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TWO-DIMENSIONAL NEUTRAL PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AND ITS STABILITY: A NEW EXPLORATION

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ABSTRACT. Investigating the qualitative properties of two-dimensional neutral integro-differential equations of fractional order is the primary goal of this article. To begin with, we consider some assumptions to establish the theoretical results. Then, the existence and uniqueness of the solution of the considered two-dimensional neutral integro-differential equation are established using Banach's and Krasnoselskii's fixed point theorems. Furthermore, we analyze the stability of the solution by considering some suitable conditions for the initial data. Additionally, we have obtained conditions for the existence of positive, maximal, and minimal solutions to the problem, followed by the Continuation theorem. The paper concludes with a few numerical examples to illustrate and validate the theoretical results.

1. Introduction

A branch of Mathematical analysis known as fractional calculus extends the idea of differentiation and integration to non-integer levels. Fractional calculus, which deals with fractional orders of differentiation and integration rather than just integer orders, finds applications in various fields such as modeling and simulating complex systems, understanding dynamical systems, enhancing control theory, representing biological phenomena, studying heat conduction, addressing non-local effects, and accounting for memory-related phenomena.

Differential and integral equations serve as highly effective tools for representing and describing physical phenomena, finding widespread application in fields like physics, engineering, and applied mathematics. The neutral differential equation is a type of differential equation that incorporates time delays in the derivatives. The neutral-type differential has applications, such as infinite-dimensional neutral functional differential equations employed in cell population model [1].

Integro-differential equations combine integral and derivative terms and are a powerful tool for describing a wide range of physical phenomena as they have been employed to model Volterra's population dynamics [2] and to model the emergence of cities and urban patterning [3].

Many researchers have established the existence and uniqueness results for differential equations, integral equations, and integro-differential equations using different fixed point theorems like Banach's fixed point theorem, which is the most popular theorem for determining uniqueness conditions for the solution [4], Krasnoselskii's, Schauder's, and Schaefer's fixed point theorems are mainly used for establishing existence conditions [4, 5, 6]. Stability analysis is a crucial part of the discussion for differential equations, integral equations, and integro-differential equations like Mittag-Leffler stability, Uniform stability, Ulam-Hyers stability, and Ulam-Hyers Rassias's stability [4, 7, 8].

Recently, two-dimensional integral and integro-differential equations have collected significant interest due to their applications in diverse fields like population dynamics, fluid mechanics, and image processing. Numerical methods like the Triangular function operational matrix method, shifted Jacobi operational matrix method, and Chebyshev integral operational matrix method have been developed to solve two-dimensional integro-differential equations. However, theoretical exploration of these equations in the two-dimensional context is less extensive than their one-dimensional counterparts. Building upon the well-established concepts in one dimension, we are now extending our efforts to explore the theoretical aspects of two-dimensional integro-differential equations.

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In this paper, we study a two-dimensional neutral integro-differential equation of fractional order of the following form:

$$(1)\qquad {}_0^C D_x^\gamma\Big[\vartheta(\zeta,\, x) - h(\zeta,\, x, \vartheta(\zeta,\, x))\Big] = f(\zeta,\, x, \vartheta(\zeta,\, x)) + {}_0I_x^\alpha\, {}_0I_\zeta^\beta K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta(\varsigma, \rho)),$$

$$(2)\qquad \vartheta(\zeta, 0) = \vartheta_0(\zeta),$$

with $\zeta \in I : [0,b]$, $x \in J : [0,T]$, $0 < \alpha, \beta, \gamma < 1$, and $\delta = \alpha + \gamma$ such that $0 < \delta < 1$, where ${}_0^C D_x^\gamma$ is Caputo fractional derivative, ${}_0I_x^\alpha$ and ${}_0I_\zeta^\beta$ are left Riemann-Liouville fractional integral.

$\vartheta \in E$, $\vartheta_0 \in \mathbb{R}$, ${}_0^C D_x^\gamma\Big[\vartheta(\zeta,\, x) - h(\zeta,\, x, \vartheta(\zeta,\, x))\Big] \in E$, $K : G \to \mathbb{R}^+$, $h : I \times J \times \mathbb{R} \to \mathbb{R}$, $f : I \times J \times \mathbb{R} \to \mathbb{R}$, and $g : I \times J \times \mathbb{R} \to \mathbb{R}$ are continuous functions, where E = The set of all mappings from $C([0,b] \times [0,T])$ to $\mathbb{R}$, E is a Banach space. Let $h(\zeta, 0, \vartheta(\zeta, 0)) = 0$.

The problem (1)-(2) we are investigating find its motivation from the following existing literature: Freedman and Kuang [9] have established sufficient conditions for positive and bounded solutions and analyzed local and global stability of the positive steady state of a class of nonlinear single species neutral differential population model,

$$\frac{d}{dt}(x(t) + \rho x(t-\tau)) = x(t) G(x(t), x(t-\tau)),$$
$$x(\theta) = \phi(\theta) \geq 0,\ \ \theta \in [-\tau, 0],$$

where $r > 0$, $0 < \rho < 1$, $G(x,y)$ is continuously differentiable function, $\phi(\theta)$ is continuously differentiable on $[-\tau, 0]$, and $\phi(0) > 0$.

Wu and Xia [10] studied neutral partial differential equations that appear in rotating waves,

$$\frac{d}{dt}(u(t,x) - bu(t-r,x)) = d\Delta[u(t,x) - bu(t-r,x)] - au(t,x) - abu(t-r,x) - g(u(t,x) - bu(t-r,x)).$$

Fu and Huang [11] studied the existence of solutions for semilinear neutral integro-differential equations of the following form,

$$\frac{d}{dt}[x(t) + F(t,x_t)] = -Ax(t) + \int_0^t \gamma(t-s) x(s) ds + G(t, x_{\rho(t,x_t)}),\ t \in [o,T],$$
$$x_0 = \varphi \in \mathscr{B}_\alpha,$$

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X, $\gamma(t)$ is a closed linear operator defined later, F, G, and ρ are given continuous functions to be specified below, and $\mathscr{B}_\alpha$ is an abstract phase space endowed with a seminorm $\|.\|_{\mathscr{B}_\alpha}$.

Andrade et al. [12] investigated the existence of mild solutions for fractional neutral integro-differential equations, which arises from an evolutionary equation,

$$D_t^\alpha(x(t) + f(t,x_t)) = Ax(t) + \int_0^t B(t-s) x(s) ds + g(t,x_t),\ t > 0,$$
$$x_0 = \varphi,\ x'(0) = x_1,$$

where $\alpha \in (1,2)$; $A, B(t)_{t \geq 0}$ are closed linear operators defined on a common domain which is dense in a Banach space X, $D_t^\alpha h(t)$ represents the Caputo derivative of order $\alpha > 0$, and $g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta}$, $t > 0$, $\beta \geq 0$. The history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathscr{B}$ defined axiomatically and $f, g : I \times \mathscr{B} \to X$ are appropriate functions.

