\mathbb{X}$ be a complex Banach space. We define $BC(\mathbb{R},\mathbb{X}) = \{f : \mathbb{R} \to \mathbb{X},\ bounded,\ continuous\ on\ \mathbb{R}\}$.

Definition 1 *Let $k \in \mathbb{R}$ and $w \in \mathbb{R}$.*

$$BPC_{w,k}(\mathbb{R},\mathbb{X}) = \left\{f \in BC(\mathbb{R},\mathbb{X}), \forall t \in \mathbb{R} : f(t+w) = e^{ikw}f(t)\right\},$$

$$ErgC(\mathbb{R},\mathbb{X},\lambda) = \left\{f \in BC(\mathbb{R},\mathbb{X}),\ \lim_{s\to+\infty} \frac{1}{\lambda([-s,s])}\int_{-s}^{s} \|f(t)\|\, d\lambda(t) = 0\right\}.$$

Remark 2 *[15]*
w and k are called the Bloch period and the Bloch wave vector respectively.

Definition 2 *[24] Let $k \in \mathbb{R}$ and $w \in \mathbb{R}$.*
$f : \mathbb{R} \to \mathbb{X}$ is called continuous measure pseudo (w,k) Bloch periodic or $f \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$, if there exist $f_1 \in BPC_{w,k}(\mathbb{R},\mathbb{X})$ and $f_2 \in ErgC(\mathbb{R},\mathbb{X},\lambda)$ such that:

$$f = f_1 + f_2.$$

Lemma 2 *[24]*
*Let $k \in \mathbb{R}$ and $w \in \mathbb{R}$ and λ satisfy **(H)**.*

$$PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda) = BPC_{w,k}(\mathbb{R},\mathbb{X}) \oplus ErgC(\mathbb{R},\mathbb{X},\lambda).$$

Lemma 3 *[24] Let $f, g \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$. If **(H)** hold, then the following properties are true:*
(1)for each $c \in \mathbb{R}$, $c{\cdot}f \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$.
(2)$f + g \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$.
(3) $f(\cdot{+}b) \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$ for every $b \in \mathbb{R}$.
(4) $PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$ is a Banach space under the supremum norm.

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2.2 Discontinuous measure pseudo Bloch periodic functions

We define

$$BCP(\mathbb{R},\mathbb{X}) = \left\{ f:\mathbb{R}\to\mathbb{X},\ bounded,\ continuous\ on\ \mathbb{R}/\mathbb{Z}, \forall j\in\mathbb{Z}: \lim_{s\to j^-} f(t) < \infty,\ \lim_{s\to j^+} f(t) < \infty \right\}$$

Definition 3 *Let $k\in\mathbb{R}$ and $w\in\mathbb{R}$.*

$$BP_{w,k}(\mathbb{R},\mathbb{X}) = \left\{ f\in BCP(\mathbb{R},\mathbb{X}), \forall t\in\mathbb{R}: f(t+w) = e^{ikw} f(t) \right\},$$

$$Erg\left(\mathbb{R},\mathbb{X},\lambda\right) = \left\{ f\in BCP(\mathbb{R},\mathbb{X}),\ \lim_{s\to+\infty} \frac{1}{\lambda([-s,s])}\int_{-s}^{s} \|f(t)\|\, d\lambda(t) = 0 \right\}.$$

Lemma 4 *1)$BCP(\mathbb{R},\mathbb{X})$ is a Banach space with the sup-norm. ([1])*
*2)If λ satisfy **(H)**, then $Erg\left(\mathbb{R},\mathbb{X},\lambda\right)$ is translation invariant. ([27], Lemma 2.1)*

Definition 4 *Let $k\in\mathbb{R}$ and $w\in\mathbb{R}$.*
$f:\mathbb{R}\to\mathbb{X}$ is called measure pseudo (w,k) Bloch periodic or $f\in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$, if there exist $f_1\in BP_{w,k}(\mathbb{R},\mathbb{X})$ and $f_2\in Erg\left(\mathbb{R},\mathbb{X},\lambda\right)$ such that:

$$f = f_1 + f_2.$$

From Lemma 4 and by the same arguments used in [24], we can state these two lemmas.

Lemma 5 *Let $k\in\mathbb{R}$ and $w\in\mathbb{R}$ and λ satisfy **(H)**.*

$$PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right) = BP_{w,k}(\mathbb{R},\mathbb{X}) \oplus Erg\left(\mathbb{R},\mathbb{X},\lambda\right).$$

Lemma 6 *Let $f,g\in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$. If **(H)** hold, then the following properties are true:*
(1)for each $c\in\mathbb{R}$, $c{\cdot}f\in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$.
(2)$f+g\in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$.
(3) $f(\cdot{+}b)\in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$ for every $b\in\mathbb{R}$.
(4) $PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$ is a Banach space under the supremum norm.

We define

$$BPC_{w,k}(\mathbb{R}{\times}\Gamma,\mathbb{X}) = \left\{ \varrho\in BC(\mathbb{R}\times\Gamma,\mathbb{X}), \varrho(t+w,x) = e^{ikw}\varrho(t,x)\ \forall t\in\mathbb{R}\ uniformly\ in\ x\in\Gamma \right\},$$

$$Erg\left(\mathbb{R}\times\Gamma,\mathbb{X},\lambda\right) = \left\{ \varsigma:\mathbb{R}\times\Gamma\to\mathbb{X}, \forall x\in\Gamma,\ \lim_{s\to+\infty}\frac{1}{\lambda([-s,s])}\int_{-s}^{s}\|\varsigma(t,x)\|\, d\lambda(t) = 0\ uniformly\ in\ x\in\Gamma \right\}$$

Definition 5 *A function $f:\mathbb{R}\times\Gamma\to\mathbb{X}$ is said to be generalized pseudo (w,k) Bloch periodic function if it admits the decomposition*

$$f = f_1 + f_2,$$

where $f_1\in BPC_{w,k}(\mathbb{R}\times\Gamma,\mathbb{X})$ and $f_2\in Erg\left(\mathbb{R}\times\Gamma,\mathbb{X},\lambda\right)$. We denote $PBP_{w,k}(\mathbb{R}\times\Gamma,\mathbb{X})$ the set of all such functions.