Yang et al. [13] have studied a class of neutral-type integral differential equations that arise in an epidemic model, Santos et al. have established [14] the existence of mild solutions for a class of partial neutral integro-differential equations, Vijayakumar and Udhayakumar [15] have explored the Sobolev-type Hilfer fractional neutral integro-differential system. Various researchers study two-dimensional integro-differential equations to model population dynamics, heat conduction in materials of fading memory, etc., where the two independent variables are considered as time and space or time and temperature [16, 17]. Based on the literature review of two-dimensional neutral integro-differential equations, we have considered the problem (1)-(2), which we believed till now no one studied this type of equation.

Following are the sections of this article: We have covered basic results in the second section. The existence and uniqueness of the solution to the considered problem are discussed in the third section. Next, the Ulam-Hyers and Ulam-Hyers Rassias stability is analyzed in the fourth section. The continuation theorem, maximal solutions, minimal solutions, and positive solutions are all covered in the fifth section. In the sixth section, we have validated the results with a few examples.

2. Preliminaries

These are the basic concepts that we require to establish the results.

Definition 1. *For* $\alpha > 0$, *the Riemann-Liouville fractional integral for two-dimensional function is defined as*

$$ {}_0I_x^{\alpha}\vartheta(\zeta,\, x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-\rho)^{\alpha-1}\vartheta(\zeta,\rho)d\rho. $$

Definition 2. *[18] For* $0<\alpha<1$, *the Caputo fractional derivative for two-dimensional function is defined as*

$$ {}_0^CD_x^{\alpha}\vartheta(\zeta,\, x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\rho)^{-\alpha}\frac{\partial}{\partial\rho}(\vartheta(\zeta,\rho))d\rho. $$

Lemma 1. *[19]* $({}_0I_x^{\alpha})({}_0^CD_x^{\alpha})\vartheta(\zeta,\, x) = \vartheta(\zeta,\, x)-\vartheta(\zeta,0)$, $0<\alpha<1$.

Theorem 2.1. *[20](Banach fixed point theorem) In a non-empty complete metric space* $J=(J,d)$, *if there exists a contraction mapping* $H: J \longrightarrow J$, *then there is a unique fixed point for* H.

Theorem 2.2. *[21](Arzela-Ascoli theorem) If* Ω *is a compact Hausdorff metric space, a subset* $Y\subset C(\Omega)$ *is said to be relatively compact if and only if it is both uniformly bounded and uniformly equicontinuous.*

Theorem 2.3. *[21](Krasnoselskii fixed point theorem) Let* N *be a closed, bounded, and convex subset of a real Banach space* J. *Consider two operators,* H_1 *and* H_2, *defined on* N. *The operators satisfy the following conditions:*

(1) $H_1(N)+H_2(N)\subset N$,
(2) H_2 *is continuous on* N *and* $H_2(N)$ *is relatively compact subset of* J,
(3) H_1 *is a strict contraction on* N, *which means there exists a constant* $\kappa\in[0,1)$ *such that* $\|H_1(n_1)-H_1(n_2)\|\leq\kappa\|n_1-n_2\|$ $\forall\, n_1,n_2\in N$. *Under these conditions, there exists an element* $n\in N$ *such that* $H_1n+H_2n=n$.

3. Existence and uniqueness

In this section, we have established the conditions for the existence and uniqueness of the solution of the problem (1)-(2).
To establish the theoretical results for the problem (1)-(2), we are following these assumptions:

(As_1) Let us assume that positive constants F_h and M_h exist for the continuous function $h: I\times J\times\mathbb{R}\to\mathbb{R}$ such that $\|h(\zeta,\, x,\vartheta_1(\zeta,\, x))-h(\zeta,\, x,\vartheta_2(\zeta,\, x))\|\leq F_h\|\vartheta_1-\vartheta_2\|$ for each $(\zeta,\, x)\in I\times J$ and for all $\vartheta_1,\vartheta_2\in E$ also $M_h=\sup_{(\zeta,\, x)\in I\times J}\|h(\zeta,\, x,0)\|$.

(As_2) Let us assume that positive constants F_f and M_f exist for the continuous function $f: I\times J\times\mathbb{R}\to\mathbb{R}$ such that $\|f(\zeta,\, x,\vartheta_1(\zeta,\, x))-f(\zeta,\, x,\vartheta_2(\zeta,\, x))\|\leq F_f\|\vartheta_1-\vartheta_2\|$ for each $(\zeta,\, x)\in I\times J$ and for all $\vartheta_1,\vartheta_2\in E$ also $M_f=\sup_{(\zeta,\, x)\in I\times J}\|f(\zeta,\, x,0)\|$.

(As_3) Let us assume that positive constants F_g and M_g exist for the continuous function $g: I\times J\times\mathbb{R}\to\mathbb{R}$

such that $\|g(\zeta,\, x, \vartheta_1(\zeta,\, x)) - g(\zeta,\, x, \vartheta_2(\zeta,\, x))\| \le F_g\|\vartheta_1 - \vartheta_2\|$ for each $(\zeta,\, x) \in I\times J$ and for all $\vartheta_1, \vartheta_2 \in E$ also $M_g = \sup_{(\zeta,\, x)\in I\times J} \|g(\zeta,\, x, 0)\|$.

(As_4) $K : G \to \mathbb{R}^+$ is continuous on D with $K_0 = \{\sup|K(\zeta,\, x, \varsigma, \rho)| : (\zeta,\, x, \varsigma, \rho) \in G\}$, where $G = \{(\zeta,\, x, \varsigma, \rho) : 0 \le \varsigma \le \zeta \le b, 0 \le \rho \le x \le T\}$.

(As_5) Let there exists a positive constant M_ϑ such that $M_\vartheta = \sup_{(\zeta,\, x)\in I\times J} \|\vartheta(\zeta, 0)\|$.

(As_6) Let $B_r = \{\vartheta \in E : \|\vartheta\| \le r\}$, B_r is a closed, bounded, and convex subset of E, where $r \ge \frac{K_1}{1-K^*}$ with $K_1 = M_\vartheta + M_h + M_f \frac{T^\gamma}{\Gamma(\gamma+1)} + K_0 M_g \frac{T^\delta}{\Gamma(\delta+1)} \frac{b^\beta}{\Gamma(\beta+1)}$ and $K^* = F_h + F_f \frac{T^\gamma}{\Gamma(\gamma+1)} + K_0 F_g \frac{T^\delta}{\Gamma(\delta+1)} \frac{b^\beta}{\Gamma(\beta+1)}$.