Now, we are interested in the sequence part.

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2.3 Measure pseudo Bloch periodic sequences

We denote U_s the set of sequences measures $\lambda : \mathbb{Z} \to +\infty$.

Lemma 7 *[28] Let λ satisfy **(H)** such that $\lambda(j) = \int_j^{j+1} d\lambda(t)$ for $j \in \mathbb{Z}$. Then $\lambda \in U_s$ and for $\kappa \in \mathbb{R}$, there exist $b_1 \geq 0$, $b_2 \geq 0$ such that, for S large enough, we have*

$$b_1 \int_{-(S+\kappa)}^{S+\kappa} d\lambda(t) \leq \sum_{j=-[S]}^{[S]} \lambda(j) \leq b_2 \int_{-(S+\kappa)}^{S+\kappa} d\lambda(t).$$

Definition 6 *Let $k \in \mathbb{R}$ and $w \in \mathbb{Z}$.*

$$BP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda) = \left\{ x : \mathbb{Z} \to \mathbb{X} : \text{ bounded}, \ \forall j \in \mathbb{Z} : x(j+w) = e^{ikw} x(j) \right\},$$

$$Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda) = \left\{ x : \mathbb{Z} \to \mathbb{X} : \text{ bounded}, \ \lim_{N \to +\infty} \frac{1}{\lambda([-N, N])} \sum_{j=-N}^{N} \|x(j)\| \lambda(j) = 0 \right\}.$$

Definition 7 *For $\lambda \in U_s$, $k \in \mathbb{R}$ and $w \in \mathbb{Z}$, $x : \mathbb{Z} \to \mathbb{X}$ a bounded sequence is called measure pseudo Bloch (w, k) periodic sequence (or $x \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, if there exist $x_1 \in BP_{w,k,seq}(\mathbb{Z}, \mathbb{X})$ and $x_2 \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$ such that:*

$$x = x_1 + x_2.$$

Lemma 8 *[27] For $\lambda \in U_s$, **(H)** hold and $j \in \mathbb{Z}$ we have: If $x \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, then $x(. - j) \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$.*

Lemma 9 *For $\lambda \in U_s$, **(H)** hold and $j \in \mathbb{Z}$ we have: If $x \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, then $x(. - j) \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$.*

Proof. For $n \in \mathbb{Z}$, $x(n-j) = x_1(n-j) + x_2(n-j)$ with $x_1(n+w-j) = x_1(n-j+w) = e^{ikw} x_1(n-j)$ and from Lemma 8, $x_2(.-j)$ is ergodic. ∎

Lemma 10 *[28] Let $\hbar : \mathbb{R} \to \mathbb{X}$ continuous, λ satisfy **(H)** such that $\lambda(j) = \int_j^{j+1} d\lambda(t)$ for $j \in \mathbb{Z}$. Assume that $x \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$ and there exist $\eta > 0$ and a finite $Sb \subset \mathbb{Z}$ such that*

$$\|\hbar(t)\| \leq \eta . max_{\{n \in Sb\}} \|x(j+n)\|, \quad t \in [j, j+1), \ j \in \mathbb{Z}.$$

Then $\hbar \in Erg(\mathbb{R}, \mathbb{X}, \lambda)$.

Lemma 11 *Let $x = x_1 + x_2 \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, with $x_1 \in BP_{w,k,seq}(\mathbb{Z}, \mathbb{X})$, $x_2 \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$ and **(H)** hold. Then $x_1(\mathbb{Z}) \subset \overline{co\, x(\mathbb{Z})}$.*

Proof. Let

$$f(t) = (x([t]+1) - x([t]))(t - [t]) + x([t]),$$

$$f_1(t) = (x_1([t]+1) - x_1([t]))(t - [t]) + x_1([t]),$$

$$f_2(t) = (x_2([t]+1) - x_2([t]))(t - [t]) + x_2([t]).$$

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We have $f = f_1 + f_2$, the functions f,f_1 and f_2 are continuous and $f(n) = x(n)$ for $n \in \mathbb{Z}$.

$$\begin{aligned} f_1(t+w) &= (x_1([t+w]+1) - x_1([t+w]))(t+w-[t+w]) + x_1([t+w]) \\ &= (e^{ikw}x_1([t]+1) - e^{ikw}x_1([t]))(t-[t]) + e^{ikw}x_1([t]) = e^{ikw}f_1(t). \end{aligned}$$

Then $f_1 \in BP_{w,k}(\mathbb{R}, \mathbb{X})$.
On other hand

$$\|f_2(t)\| \leq 2.max\{\|x_2(n)\|, \|x_2(n+1)\|\} \leq 2.max_{\{j\in[[0,1]]\}}\{\|x_2(n+j)\|\}, \quad t \in [n, n+1),\ n \in \mathbb{Z}.$$

Using lemma 10, we check that $f_2 \in Erg(\mathbb{R}, \mathbb{X}, \lambda)$. Hence $f \in PBP_{w,k}(\mathbb{R}, \mathbb{X})$.

From [24] lemma 3.4, we deduce that $f_1(\mathbb{R}) \subset \overline{f(\mathbb{R})} = \overline{co\ x(\mathbb{Z})}$.
Since $f(n) = x(n)$ ($f_1(n) = x_1(n)$) for $n \in \mathbb{Z}$, we can conclude that $x_1(\mathbb{Z}) \subset \overline{co\ x(\mathbb{Z})}$. ■

Lemma 12 *For $\lambda \in U_s$, **(H)** hold, $k \in \mathbb{R}$ and $w \in \mathbb{Z}$, $x : \mathbb{Z} \to \mathbb{X}$*

$$PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda) = BP_{w,k,seq}(\mathbb{Z}, \mathbb{X}) \oplus Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda).$$

Proof. *Let $x \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$.*
We assume that exists two decompositions of x : $x = x_{11} + x_{21} = x_{12} + x_{22}$. Then $(x_{11} - x_{12}) + (x_{21} - x_{22}) = 0$ with $x_{11} - x_{12} \in BP_{w,k,seq}(\mathbb{Z}, \mathbb{X})$ and $x_{21} - x_{22} \in Erg_{seq}(\mathbb{Z}, \mathbb{X}, \lambda)$. From Lemma 11, we have $x_{11} = x_{12}$ and $x_{21} = x_{22}$ then the decomposition of x is unique.
■

Lemma 13 *i) If **(H)** hold, $w \in \mathbb{Z}$ and $x(n) \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, $(n \in \mathbb{Z})$, then there exists a function $f \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$ such that: $f(n) = x(n)$, for $n \in \mathbb{Z}$.*

Proof. i) For $w \in \mathbb{Z}$ and $t \in \mathbb{R}$, we take $f(t) = x([t]) = x_1([t]) + x_2([t]) = f_1(t) + f_2(t)$.
Then,

$$f_1(t+w) = x_1([t+w]) = x_1([t]+w) = e^{ikw}x_1([t]) = e^{ikw}f_1(t),$$

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and

$$\frac{1}{\lambda([-s,s])}\int_{-s}^{s}\|f_2(t)\|\,d\lambda(t)$$

$$=\frac{1}{\lambda([-s,s])}\int_{-s}^{s}\|x_2([t])\|\,d\lambda(t)$$

$$\leq\frac{1}{\lambda([-s,s])}\sum_{j=-[s+1]}^{[s+1]}\int_{j}^{j+1}\|x_2([t])\|\,d\lambda(t)$$

$$\leq\frac{1}{\lambda([-s,s])}\sum_{j=-[s]-1}^{[s]+1}\int_{j}^{j+1}\|x_2(j)\|\,d\lambda(t)$$

$$\leq\frac{\lambda([-[s]-1,[s]+1])}{\lambda([-s,s])}.\frac{1}{\lambda([-[s]-1,[s]+1])}\sum_{j=-[s]-1}^{[s]+1}\|x_2(j)\|\,\lambda(j)$$

$$\leq\frac{\beta}{\lambda([-[s]-1,[s]+1])}\sum_{j=-[s]-1}^{[s]+1}\|x_2(j)\|\,\lambda(j).$$

Hence $f\in PBP_{w,k}(\mathbb{R},\mathbb{X},\lambda)$. ■

Remark 3 *The proof of the previous lemma can be found by considering the following continuous function $f(t)=(x([t]+1)-x([t]))(t-[t])+x([t])$. Therefore, we also have: if $x(n)\in PBP_{w,k,seq}(\mathbb{Z},\mathbb{X},\lambda)$, $(n\in\mathbb{Z})$, then there exists a function $f\in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$ such that: $f(n)=x(n)$, for $n\in\mathbb{Z}$. (for the ergodic part, see [16] Proposition 2.2)*

From Lemma 12 in Assel, Hammami and Miraoui [3], we can mention this lemma :

Lemma 14 *If **(H)** hold and $g\in Erg(\mathbb{R},\mathbb{X},\lambda)$, then*

$$\lim_{N\to+\infty}\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\int_{j}^{j+1}\|g(u)\|\,du\lambda(j)=0.$$

Lemma 15 *If **(H)** hold and $f\in PBP_{w,k}(\mathbb{R},\mathbb{X},\lambda)$, then for $a\in\mathbb{R}$,*

$$I_a(j)=\int_{j}^{j+a}f(u)du\in PBP_{w,k,seq}(\mathbb{Z},\mathbb{X},\lambda).$$

Proof. Without loss of generality, we assume that $a>0$.
$I_a(j)=\int_j^{j+a}f_1(u)du+\int_j^{j+a}f_2(u)du$, with $f_1\in BP(\mathbb{R},\mathbb{X},\lambda)$ and $f_2\in Erg(\mathbb{R},\mathbb{X},\lambda)$.

For $j\in\mathbb{Z}$, $\int_{j+w}^{j+w+a}f_1(u)du=\int_j^{j+a}f_1(u+w)du=e^{ikw}\int_j^{j+a}f_1(u)du$.

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$$\lim_{N\to+\infty}\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\left\|\int_{j}^{j+a}f_2(u)du\right\|\lambda(j)$$

$$\leq \lim_{N\to+\infty}\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\int_{j}^{j+a}\|f_2(u)\|\,du\lambda(j)$$

$$\leq \lim_{N\to+\infty}\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\sum_{l=0}^{[a]}\int_{j+l}^{j+l+1}\|f_2(u)\|\,du\lambda(j)$$

$$\leq \textstyle\sum_{l=0}^{[a]}\displaystyle\lim_{N\to+\infty}\frac{1}{\lambda([-N,N])}\sum_{j=-N-l}^{N+l}\int_{j}^{j+1}\|f_2(u)\|\,du\lambda(j-l)$$

$$\leq \textstyle\sum_{l=0}^{[a]}\displaystyle\lim_{N\to+\infty}\frac{\alpha\beta}{\lambda([-N-l,N+l])}\sum_{j=-N-l}^{N+l}\int_{j}^{j+1}\|f_2(u)\|\,du\lambda(j).$$

■

3 Main results

3.1 Difference equation

We consider the differential equations

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + g(t), \tag{7}$$

$$x'(t) = M(t)x(t) + M_b(t)x([t]) + f(t, x(t), x([t])), \tag{8}$$

where $M(t)$ generates an exponentially stable evolutionary process, $M_b(t)$ is a bounded linear operator, $[.]$ is the greatest integer function, $g : \mathbb{R} \to \mathbb{X}$ is a measure pseudo Bloch (w, k)-periodic function and $f : \mathbb{R}\times\mathbb{X}\times\mathbb{X} \to \mathbb{X}$ is bounded continuous.

Definition 8 *A solution of equation* (7) *(or* (8)*) on* $\mathbb{R}$ *is a function* $x(.)$ *that satisfies :*
1) $x(.)$ *is continuous on* $\mathbb{R}$.
2) The derivative $x'(.)$ *exists at each point* $t \in \mathbb{R}$*, with possible exception of the points* $[t]$*, where one -sided derivatives exists.*
3) System (7) *(or* (8)*) is satisfied on each interval* $[j, j + 1[$ *with* $j \in \mathbb{Z}$.