Firstly, we will transform the considered two-dimensional neutral integro-differential equation into the corresponding integral equation.

$$\begin{aligned}\vartheta(\zeta,\, x) - h(\zeta,\, x, \vartheta(\zeta,\, x)) - \vartheta(\zeta, 0) + h(\zeta, 0, \vartheta(\zeta, 0)) &= {}_0I_x^\gamma f(\zeta, \rho, \vartheta(\zeta, \rho)) \\ &\quad + {}_0I_x^{\gamma+\alpha}\, {}_0I_\zeta^\beta K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta(\varsigma, \rho)),\end{aligned}$$

this gives

$$\vartheta(\zeta,\, x) = \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta(\zeta,\, x)) + {}_0I_x^\gamma f(\zeta, \rho, \vartheta(\zeta, \rho)) + {}_0I_x^{\gamma+\alpha}\, {}_0I_\zeta^\beta K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta(\varsigma, \rho)). \tag{3}$$

Theorem 3.1. *Assume that assumptions $(As_1) - (As_6)$ are satisfied. If*

$$F_h + F_f \frac{T^\gamma}{\Gamma(\gamma+1)} < 1,$$

then problem (1)-(2) has at least one solution.

Proof. The operator Λ is defined as the sum of the two operators Λ_1 and Λ_2 according to the following equations:

$$\Lambda\vartheta(\zeta,\, x) = \Lambda_1\vartheta(\zeta,\, x) + \Lambda_2\vartheta(\zeta,\, x), \tag{4}$$

where

$$\Lambda_1\vartheta(\zeta,\, x) = \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta(\zeta,\, x)) + \frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} f(\zeta, \rho, \vartheta(\zeta, \rho)) d\rho \tag{5}$$

and

$$\Lambda_2\vartheta(\zeta,\, x) = \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \int_0^x \int_0^\zeta (x-\rho)^{\delta-1} (\zeta-\varsigma)^{\beta-1} K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta(\varsigma, \rho)) d\varsigma d\rho. \tag{6}$$

STEP 1: We will show that $\Lambda_1\vartheta + \Lambda_2\mu \in B_r, \forall\ \vartheta, \mu \in B_r$.

$$\begin{aligned}\|\Lambda_1\vartheta(\zeta,\, x)\| &\le \|\vartheta_0(\zeta)\| + \|h(\zeta,\, x, \vartheta(\zeta,\, x))\| + \frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} \|f(\zeta, \rho, \vartheta(\zeta, \rho))\| d\rho \\ &\le M_\vartheta + F_h\|\vartheta(\zeta,\, x)\| + M_h + F_f\|\vartheta(\zeta,\, x)\| \frac{x^\gamma}{\Gamma(\gamma+1)} + M_f \frac{x^\gamma}{\Gamma(\gamma+1)}\end{aligned}$$

and

$$\begin{aligned}\|\Lambda_2\mu(\zeta,\, x)\| &\le \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \int_0^x \int_0^\zeta (x-\rho)^{\delta-1} (\zeta-\varsigma)^{\beta-1} \|K(\zeta,\, x, \varsigma, \rho)\| \|g(\varsigma, \rho, \mu(\varsigma, \rho))\| d\varsigma d\rho \\ &\le K_0 F_g \frac{x^\delta}{\Gamma(\delta+1)} \frac{\zeta^\beta}{\Gamma(\beta+1)} \|\mu(\zeta,\, x)\| + K_0 M_g \frac{x^\delta}{\Gamma(\delta+1)} \frac{\zeta^\beta}{\Gamma(\beta+1)}.\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\Lambda_1\vartheta(\zeta,\,x)+\Lambda_2\mu(\zeta,\,x)\| &\le \|\Lambda_1\vartheta(\zeta,\,x)\|+\|\Lambda_2\mu(\zeta,\,x)\|\\
&\le M_\vartheta+F_h\|\vartheta(\zeta,\,x)\|+M_h+F_f\|\vartheta(\zeta,\,x)\|\frac{x^\gamma}{\Gamma(\gamma+1)}+M_f\frac{x^\gamma}{\Gamma(\gamma+1)}\\
&\quad+K_0F_g\frac{x^\delta}{\Gamma(\delta+1)}\frac{\zeta^\beta}{\Gamma(\beta+1)}\|\mu(\zeta,\,x)\|+K_0M_g\frac{x^\delta}{\Gamma(\delta+1)}\frac{\zeta^\beta}{\Gamma(\beta+1)}\\
&\le M_\vartheta+F_hr+M_h+F_f\frac{T^\gamma}{\Gamma(\gamma+1)}r+M_f\frac{T^\gamma}{\Gamma(\gamma+1)}\\
&\quad+K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}r+K_0M_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\\
&\le\left[M_\vartheta+M_h+M_f\frac{T^\gamma}{\Gamma(\gamma+1)}+K_0M_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right]\\
&\quad+\left[F_h+F_f\frac{T^\gamma}{\Gamma(\gamma+1)}+K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right]r\\
&\le K_1+K^*r\\
&\le r.
\end{aligned}$$

STEP 2: Λ_1 is a contraction mapping.

$$\Lambda_1\vartheta(\zeta,\,x)=\vartheta_0(\zeta)+h(\zeta,\,x,\vartheta(\zeta,\,x))+\frac{1}{\Gamma(\gamma)}\int_0^x(x-\rho)^{\gamma-1}f(\zeta,\rho,\vartheta(\zeta,\rho))d\rho. \tag{7}$$

Consider $\vartheta_1,\vartheta_2\in B_r$. Accordingly,

$$\begin{aligned}
\|\Lambda_1\vartheta_1(\zeta,\,x)-\Lambda_1\vartheta_2(\zeta,\,x)\| &\le F_h\|\vartheta_1-\vartheta_2\|+\frac{1}{\Gamma(\gamma)}\int_0^x(x-\rho)^{\gamma-1}F_f\|\vartheta_1-\vartheta_2\|d\rho\\
&\le F_h\|\vartheta_1-\vartheta_2\|+F_f\frac{x^\gamma}{\Gamma(\gamma+1)}\|\vartheta_1-\vartheta_2\|\\
&\le\left[F_h+F_f\frac{T^\gamma}{\Gamma(\gamma+1)}\right]\|\vartheta_1-\vartheta_2\|\\
&\le K^{**}\|\vartheta_1-\vartheta_2\|,
\end{aligned}$$

where $K^{**}=\left[F_h+F_f\frac{T^\gamma}{\Gamma(\gamma+1)}\right]<1$. This means Λ_1 exhibits the property of being a contraction mapping.

STEP 3: Λ_2 is a continuous function and the set Λ_2B_r is relatively compact within the space E.

(i) Λ_2 is continuous.
In the given argument, it is assumed that the sequence $\{\vartheta_i\}$ converges to ϑ, where $\vartheta_i\in B_r\ \forall\ i\in\mathbb{N}$ (the natural numbers). This implies that as n approaches infinity, the norm of the difference between ϑ_n and ϑ tends to zero, which means, $\lim_{n\to\infty}\|\vartheta_n-\vartheta\|=0$. From this assumption, it is claimed that $\lim_{n\to\infty}\Lambda_2\vartheta_n=\Lambda_2\vartheta$. To prove this claim, the following inequality is derived:

$$\begin{aligned}
\|\Lambda_2\vartheta_n-\Lambda_2\vartheta\| &\le \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta(x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}\|K(\zeta,\,x,\varsigma,\rho)\|\\
&\quad\|g(\varsigma,\rho,\vartheta_n(\varsigma,\rho))-g(\varsigma,\rho,\vartheta(\varsigma,\rho))\|d\varsigma d\rho\\
&\le\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta(x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}K_0F_g\|\vartheta_n-\vartheta\|d\varsigma d\rho.
\end{aligned}$$

Finally, it is concluded that $\|\Lambda_2\vartheta_n-\Lambda_2\vartheta\|\to0$ whenever $\vartheta_n\to\vartheta$.

(ii) $\Lambda_2 B_r$ is uniformly bounded.

$$\|\Lambda_2 \vartheta(\zeta,\, x)\| \le r^*,$$

where $r^* = \frac{T^\delta}{\Gamma(\delta+1)} \frac{b^\beta}{\Gamma(\beta+1)} K_0[F_g r + M_g]$. This implies $\Lambda_2 B_r \subset B_{r^*}$ for any $\vartheta \in B_r$.

(iii) $\Lambda_2 B_r$ is uniformly equicontinuous.
Let $(\zeta_1, x_1), (\zeta_2, x_2) \in I \times J$ and $\vartheta \in B_r$, then we have

$$\begin{aligned}
&\Big\|\Lambda_2\vartheta(\zeta_1,x_1) - \Lambda_2\vartheta(\zeta_2,x_2)\Big\| \\
&= \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \Bigg\| \int_0^{x_1}\int_0^{\zeta_1} (x_1-\rho)^{\delta-1}(\zeta_1-\varsigma)^{\beta-1} K(\zeta_1,x_1,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\varsigma\, d\rho \\
&\quad - \int_0^{x_2}\int_0^{\zeta_2} (x_2-\rho)^{\delta-1}(\zeta_2-\varsigma)^{\beta-1} K(\zeta_2,x_2,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\varsigma\, d\rho \Bigg\| \\
&\le \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \Bigg\| \int_0^{\zeta_1}\int_0^{x_1} (x_1-\rho)^{\delta-1}(\zeta_1-\varsigma)^{\beta-1} K(\zeta_1,x_1,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\rho\, d\varsigma \\
&\quad - \int_0^{\zeta_1}\int_0^{x_2} (x_2-\rho)^{\delta-1}(\zeta_1-\varsigma)^{\beta-1} K(\zeta_1,x_2,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\rho\, d\varsigma \Bigg\| \\
&\quad + \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \Bigg\| \int_0^{x_2}\int_0^{\zeta_1} (x_2-\rho)^{\delta-1}(\zeta_1-\varsigma)^{\beta-1} K(\zeta_1,x_2,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\varsigma\, d\rho \\
&\quad - \int_0^{x_2}\int_0^{\zeta_2} (x_2-\rho)^{\delta-1}(\zeta_2-\varsigma)^{\beta-1} K(\zeta_2,x_2,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))\, d\varsigma\, d\rho \Bigg\| \\
&\le \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \int_0^{\zeta_1} (\zeta_1-\varsigma)^{\beta-1}(K_0F_g + M_g) \\
&\quad \left\{ \int_0^{x_1} \{(x_1-\rho)^{\delta-1} - (x_2-\rho)^{\delta-1}\} d\rho + \int_{x_1}^{x_2} (x_2-\rho)^{\delta-1} d\rho \right\} d\varsigma \\
&\quad + \frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \int_0^{x_2} (x_2-\rho)^{\delta-1}(K_0F_g + M_g) \\
&\quad \left\{ \int_0^{\zeta_1} \{(\zeta_1-\varsigma)^{\beta-1} - (\zeta_2-\varsigma)^{\beta-1}\} d\varsigma + \int_{\zeta_1}^{\zeta_2} (\zeta_2-\varsigma)^{\beta-1} d\varsigma \right\} d\rho \\
&\le \frac{\zeta_1^\beta}{\Gamma(\delta+1)} \frac{(K_0F_g + M_g)}{\Gamma(\beta+1)} [2(x_2-x_1)^\delta + x_1^\delta - x_2^\delta] \\
&\quad + \frac{x_2^\delta}{\Gamma(\delta+1)} \frac{(K_0F_g + M_g)}{\Gamma(\beta+1)} [2(\zeta_2-\zeta_1)^\beta + \zeta_1^\beta - \zeta_2^\beta],
\end{aligned}$$

$\|\Lambda_2\vartheta(\zeta_1,x_1) - \Lambda_2\vartheta(\zeta_2,x_2)\| \to 0$ whenever $\zeta_1 \to \zeta_2$, $x_1 \to x_2$.

Thus, $\Lambda_2 B_r$ is uniformly equicontinuous. By the Theorem 2.2, $\Lambda_2 B_r$ becomes relatively compact.

Since each of the criteria of the Theorem 2.3 has been fulfilled, the problem (1)-(2) has at least one solution.

Theorem 3.2. *Suppose that assumptions $(As_1) - (As_6)$ are satisfied. If*

$$F_h + F_f \frac{T^\gamma}{\Gamma(\gamma+1)} + K_0 F_g \frac{T^\delta}{\Gamma(\delta+1)} \frac{b^\beta}{\Gamma(\beta+1)} < 1$$

then problem (1)-(2) has a unique solution.