For the function $\Im(u) = Ev(t, u)x(u)$ with $u \leq t$, we have

$$\begin{aligned}\frac{d\Im}{du}(u) &= \frac{dEv(t,u)}{du}x(u) + Ev(t,u)x'(u)\\ &= -M(u)Ev(t,u)x(u) + Ev(t,u)M(u)x(u) + Ev(t,u)M_b(u)x([u]) + Ev(t,u)g(u)\\ &= Ev(t,u)M_b(u)x([u]) + Ev(t,u)g(u).\end{aligned}$$

Then

$$\int_{[t]}^{t} \frac{d\Im(u)}{du} du = \int_{[t]}^{t} Ev(t,u)M_b(u)x([u]) + Ev(t,u)g(u)du.$$

Therefore

$$Ev(t,t)x(t) - T(t-[t])x([t]) = \int_{[t]}^{t} Ev(t,u)M_b x([u]) + Ev(t,u)g(u))du.$$

Hence, the solution of the equation (7) is : $\forall t \in [[t],[t]+1[$:

$$x(t) = Ev(t,[t])x([t]) + \int_{[t]}^{t} Ev(t,u)\left[M_b(u)x([t]) + g(u)\right] du. \tag{9}$$

If we denote n=[t] and we use the continuity of solution, then we obtain :

$$x(n+1) = Ev(n+1,n)x(n) + \int_{n}^{n+1} Ev(n+1,u)\left[M_b(u).x(n) + g(u)\right] du.$$

We find the equation's form:

$$x(n+1) = Q(n).x(n) + l(n), n \in \mathbb{Z} \tag{10}$$

where

$$Q(n) = Ev(n+1,n) + \int_{n}^{n+1} Ev(n+1,u)M_b(u)du.$$

$$l(n) = \int_{n}^{n+1} Ev(n+1,u)g(u)du.$$

We assume :
C) $M(t)$ generates an exponentially stable evolutionary process $(Ev(t,u))_{t\geq u}$ in $\mathbb{X}$, a family of bounded linear operators, satisfy the following hypothesis :
(1) $\forall t \geq 0$, $Ev(t,t) = I$, I is the identity operator.
(2) $\forall t \geq u \geq r$, $Ev(t,u)Ev(u,r) = Ev(t,r)$.
(3) The map $(t,u) \to Ev(t,u)x$ is continuous for every fixed $x \in \mathbb{X}$.
(4) $\forall t \geq u$, $Ev(t+w,u+w) = Ev(t,u)$.
(5) $\exists m_{Ev} > 0$ and $\delta > 0$, such that $\|Ev(t,u)\| \leq m_{Ev}e^{-\delta(t-u)}$, $\forall t \geq u$.

3.2 Solution study

Lemma 16 *If **(H)**, **(C)** hold and* $g \in PBP_{w,k}(\mathbb{R},\mathbb{X},\lambda)$, *then* $l(n) \in PBP_{w,k,seq}(\mathbb{Z},\mathbb{X},\lambda)$.

Proof.
If $g \in PBP_{w,k}(\mathbb{R},\mathbb{X},\lambda)$, then we can write $g = g_1 + g_2$ with $g_1 \in BP_{w,k}(\mathbb{R},\mathbb{X},\lambda)$ and $g_2 \in Erg(\mathbb{R},\mathbb{X},\lambda)$.
Hence,

$$l(n) = \int_{n}^{n+1} Ev(n+1,s)g(s)ds = \int_{n}^{n+1} Ev(n+1,s)g_1(s)ds + \int_{n}^{n+1} Ev(n+1,s)g_2(s)ds$$

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We have

$$\int_{n+w}^{n+w+1} Ev(n+w+1,s)g_1(s)ds = \int_n^{n+1} Ev(n+w+1,s+w)g_1(s+w)ds = e^{ikw}\int_n^{n+1} Ev(n+1,s)g_1(s)ds.$$

On other hand

$$\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\left\|\int_j^{j+1} Ev(j+1,s)g_2(s)ds\right\|\lambda(j)$$
$$\leq \frac{m_{Ev}}{\lambda([-N,N])}\sum_{j=-N}^{N}\int_j^{j+1}\|g_2(s)\|\,ds\lambda(j).$$

Then, $l(n) = \int_n^{n+1} Ev(n+1,u)g(u)du \ \in PBP_{w,k,seq}\left(\mathbb{Z},\mathbb{X},\lambda\right).$ ■

Lemma 17 *If **(H)**, **(C)** hold, $g \in PBP_{w,k}\left(\mathbb{R},\mathbb{X},\lambda\right)$, M_b is w periodic and $sup_{p\in\mathbb{Z}}(\|Q(p)\|) < 1$, then*

$$x(n) = l(n-1) + \sum_{k=-\infty}^{n-2}\Bigg(\prod_{j=k+1}^{n-1} Q(j)\Bigg)l(k)$$

is a measure pseudo Bloch periodic solution sequence of difference system (10). ($x(n) \in PBP_{w,k,seq}\left(\mathbb{Z},\mathbb{X},\lambda\right)$).

Proof If we have

$$x(n) = l(n-1) + \sum_{p=-\infty}^{n-2}\prod_{j=p+1}^{n-1} Q(j)l(p),$$

then

$$\begin{aligned}
x(n+1) &= l(n) + \sum_{p=-\infty}^{n-1}\prod_{j=p+1}^{n} Q(j)l(p)\\
&= l(n) + \Bigg(\sum_{p=-\infty}^{n-2}\prod_{j=p+1}^{n} Q(j)l(p) + (Q(n-1)l(n-1))\Bigg)\\
&= l(n) + \Bigg(\sum_{p=-\infty}^{n-2} Q_n\prod_{j=p+1}^{n-1} Q(j)l(p) + (Q(n-1)l(n-1))\Bigg)\\
&= l(n) + Q(n)\Bigg(\sum_{p=-\infty}^{n-2}\prod_{j=p+1}^{n-1} Q(j)l(p) + (Q(n-1)l(n-1))\Bigg)\\
&= l(n) + \Bigg(Q(n)x(n)\Bigg).
\end{aligned}$$

Hence, $x(n) = l(n-1) + \sum_{p=-\infty}^{n-2}\prod_{j=p+1}^{n-1} Q(j)l(p)$ is a solution of the equation (10).