Proof. Let us consider an operator $\Lambda : B_r \to B_r$, define as

$$\Lambda(\vartheta) = \vartheta, \tag{8}$$

$$\begin{aligned}\Lambda\vartheta(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta(\zeta,\, x)) + {}_0I_x^{\gamma} f(\zeta,\rho,\vartheta(\zeta,\rho))\\ &\quad + {}_0I_x^{\gamma+\alpha}\, {}_0I_\zeta^{\beta} K(\zeta,\, x, \varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)),\end{aligned} \tag{9}$$

$$\begin{aligned}\Lambda\vartheta(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta(\zeta,\, x)) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1} f(\zeta,\rho,\vartheta(\zeta,\rho))d\rho\\ &\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho))d\varsigma d\rho,\end{aligned} \tag{10}$$

where $\delta = \gamma + \alpha$. For $\vartheta \in B_r$, we have

$$\begin{aligned}\|\Lambda\vartheta\| &\le \|\vartheta_0(\zeta)\| + \|h(\zeta,\, x, \vartheta(\zeta,\, x))\| + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}\|f(\zeta,\rho,\vartheta(\zeta,\rho))\|d\rho\\ &\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}\|K(\zeta,\, x,\varsigma,\rho)\|\|g(\varsigma,\rho,\vartheta(\varsigma,\rho))\|d\varsigma d\rho\\ &\le M_\vartheta + F_h\|\vartheta(\zeta,\, x)\| + M_h + F_f\|\vartheta(\zeta,\, x)\|\frac{x^\gamma}{\Gamma(\gamma+1)} + M_f\frac{x^\gamma}{\Gamma(\gamma+1)}\\ &\quad + K_0F_g\frac{x^\delta}{\Gamma(\delta+1)}\frac{\zeta^\beta}{\Gamma(\beta+1)}\|\vartheta(\zeta,\, x)\| + K_0M_g\frac{x^\delta}{\Gamma(\delta+1)}\frac{\zeta^\beta}{\Gamma(\beta+1)}\\ &\le M_\vartheta + F_h r + M_h + F_f r\frac{T^\gamma}{\Gamma(\gamma+1)} + M_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}r + K_0M_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\\ &\le \left[M_\vartheta + M_h + M_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0M_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right] + \left[F_h + F_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right]r\\ &\le K_1 + K^* r\\ &\le r.\end{aligned}$$

This concludes that $\Lambda B_r \subset B_r$.

Now, consider $\vartheta_1, \vartheta_2 \in B_r$ such that

$$\begin{aligned}\Lambda\vartheta_1(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta_1(\zeta,\, x)) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1} f(\zeta,\rho,\vartheta_1(\zeta,\rho))d\rho\\ &\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_1(\varsigma,\rho))d\varsigma d\rho\end{aligned} \tag{11}$$

and

$$\begin{aligned}\Lambda\vartheta_2(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta_2(\zeta,\, x)) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1} f(\zeta,\rho,\vartheta_2(\zeta,\rho))d\rho\\ &\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_2(\varsigma,\rho))d\varsigma d\rho.\end{aligned} \tag{12}$$

Then we have,

$$\begin{aligned}\|\Lambda\vartheta_1(\zeta,\, x) - \Lambda\vartheta_2(\zeta,\, x)\| &\le \|h(\zeta,\, x, \vartheta_1(\zeta,\, x)) - h(\zeta,\, x, \vartheta_2(\zeta,\, x))\|\\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}\|f(\zeta,\rho,\vartheta_1(\zeta,\rho)) - f(\zeta,\rho,\vartheta_2(\zeta,\rho))\|d\rho\\ &\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1} K(\zeta,\, x,\varsigma,\rho)\\ &\quad \|g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) - g(\varsigma,\rho,\vartheta_2(\varsigma,\rho))\|d\varsigma d\rho\end{aligned}$$

$$\begin{aligned}
&\leq F_h\|\vartheta_1-\vartheta_2\| + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}F_f\|\vartheta_1-\vartheta_2\|d\rho \\
&\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}K_0F_g\|\vartheta_1-\vartheta_2\|d\varsigma d\rho \\
&\leq F_h\|\vartheta_1-\vartheta_2\| + F_f\frac{x^\gamma}{\Gamma(\gamma+1)}\|\vartheta_1-\vartheta_2\| + K_0F_g\frac{x^\delta}{\Gamma(\delta+1)}\frac{\zeta^\beta}{\Gamma(\beta+1)}\|\vartheta_1-\vartheta_2\| \\
&\leq \left[F_h + F_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right]\|\vartheta_1-\vartheta_2\| \\
&\leq K^*\|\vartheta_1-\vartheta_2\|,
\end{aligned}$$

where

$$K^* = \left[F_h + F_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0F_g\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right] < 1.$$

In consideration of the fact that Λ is a contraction mapping, the problem of (1)-(2) has a unique solution according to the Theorem 2.1.

4. Stability analysis

In the current section, we have discussed the Ulam-Hyer and Ulam-Hyer Rassias stability for the problem (1)-(2).

Definition 3. *The problem (1)-(2) is said to be Ulam-Hyers stable if, for any given positive ε, whenever there exists a function $\vartheta(\zeta, x)$ satisfies the inequality*

$$\left|{}_0^CD_x^\gamma\Big[\vartheta(\zeta, x) - h(\zeta, x, \vartheta(\zeta, x))\Big] - f(\zeta, x, \vartheta(\zeta, x)) - {}_0I_x^\alpha{}_0I_\zeta^\beta K(\zeta, x, \varsigma, \rho)g(\varsigma, \rho, \vartheta(\varsigma, \rho))\right| < \varepsilon \tag{13}$$

then, there must exist a solution $\mu(\zeta, x)$ of problem (1)-(2), which satisfies

$$|\vartheta(\zeta, x) - \mu(\zeta, x)| < k_f\varepsilon, \; k_f \in \mathbb{R}. \tag{14}$$

Definition 4. *The problem (1)-(2) is said to be Ulam-Hyers Rassias stable if, for any given positive ε_ψ, whenever there exists a function $\vartheta(\zeta, x)$ satisfies the inequality*

(15)
$$\left|{}_0^CD_x^\gamma\Big[\vartheta(\zeta, x) - h(\zeta, x, \vartheta(\zeta, x))\Big] - f(\zeta, x, \vartheta(\zeta, x)) - {}_0I_x^\alpha{}_0I_\zeta^\beta K(\zeta, x, \varsigma, \rho)g(\varsigma, \rho, \vartheta(\varsigma, \rho))\right| < \varepsilon_\psi\psi(\zeta, x)$$

then, there must exist a solution $\mu(\zeta, x)$ of problem (1)-(2), which satisfies

$$|\vartheta(\zeta, x) - \mu(\zeta, x)| < k_f\varepsilon_\psi\psi(\zeta, x), \; k_f \in \mathbb{R}. \tag{16}$$

Theorem 4.1. *Assume that assumptions $(As_1)-(As_6)$ are satisfied. If $T^\gamma F_f < (1-F_h)\Gamma(\gamma+1)$ then the problem (1)-(2) is Ulam-Hyers stable.*