Now, we verify that $x(n) \in PBP_{w,k,seq}\left(\mathbb{Z},\mathbb{X},\lambda\right)$. Clearly

$$x(n) = x_1(n) + x_2(n),$$

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with

$$x_1(n) = l_1(n-1) + \sum_{p=-\infty}^{n-2} \prod_{j=p+1}^{n-1} Q(j) l_1(p)$$

and

$$x_2(n) = l_2(n-1) + \sum_{p=-\infty}^{n-2} \prod_{j=p+1}^{n-1} Q(j) l_2(p).$$

We have

$$\begin{aligned} x_1(n+w) &= l_1(n+w-1) + \sum_{p=-\infty}^{n+w-2} \prod_{j=p+1}^{n+w-1} Q(j) l_1(p) \\ &= e^{ikw} l_1(n-1) + \sum_{p=-\infty}^{n-2} \prod_{j=p+1}^{n-1} Q(j+w) l_1(p+w) \\ &= e^{ikw} l_1(n-1) + \sum_{p=-\infty}^{n-2} \left(\prod_{j=p+1}^{n-1} Q(j+w) \right) e^{ikw} l_1(p) \\ &= e^{ikw} l_1(n-1) + e^{ikw} \sum_{p=-\infty}^{n-2} \left(\prod_{j=p+1}^{n-1} Q(j) \right) l_1(p) \\ &= e^{ikw} x_1(n). \end{aligned}$$

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Now, we prove that x_2 is the ergodic perturbation of x .

$$\frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\|x_2(j)\|\,\lambda(j)$$

$$= \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\left\|l_2(j-1)+\sum_{q=-\infty}^{j-2}(\prod_{p=q+1}^{j-1}Q(p))l_2(q)\right\|\lambda(j)$$

$$\leq \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\|l_2(j-1)\|\,\lambda(j)$$

$$+ \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\left\|\sum_{q=-\infty}^{j-2}(\prod_{p=q+1}^{j-1}Q(p))(l_2(q))\right\|\lambda(j)$$

$$\leq \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\|l_2(j-1)\|\,\lambda(j)$$

$$+ \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\sum_{q=-\infty}^{j-2}(sup_{p\in\mathbb{Z}}\|Q(p)\|)^{j-q-1}\|l_2(q))\|\,\lambda(j)$$

$$\leq \frac{1}{\lambda([-N,N])}\sum_{j=-N-1}^{N-1}\|l_2(j)\|\,\lambda(j+1)$$

$$+ \frac{1}{\lambda([-N,N])}\sum_{j=-N}^{N}\sum_{q=1}^{+\infty}(sup_{p\in\mathbb{Z}}\|Q(p)\|)^{q}\|l_2(j-1-q))\|\,\lambda(j).$$

$$\leq \frac{1}{\lambda([-N,N])}\sum_{j=-N-1}^{N+1}\|l_2(j-1)\|\,\lambda(j)$$

$$+ \frac{1}{\lambda([-N,N])}\sum_{q=1}^{+\infty}(sup_{p\in\mathbb{Z}}\|Q(p)\|)^{q}\sum_{j=-N-(1+q)}^{N-(1+q)}\|l_2(j))\|\,\lambda(j+1+q).$$

$$\leq \frac{\lambda([-N-1,N+1])}{\lambda([-N,N])}\frac{1}{\lambda([-N-1,N+1])}\sum_{j=-N-1}^{N+1}\|l_2(j)\|\,\lambda(j+1)$$

$$+ \sum_{q=1}^{+\infty}(sup_{p\in\mathbb{Z}}\|Q(p)\|)^{q}\frac{\lambda([-(N+q+1),N+q+1])}{\lambda([-N,N])}.\frac{\alpha}{\lambda([-(N+q+1),N+q+1])}\sum_{j=-N-(1+q)}^{N+(1+q)}\|l_2(j))\|\,\lambda(j)$$

$$\leq \frac{\alpha\beta}{\lambda([-N-1,N+1])}\sum_{j=-N-1}^{N+1}\|l_2(j)\|\,\lambda(j)$$

$$+ \sum_{q=1}^{+\infty}(sup_{p\in\mathbb{Z}}\|Q(p)\|)^{q}.\frac{\alpha\beta}{\lambda([-(N+q+1),N+q+1])}\sum_{j=-N-(1+q)}^{N+(1+q)}\|l_2(j))\|\,\lambda(j).$$

From lemma 16 and the limit theorem of series, we can find the result. ∎

Theorem 1 *If* ***(H)***, ***(C)*** *hold,* $g \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$*,* M_b *is* w *periodic and* $sup_{p\in\mathbb{Z}}(\|Q(p)\|) < 1$*, then*
the system (7) has a measure pseudo Bloch periodic solution. $\Big(x(.) \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)\Big)$.

Proof. We have

$$\begin{aligned} x(t) &= Ev(t,[t])x([t]) + \int_{[t]}^{t} Ev(t,u)[M_b(u)x([t]) + g(u)]du \\ &= Ev(t,[t])x_1([t]) + \int_{[t]}^{t} Ev(t,u)[M_b(u)x_1([t]) + g_1(u)]du \\ &+ Ev(t,[t])x_2([t]) + \int_{[t]}^{t} Ev(t,u)[M_b(u)x_2([t]) + g_2(u)]du. \end{aligned}$$

We denote

$$x_1(t) = Ev(t,[t])x_1([t]) + \int_{[t]}^{t} Ev(t,u)[M_b(u)x_1([t]) + g_1(u)]du,$$

and

$$x_2(t) = Ev(t,[t])x_2([t]) + \int_{[t]}^{t} Ev(t,u)[M_b(u)x_2([t]) + g_2(u)]du.$$

We have

$$\begin{aligned} x_1(t+\omega) &= Ev(t+\omega,[t]+\omega)x_1([t]+\omega) \\ &+ \int_{[t]}^{t} Ev(t+\omega,u+\omega)M_b(u+\omega)x_1([t]+\omega)du \\ &+ \int_{[t]}^{t} Ev(t+\omega,u+\omega)g_1(u+\omega)du \\ &= e^{ik\omega}x_1(t). \end{aligned}$$