Proof. For a given $\varepsilon > 0$, if the inequality (13) is satisfied, then there exist a function $\phi(\zeta,x)$ satisfying $|\phi(\zeta,x)| < \varepsilon$, which can be written as

$${}_0^CD_x^\gamma\Big[\vartheta(\zeta,x) - h(\zeta, x, \vartheta(\zeta,x))\Big] - f(\zeta, x, \vartheta(\zeta,x)) - {}_0I_x^\alpha{}_0I_\zeta^\beta K(\zeta, x, \varsigma, \rho)g(\varsigma, \rho, \vartheta(\varsigma, \rho)) = \phi(\zeta,x).$$

Therefore, we have

$$\begin{aligned}
&|\vartheta(\zeta,x) - \vartheta_0(\zeta) - h(\zeta, x, \vartheta(\zeta,x)) - {}_0I_x^\gamma f(\zeta, \rho, \vartheta(\zeta, \rho)) - {}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta, x, \varsigma, \rho)g(\varsigma, \rho, \vartheta(\varsigma, \rho))| \\
&= |{}_0I_x^\gamma\phi(\zeta,x)| \leq \frac{x^\gamma}{\Gamma(\gamma+1)}\varepsilon \leq \frac{T^\gamma}{\Gamma(\gamma+1)}\varepsilon.
\end{aligned}$$

Now, let $\mu(\zeta, x)$ be the solution of problem (1)-(2), satisfying $\mu(\zeta,0)=\vartheta(\zeta,0)=\vartheta_0(\zeta)$. We have,

$$\begin{aligned}
|\vartheta(\zeta, x)-\mu(\zeta, x)| &= |\vartheta(\zeta,x)-\vartheta_0(\zeta)-h(\zeta, x,\mu(\zeta,x))-{}_0I_x^{\gamma}f(\zeta,\rho,\mu(\zeta,\rho)) \\
&\quad -{}_0I_x^{\gamma+\alpha}{}_0I_\zeta^{\beta}K(\zeta, x,\varsigma,\rho)g(\varsigma,\rho,\mu(\varsigma,\rho))| \\
&\leq |\vartheta(\zeta,x)-\vartheta_0(\zeta)-h(\zeta, x,\vartheta(\zeta,x))-{}_0I_x^{\gamma}f(\zeta,\rho,\vartheta(\zeta,\rho)) \\
&\quad -{}_0I_x^{\gamma+\alpha}{}_0I_\zeta^{\beta}K(\zeta, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho))|+|h(\zeta, x,\vartheta(\zeta,x))-h(\zeta, x,\mu(\zeta,x))| \\
&\quad +|{}_0I_x^{\gamma}f(\zeta,\rho,\vartheta(\zeta,\rho))-{}_0I_x^{\gamma}f(\zeta,\rho,\mu(\zeta,\rho))| \\
&\quad +|{}_0I_x^{\gamma+\alpha}{}_0I_\zeta^{\beta}K(\zeta, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho)) \\
&\quad -{}_0I_x^{\gamma+\alpha}{}_0I_\zeta^{\beta}K(\zeta, x,\varsigma,\rho)g(\varsigma,\rho,\mu(\varsigma,\rho))| \\
&\leq \frac{T^{\gamma}}{\Gamma(\gamma+1)}\varepsilon+F_h|\vartheta(\zeta, x)-\mu(\zeta, x)|+{}_0I_x^{\gamma}F_f|\vartheta(\zeta, x)-\mu(\zeta, x)|+ \\
&\quad {}_0I_x^{\delta}{}_0I_\zeta^{\beta}K_0F_g|\vartheta(\zeta, x)-\mu(\zeta, x)|.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\vartheta(\zeta, x)-\mu(\zeta, x)|[1-F_h] &\leq \frac{T^{\gamma}}{\Gamma(\gamma+1)}\varepsilon+{}_0I_x^{\gamma}F_f|\vartheta(\zeta, x)-\mu(\zeta, x)|+ \\
&\quad {}_0I_x^{\delta}{}_0I_\zeta^{\beta}K_0F_g|\vartheta(\zeta, x)-\mu(\zeta, x)|
\end{aligned}$$

$$\begin{aligned}
|\vartheta(\zeta, x)-\mu(\zeta, x)|[1-F_h] &\leq \frac{T^{\gamma}}{\Gamma(\gamma+1)}\varepsilon+\frac{T^{\gamma}}{\Gamma(\gamma+1)}F_f|\vartheta(\zeta, x)-\mu(\zeta, x)| \\
&\quad +{}_0I_x^{\delta}{}_0I_\zeta^{\beta}K_0F_g|\vartheta(\zeta, x)-\mu(\zeta, x)|
\end{aligned}$$

$$|\vartheta(\zeta, x)-\mu(\zeta, x)|\left[1-F_h-\frac{T^{\gamma}}{\Gamma(\gamma+1)}F_f\right] \leq \frac{T^{\gamma}}{\Gamma(\gamma+1)}\varepsilon+{}_0I_x^{\delta}\frac{b^{\beta}}{\Gamma(\beta+1)}K_0F_g|\vartheta(\zeta, x)-\mu(\zeta, x)|$$

$$\begin{aligned}
|\vartheta(\zeta, x)-\mu(\zeta, x)| &\leq \frac{T^{\gamma}}{[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}\varepsilon \\
&\quad +\frac{b^{\beta}K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}{}_0I_x^{\delta}|\vartheta(\zeta, x)-\mu(\zeta, x)|.
\end{aligned}$$

Now, by using Gronwall's inequality [22], we get

$$|\vartheta(\zeta, x)-\mu(\zeta, x)| \leq \frac{T^{\gamma}}{[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}E_{\delta}\left(\frac{T^{\delta}b^{\beta}K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}\right)\varepsilon,$$

where E_δ is the Mittag-Leffler function. Therefore, $|\vartheta(\zeta, x)-\mu(\zeta, x)|<k_f\varepsilon$ with

$$k_f=\frac{T^{\gamma}}{[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}E_{\delta}\left(\frac{T^{\delta}b^{\beta}K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1)-T^{\gamma}F_f]}\right).$$

Hence, the problem (1)-(2) is Ulam-Hyers stable.