On other hand,

$$\begin{aligned} \frac{1}{\lambda([-s,s])}\int_{-s}^{s} \|x_2(t)\|\, d\lambda(t) &\le \frac{1}{\lambda([-s,s])}\sum_{[-s]-1}^{[s]+1}\int_{j}^{j+1} \|x_2(t)\|\, d\lambda(t) \\ &\le J_1 + J_2 + J_3, \end{aligned}$$

where

$$J_1 = \frac{1}{\lambda([-s,s])}\sum_{[-s]-1}^{[s]+1}\int_{j}^{j+1} \|Ev(t,[t])x_2([t]))\|\, d\lambda(t),$$

$$J_2 = \frac{1}{\lambda([-s,s])}\sum_{[-s]-1}^{[s]+1}\int_{j}^{j+1}\int_{[t]}^{t} \|Ev(t,u)(M_b(u)x_2([t]))\|\, dud\lambda(t),$$

$$J_3 = \frac{1}{\lambda([-s,s])}\sum_{[-s]-1}^{[s]+1}\int_{j}^{j+1}\int_{[t]}^{t} \|Ev(t,u)(g_2(u))\|\, dud\lambda(t).$$

First, we start by J_1.

$$J_1 \leq \frac{1}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \|Ev(t,[t])\| \, \|x_2([t])\| \, d\lambda(t)$$
$$\leq \frac{m_{Ev}}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \|x_2([t])\| \, d\lambda(t)$$
$$\leq \frac{m_{Ev}}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \|x_2(j)\| \, \lambda(j).$$

Now, we turn to J_2.

$$J_2 \leq \frac{1}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \int_j^{j+1} \|Ev(t,u)(M_b(u)x_2([t]))\| \, du d\lambda(t)$$
$$\leq \frac{m_{Ev} \|M_b\|}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \|x_2(j)\| \, \lambda(j).$$

We finish by J_3.

$$J_3 \leq \frac{1}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \int_{[t]}^{t} \|Ev(t,u)\| \, \|g_2(u)\| \, du d\lambda(t)$$
$$\leq \frac{1}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \int_j^{j+1} \|Ev(t,u)\| \, \|g_2(u)\| \, du d\lambda(t)$$
$$\leq \frac{m_{Ev}}{\lambda([-s,s])} \sum_{[-s]-1}^{[s]+1} \int_j^{j+1} \|g_2(u)\| \, du \lambda(j).$$

From Lemma 14 and Lemma 16, we prove that the solution $x(.) \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$. ■

Lemma 18 *Let* $f \in PBP_{w,k}(\mathbb{R}\times\Gamma, \mathbb{X})$, $x \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$ *and* $f = f_1 + f_2$, $x = x_1 + x_2$, $\Gamma \subset \mathbb{X} \times \mathbb{X}$ *and* $x_1(\mathbb{R}) \times x_1(\mathbb{Z}) \subset \Gamma$. *If* f *verify*

(A.1) $\exists L_1 > 0$, *such that :*

For all $\phi_1, \phi_2, \xi_1, \xi_2 \in \mathbb{X}$ *and* $t \in \mathbb{R}$

$$\|f(t,\phi_1,\xi_1) - f(t,\phi_2,\xi_2)\| \leq L_1(\|\phi_1 - \phi_2\| + \|\xi_1 - \xi_2\|),$$

then $f_2(., x_1(.), x_1([.])) \in Erg(\mathbb{R}, \mathbb{X}, \lambda)$.

Proof. We have $x_1(\mathbb{R})$ is bounded and f_1 is uniformly continuous, then
$\forall \epsilon > 0$, there exist finite open balls B_p with center $(\tau_1^{(p)}, \tau_2^{(p)}) \in \Gamma$, $p = 1, 2, ..., q$, and

radius small enough such that $x_1(\mathbb{R}) \times x_1(\mathbb{Z}) \subset \cup_{p=1}^{q} B_p$ and
$\left\| f_1(t, \tau_1, \tau_2 - f_1(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| < \frac{\epsilon}{3}$ for $\tau = (\tau_1, \tau_2) \in B_p, \quad t \in \mathbb{R}$.
We have $\mathbb{R} = \cup_{p=1}^{q} F_p$ where F_p is the open set $F_p = \{t \in \mathbb{R}, (x_1(t), x_1([t])) \in B_p\}$.
Let $E_1 = F_1$, $E_p = F_p \setminus \cup_{i=1}^{p-1} F_i$, $2 \leq p \leq q$. Hence $E_p \cap E_i = \varnothing$, if $p \neq i$, $1 \leq p, i \leq q$.
Since $f_2(., \tau_1^{(p)}, \tau_2^{(p)}) \in Erg\,(\mathbb{R}, \mathbb{X}, \lambda)$, then there exists $s_0 > 0$ such that

$$\sum_{p=1}^{q} \frac{1}{\lambda([-s, s])} \int_{-s}^{s} \left\| f_2(., \tau_1^{(p)}, \tau_2^{(p)}) \right\| d\lambda(t) < \frac{\epsilon}{3},\ s > s_0.$$

We have

$$\frac{1}{\lambda([-s, s])} \int_{-s}^{s} \| f_2(t, x_1(t), x_1([t])) \| \, d\lambda(t)$$

$$\leq \frac{1}{\lambda([-s, s])} \sum_{p=1}^{q} \int_{[-s,s] \cap E_p} \left\| f_2(t, x_1(t), x_1([t])) - f_2(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| + \left\| f_2(., \tau_1^{(p)}, \tau_2^{(p)}) \right\| d\lambda(t)$$

$$\leq \frac{1}{\lambda([-s, s])} \sum_{p=1}^{q} \int_{[-s,s] \cap E_p} \left\| f(t, x_1(t), x_1([t]) - f(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\|$$

$$+ \left\| f_1(t, x_1(t), x_1([t]) - f_1(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| + \left\| f_2(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| d\lambda(t)$$

Using the hypothesis **(A.1)**, we can write

$$\frac{1}{\lambda([-s, s])} \int_{-s}^{s} \| f_2(t, x_1(t), x_1([t])) \| \, d\lambda(t)$$

$$\leq \frac{1}{\lambda([-s, s])} \sum_{p=1}^{q} \int_{[-s,s] \cap E_p} L_1(\left\| x_1(t) - \tau_1^{(p)} \right\| + \left\| x_1([t]) - \tau_2^{(p)} \right\|$$

$$+ \left\| f_1(t, x_1(t), x_1([t]) - f_1(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| + \left\| f_2(t, \tau_1^{(p)}, \tau_2^{(p)}) \right\| d\lambda(t)$$

From previous increases, we have :
For $s > s_0$, $\frac{1}{\lambda([-s, s])} \int_{-s}^{s} \| f_2(t, x_1(t), x_1([t])) \| \, d\lambda(t) < \epsilon$. Then, we obtain the result. ■

Lemma 19 *Let $f = f_1 + f_2 \in PBP\,(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$.*

*If **(H)** hold and f satisfies the hypothesis*

(A.1) *$\exists L_1 > 0$, such that :*

For all $\phi_1, \phi_2, \xi_1, \xi_2 \in \mathbb{X}$ and $t \in \mathbb{R}$

$$\| f(t, \phi_1, \xi_1) - f(t, \phi_2, \xi_2) \| \leq L_1(\| \phi_1 - \phi_2 \| + \| \xi_1 - \xi_2 \|).$$

(A.2) $f_1(t+w, e^{ikw}x, e^{ikw}y) = e^{ikw} f_1(t,x,y)$ *for all* $(t,x,y) \in \mathbb{R} \times \mathbb{X} \times \mathbb{X}$.

Then for each $x \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$ *such that* $x(n) \in PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, *we have* $f(., x(.), x([.])) \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$.

Proof.
We have $f = f_1 + f_2$ and $x = x_1 + x_2$, then we can write
$f(t, x(t), x([t])) = f_1(t, x_1(t), x_1([t])) + f(t, x(t), x([t])) - f(t, x_1(t), x_1([t])) + f_2(t, x_1(t), x_1([t])) = F_1(t) + F_2(t) + F_3(t)$,
where $F_1(t) = f_1(t, x_1(t), x_1([t]))$, $F_2(t) = f(t, x(t), x([t])) - f(t, x_1(t), x_1([t]))$ and $F_3(t) = f_2(t, x_1(t), x_1([t]))$.

We have $F_1(t+w) = f_1(t+w, x_1(t+w), x_1([t+w])) = f_1(t+w, e^{ikw}x_1(t), e^{ikw}x_1([t])) = e^{ikw} f_1(t, x_1(t), x_1([t])) = e^{ikw} F_1(t)$. Therefore $F_1 \in BP_{w,k}(\mathbb{R}, \mathbb{X})$.

It remains to prove that F_2 and F_3 are ergodic.
From the hypothesis **(A.1)**, we have

$$\lim_{s\to\infty} \frac{1}{\lambda([-s,s])} \int_{-s}^{s} \|f(t, x(t), x([t])) - f(t, x_1(t), x_1([t]))\| \, d\lambda(t)$$

$$\leq \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \int_{-s}^{s} \|x(t) - x_1(t)\| \, d\lambda(t) + \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \int_{-s}^{s} \|x([t]) - x_1([t])\| \, d\lambda(t)$$

$$\leq \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \int_{-s}^{s} \|x_2(t)\| \, d\lambda(t) + \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \int_{-s}^{s} \|x_2([t])\| \, d\lambda(t)$$

$$\leq \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \int_{-s}^{s} \|x_2(t)\| \, d\lambda(t) + \lim_{s\to\infty} \frac{L_1}{\lambda([-s,s])} \sum_{-[s]-1}^{[s]+1} \|x_2(j)\| \, \lambda(j).$$

Since x_2 is ergodic, hence F_2 is ergodic.
From Lemma 18, we can deduce that $F_3 \in Erg(\mathbb{R}, \mathbb{X}, \lambda)$. Finally, we can confirm that $f(., x(.), x([.])) \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$.
■

Theorem 2 *Let* $f \in PBP_{w,k}(\mathbb{R} \times \mathbb{X} \times \mathbb{X}, \mathbb{X})$. *If* ***(H)***, ***(C)***, *(A.1) and (A.2) hold,* M_b *is* ω *periodic and* $sup_{p\in\mathbb{Z}}(\|Q_p\|) < 1$, *then there exist* $L > 0$ *such that* $\forall\ 0 < L_1 < L$, *the equation (8) has a unique continuous measure pseudo Bloch periodic solution.*

Proof. From Lemma 19, if $x \in PBPC_{w,k}(\mathbb{R}, \mathbb{X}, \lambda) \cap PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$, then $f(., x(.), x([.])) \in PBP_{w,k}(\mathbb{R}, \mathbb{X}, \lambda)$.
Hence, by Theorem 1 there exists a measure pseudo Bloch periodic solution of system (8).
We consider the application
$K : PBPC_{w,k}(\mathbb{R}, \mathbb{X}, \lambda) \cap PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda) \to PBPC_{w,k}(\mathbb{R}, \mathbb{X}, \lambda) \cap PBP_{w,k,seq}(\mathbb{Z}, \mathbb{X}, \lambda)$

defined by :

$$Kx(t) = Ev(t,[t])x([t]) + \int_{[t]}^{t} Ev(t,u)\left[M_b(u)x([t]) + f(u,x(u),x([u])\right] du. \qquad (11)$$

For $x, y \in PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$ such that $x(n), y(n) \in PBP_{w,k,seq}(\mathbb{Z},\mathbb{X},\lambda)$, it is easy to verify that $Kx - Ky$ is a solution of this equation

$$\phi'(t) = M(t)\phi(t) + M_b(t)\phi([t]) + f(t,x(t),x([t]) - f(t,y(t),y([t]).$$