Theorem 4.2. *Assume that assumptions $(As_1)-(As_6)$ are satisfied. If $T^{\gamma}F_f<(1-F_h)\Gamma(\gamma+1)$ then the problem (1)-(2) is Ulam-Hyers Rassias stable.*

Proof. Let $\vartheta_1(\zeta,\, x)$ satisfy the problem (1)-(2). For a given $\varepsilon_\psi > 0$, if the inequality (15) holds, then there exist a function $\psi(x,t)$ such that

$$\begin{aligned}&|\vartheta_1(\zeta,x) - \vartheta_0(\zeta) - h(\zeta,\, x, \vartheta_1(\zeta,x)) - {}_0I_x^\gamma f(\zeta,\rho,\vartheta_1(\zeta,\rho)) - {}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta_1(\varsigma,\rho))|\\ &\le \frac{T^\gamma}{\Gamma(\gamma+1)}\varepsilon_\psi\psi(\zeta,\, x).\end{aligned}$$

Now, let $\mu_1(\zeta,\, x)$ be the solution of problem (1)-(2), satisfying $\mu_1(\zeta,0) = \vartheta(_1\zeta,0) = \vartheta_0(\zeta)$. We have,

$$\begin{aligned}|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)| &= |\vartheta_1(\zeta,x) - \vartheta_0(\zeta) - h(\zeta,\, x,\mu_1(\zeta,\, x)) - {}_0I_x^\gamma f(\zeta,\rho,\mu_1(\zeta,\rho))\\ &\quad - {}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\mu_1(\varsigma,\rho))|\\ &\le |\vartheta_1(\zeta,x) - \vartheta_0(\zeta) - h(\zeta,\, x,\vartheta_1(\zeta,x)) - {}_0I_x^\gamma f(\zeta,\rho,\vartheta_1(\zeta,\rho))\\ &\quad - {}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta_1(\varsigma,\rho))| + |h(\zeta,\, x,\vartheta_1(\zeta,x)) - h(\zeta,\, x,\mu_1(\zeta,x))|\\ &\quad + |{}_0I_x^\gamma f(\zeta,\rho,\vartheta_1(\zeta,\rho)) - {}_0I_x^\gamma f(\zeta,\rho,\mu_1(\zeta,\rho))|\\ &\quad + |{}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) - {}_0I_x^{\gamma+\alpha}{}_0I_\zeta^\beta K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\mu_1(\varsigma,\rho))|\\ &\le \frac{T^\gamma}{\Gamma(\gamma+1)}\varepsilon_\psi\psi(\zeta,\, x) + F_h|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)| + {}_0I_x^\gamma F_f|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)| +\\ &\quad {}_0I_x^\delta{}_0I_\zeta^\beta K_0F_g|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|.\end{aligned}$$

So,

$$\begin{aligned}|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|[1-F_h] &\le \frac{T^\gamma}{\Gamma(\gamma+1)}\varepsilon_\psi\psi(\zeta,\, x) + \frac{T^\gamma}{\Gamma(\gamma+1)}F_f|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|\\ &\quad + {}_0I_x^\delta{}_0I_\zeta^\beta K_0F_g|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|\end{aligned}$$

$$|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|\left[1 - F_h - \frac{T^\gamma}{\Gamma(\gamma+1)}F_f\right] \le \frac{T^\gamma}{\Gamma(\gamma+1)}\varepsilon_\psi\psi(\zeta,\, x) + {}_0I_x^\delta\frac{b^\beta}{\Gamma(\beta+1)}K_0F_g|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)|$$

$$\begin{aligned}|\vartheta_1(\zeta,x) - \mu_1(\zeta,x)| &\le \frac{T^\gamma}{[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}\varepsilon_\psi\psi(\zeta,x)\\ &\quad + \frac{b^\beta K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}{}_0I_x^\delta|\vartheta_1(\zeta,x) - \mu_1(\zeta,x)|.\end{aligned}$$

Now, by using Gronwall's inequality [22], we get

$$|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)| \le \frac{T^\gamma}{[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}E_\delta\left(\frac{T^\delta b^\beta K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}\right)\varepsilon_\psi\psi(\zeta,\, x),$$

where E_δ is the Mittag-Leffler function. Therefore, $|\vartheta_1(\zeta,\, x) - \mu_1(\zeta,\, x)| < k_f\varepsilon_\psi\psi(\zeta,\, x)$ with

$$k_f = \frac{T^\gamma}{[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}E_\delta\left(\frac{T^\delta b^\beta K_0F_g\Gamma(\gamma+1)}{\Gamma(\beta+1)[(1-F_h)\Gamma(\gamma+1) - T^\gamma F_f]}\right).$$

Thus, the problem (1)-(2) is Ulam-Hyers Rassias stable.

5. Positive solutions, maximal and minimal solutions, and continuation theorem

In the current section, we have established the conditions for the existence of positive solutions, maximal and minimal solutions, and continuation theorem for the problem (1)-(2).

5.1. *Positive solutions.* Assumptions

AP1: The functions $h : I \times J \times \mathbb{R}^+ \to \mathbb{R}^+$, $f : I \times J \times \mathbb{R}^+ \to \mathbb{R}^+$, $g : I \times J \times \mathbb{R}^+ \to \mathbb{R}^+$ and $K : G \to \mathbb{R}^+$.
AP2: There exist $m_1, m_2, m_3, M_1, M_2, M_3 > 0$ such that $m_1 \le h \le M_1$, $m_2 \le f \le M_2$, and $m_3 \le g \le M_3$, for every $(x,t) \in I \times J$. Let $m = \min\{m_1, m_2, m_3\}$ and $M = \max\{M_1, M_2, M_3\}$.

Let $D \subset E$ be a cone defined by $D = \{\vartheta \in E : \vartheta(\zeta,\ x) \ge 0, \|(\zeta,\ x)\| \le p\}$. Then (E,D) forms an ordered Banach space. We have the following theorem if we assume that $\Lambda : D \to D$ be the operator defined as in the equation (10).

Theorem 5.1. *Assume that assumptions AP1 and AP2 are satisfied. Then Λ is completely continuous.*

Proof. According to Theorem 3.1, the operator Λ is bounded mapping. We will demonstrate the continuity of $\Lambda : D \to D$. Let $\vartheta \in D$, where $\|\vartheta\| \le r$. Let $\tilde{D} = \{\tilde{\vartheta} \in D : \left\|\vartheta - \tilde{\vartheta}\right\| \le \tilde{r}\}$. Then $\left\|\tilde{\vartheta}\right\| \le r + \tilde{r} := r_0$, $\forall\ \tilde{\vartheta} \in \tilde{D}$.

Since h, f, g, and k are continuous on $I \times J$, then it uniformly continuous there. Therefore, for given $\varepsilon > 0$, there exist $r_1 > 0$ $(r_1 < \tilde{r})$ such that

$$\left\|h(\zeta,\ x, \vartheta(\zeta,\ x)) - h(\zeta,\ x, \tilde{\vartheta}(\zeta,\ x))\right\| < \frac{\varepsilon}{\tilde{K}},$$

$$\left\|f(\zeta,\ x, \vartheta(\zeta,\ x)) - f(\zeta,\ x, \tilde{\vartheta}(\zeta,\ x))\right\| < \frac{\varepsilon}{\tilde{K}},$$

and

$$\left\|g(\zeta,\ x, \vartheta(\zeta,\ x)) - g(\zeta,\ x, \tilde{\vartheta}(\zeta,\ x))\right\| < \frac{\varepsilon}{\tilde{K}},$$

for $\left\|\vartheta - \tilde{\vartheta}\right\| < r_1$, $(\zeta,\ x) \in I \times J$. If $\left\|\vartheta - \tilde{\vartheta}\right\| < r_1$, then $\tilde{\vartheta} \in \tilde{D}$ and $\left\|\tilde{\vartheta}\right\| < r_0$. As $\tilde{\vartheta} \in \tilde{D} \subset D$, $\left\|\tilde{\vartheta}\right\| \le r_0$. Similarly $\|\vartheta\| \le r_0$. Since we have $\left\|\Lambda\vartheta - \Lambda\tilde{\vartheta}\right\| < \varepsilon$, it follows that Λ is continuous. Consequently, Λ has a fixed point.