Then

$$Kx(n+1) - Ky(n+1) = Q(n)(Kx(n) - Ky(n)) + H(n), n \in \mathbb{Z} \qquad (12)$$

and

$$Kx(n) - Ky(n) = H(n-1) + \sum_{k=-\infty}^{n-2} \left(\prod_{j=k+1}^{n-1} Q(j)\right)H(k),$$

where $H(n) = \int_n^{n+1} Ev(n+1,u)(f(u,x(u),x([u]) - f(u,y(u),y([u]))du$.
Hence, from (**(A.1)**) and (**(C.5)**)

$$\|Kx([t]) - Ky([t])\| \leq \frac{m_{Ev}}{\delta}\left(\frac{1}{1 - sup_{p\in\mathbb{Z}}(\|Q_p\|)}\right)2L_1 \|x-y\|_\infty .$$

We recall that

$$Kx(t) - Ky(t) = Ev(t,[t])(x([t]) - y([t])) + \int_{[t]}^{t} Ev(t,u)M_b(u)(x([t] - y([t]))du$$
$$+ \int_{[t]}^{t} Ev(t,u)(f(u,x(u),x([u]) - f(u,y(u),y([u]))du.$$

Therefore

$$\|Kx(t) - Ky(t)\| \leq (m_{Ev} + \frac{m_{Ev}\|M_b\|}{\delta}) \|x([t]) - y([t])\| + \frac{m_{Ev}}{\delta} \|f(u,x(u),x([u]) - f(u,y(u),y([u])\| .$$

We deduce that :

$$\|Kx(t) - Ky(t)\| \leq 2\frac{m_{Ev}}{\delta}((m_{Ev} + \frac{m_{Ev}\|M_b\|}{\delta})(\frac{1}{1 - sup_{p\in\mathbb{Z}}(\|Q(p)\|)}) + 1)L_1 \|x-y\|_\infty .$$

We denote $a_0 = \dfrac{m_{Ev}}{\delta} > 0$, $a_1 = \dfrac{1}{1 - sup_{p\in\mathbb{Z}}(\|Q(p)\|)} > 0$ and $a_2 = \|M_b\| + \delta > 0$.
Hence , if $0 < L_1 < L = \dfrac{1}{2(a_0)^2 a_1 a_2 + 2a_0}$, then
G is a contraction mapping on $PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda) \cap PBP_{w,k,seq}(\mathbb{Z},\mathbb{X},\lambda)$ which is a closed subset of the Banach space $PBPC_{w,k}(\mathbb{R},\mathbb{X},\lambda)$.
By the Banach-Picard fixed point theorem, we conclude that the equation (8) has a unique continuous measure pseudo Bloch periodic solution. ■

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4 Example

We consider, for $t \in \mathbb{R}$, $x \in [0,\pi]$ the following differential equation :

$$\begin{cases} \frac{\partial \varphi}{\partial t}(t,x) = \frac{\partial^2 \varphi}{\partial x^2}(t,x) + \nu(t,x)\varphi(t,x) + z\varphi([t],x) + g(t,\varphi(t,x),\varphi([t],x)) \\ \varphi(t,0) = \varphi(t,\pi) = 0, \end{cases} \tag{13}$$

where $\nu : \mathbb{R} \times [0,\pi] \to \mathbb{R}$, $\nu(t+\omega,x) = \nu(t,x)$ and $\exists c > 0$, such that $\nu(t,x) \leq -c$.

Let $(\mathbb{X}, \|.\|) = (L^2[0,\pi], \|.\|_2\ )$ and we take the bounded linear operator $M_b : L^2[0,\pi] \to L^2[0,\pi]$ defined by $M_b(t)x = zx$, where z is real constant.
We define the operator M by :

$$M\phi = \phi''$$
$$D(M) = \left\{\phi \in L^2[0,\pi], \phi'' \in L^2[0,\pi], \phi(0) = \phi(\pi) = 0\right\}.$$

The corresponding semi group T on $L^2[0,\pi]$ satisfies: for $t \geq 0$, $\|T(t)\| \leq e^{-t}$ ([11]).
We define $M(t)$ as follows :

$$D(M(t)) = D(M),$$
$$M(t) = M + \nu(t,.).$$

$M(t)$ generate the w periodic evolution process Ev defined by $Ev(t,u) = T(t-u)e^{\int_u^t \nu(s,x)ds}$.
Then, we have $\|Ev(t,u)\| \leq e^{-(c+1)(t-u)}$ ([12, 26]).

On the other side, from [5] we can say : if f can be written in the following form :
$f(t,x_1,y_1) = f_1(t,x_1,y_1) + f_2(t,x_1,y_1) = \frac{1}{5}(B(t)(x_1+y_1) + e^{-t^2}(\frac{x_1}{1+\mid x_1 \mid} + \frac{y_1}{1+\mid y_1 \mid}))$,
where B is a bounded continuous w periodic function, then we have :

$$f_1(t+w, e^{ikw}x_1, e^{ikw}y_1) = B(t+w)(e^{ikw}x_1 + e^{ikw}y_1) = e^{ikw}B(t)(x_1+y_1) = e^{ikw}f_1(t,x_1,y_1),$$

and

$$\|f(t,x_1(t,.),y_1(t,.)) - f(t,x_2(t,.),y_2(t,.))\|_2 \leq \frac{1}{5}(\mid B(t) \mid +1)(\|x_1(t,.) - x_2(t,.)\|_2 + \|y_1(t,.) - y_2(t,.)\|_2).$$

Subsequently f verify the hypothesis of lemma (19).

The system (13) can be seen as an application of the equation (8) with $\varphi(t) = \varphi(t,.)$.

$$sup_{p\in\mathbb{Z}} \|Q_p\| \leq m_E(e^{-\delta} + \frac{1-e^{-\delta}}{\delta}\|M_b\|). \tag{14}$$

For this example, we have $m_{Ev} = 1$, $\delta = c+1$ and $\|M_b\| = \mid z \mid$.
If $\mid z \mid < \delta$, then $sup_{p\in\mathbb{Z}} \|Q_p\| < 1$ and $L_1 < L = \frac{(1-e^{-\delta})(\delta^2 - \delta \mid z \mid)}{4\delta - 2\delta e^{-\delta} + 2e^{-\delta} \mid z \mid}$, $(L > 0)$.
Then, the system (13) has an unique continuous measure pseudo-asymptotically Bloch periodic solution.

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