Theorem 5.2. *Assume that assumptions AP1 and AP2 are satisfied. Then the problem (1)-(2) has at least one positive solution.*

Proof. Let $D_1 = \{\vartheta \in E : \|\vartheta\| \le K_1 + \tilde{K}rM\}$ and $D_2 = \{\vartheta \in E : \|\vartheta\| \le K_1 + \tilde{K}rm\}$. For $\vartheta \in D \cap \partial D_2$, we have $0 \le \vartheta(\zeta,\ x) \le K_1 + \tilde{K}rM, (\zeta,\ x) \in I \times J$. Since $h(\zeta,\ x, \vartheta(\zeta,\ x)) \le M$, $f(\zeta,\ x, \vartheta(\zeta,\ x)) \le M$, and $g(\zeta,\ x, \vartheta(\zeta,\ x)) \le M$, we have

$$\begin{aligned}
\Lambda\vartheta(\zeta,\ x) &= \vartheta_0(\zeta) + h(\zeta,\ x, \vartheta(\zeta,\ x)) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1} f(\zeta,\ x, \vartheta(\zeta,\ x))d\rho \\
&\quad + \frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho))d\varsigma d\rho \\
\|\Lambda\vartheta(\zeta,\ x)\| &\le \left[M_\vartheta + M_h + M_f\frac{T^\gamma}{\Gamma(\gamma+1)} + K_0 M_g \frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right] \\
&\quad + \left[F_h + F_f\frac{T^\gamma}{\Gamma(\gamma+1)} + + K_0 F_g \frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right] r \\
&\le K_1 + \tilde{K}rM,
\end{aligned}$$

where $\tilde{K} = \left[1 + \frac{T^\gamma}{\Gamma(\gamma+1)} + + K_0\frac{T^\delta}{\Gamma(\delta+1)}\frac{b^\beta}{\Gamma(\beta+1)}\right]$.

Hence, $\|\Lambda\vartheta\| \le \|\vartheta\|$. On the other hand, for $\vartheta \in D \cap \partial D_1$, we have $0 \le \vartheta(\zeta,\ x) \le K_1 + \tilde{K}rm$, $(\zeta,\ x) \in I \times J$. As a result of the fact that $h(\zeta,\ x, \vartheta(\zeta,\ x)) \ge m$, $f(\zeta,\ x, \vartheta(\zeta,\ x)) \ge m$, and $g(\zeta,\ x, \vartheta(\zeta,\ x)) \ge m$, we have $\|\Lambda\vartheta\| \ge K_1 + \tilde{K}rm = \|\vartheta\|$, (see Theorem 1.2 [23]).

Consequently, the operator Λ has a fixed point in $D \cap (\bar{D}_2 \backslash D_1)$. Thus the problem (1)-(2) has at least one positive solution.

Theorem 5.3. *Let $h : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$, $f : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$, and $g : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$ be continuous and increasing functions for each $(\zeta,\ x)\in I\times J$. Let there exist c_0 and d_0 satisfying ${}_0^C D_x^\gamma c_0 \le c_0$, ${}_0^C D_x^\gamma d_0 \ge d_0$ and $0\le c_0\le d_0$, $(\zeta,\ x)\in I\times J$. Then problem (1)-(2) has a positive solution.*

Proof. Let $c,\ d\in D$ such that $c<d$, then we have

$$\begin{aligned}\Lambda c(\vartheta,x) &= \vartheta_0(\zeta)+h(\zeta,\ x,c(\zeta,\ x))+\frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}f(\zeta,\ x,c(\zeta,\rho))d\rho\\ &\quad+\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,c(\varsigma,\rho))d\varsigma d\rho\\ &\le \vartheta_0(\zeta)+h(\zeta,\ x,d(\zeta,\ x))+\frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}f(\zeta,\ x,d(\zeta,\rho))d\rho\\ &\quad+\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,d(\varsigma,\rho))d\varsigma d\rho\\ &= \Lambda d(\zeta,\ x).\end{aligned}$$

Therefore $\Lambda c(\zeta,\ x)\le \Lambda d(\zeta,\ x)$, $\forall\,(\zeta,\ x)\in I\times J$, which gives $\Lambda c\le\Lambda d$. As there exist c_0, d_0 such that $0\le c_0\le d_0$, with $\Lambda c_0\le c_0$, $\Lambda d_0\ge d_0$, (see Theorem 1.3 [23]) Λ is compact and has a fixed point in $\langle c,d\rangle$.

Therefore, according to Theorem 1.3 [23] $\Lambda : \langle c_0,d_0\rangle \to \langle c_0,d_0\rangle$ is compact. Hence Λ has a fixed point $e\in\langle c,d\rangle$, which is the positive solution. This supports the argument.

5.2. *Maximal and minimal solutions theorems for the problem (1)-(2).* In the current section, we investigate the existence of both maximal and minimal solutions for the problem (1)-(2).

Definition 5. *Let $l(\zeta,\ x)$ be a solution of problem (1)-(2) in $I\times J$. If the inequality $\vartheta(\zeta,\ x)\le l(\zeta,\ x)$, $(\zeta,\ x)\in I\times J$, holds for every solution of problem (1)-(2) define on $(\zeta,\ x)\in I\times J$, then $l(\zeta,\ x)$ is said to be a maximal solution of problem (1)-(2).*

Definition 6. *Let $\tilde{l}(\zeta,\ x)$ be a solution of problem (1)-(2) in $I\times J$. If the inequality $\vartheta(\zeta,\ x)\ge \tilde{l}(\zeta,\ x)$, $(\zeta,\ x)\in I\times J$, holds for every solution of problem (1)-(2) define on $(\zeta,\ x)\in I\times J$, then $\tilde{l}(\zeta,\ x)$ is said to be a minimal solution of problem (1)-(2).*

Theorem 5.4. *Suppose $h : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$, $f : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$, $g : I\times J\times\mathbb{R}^+ \to \mathbb{R}^+$, and $K : G\to\mathbb{R}^+$ are continuous and non-decreasing functions defined on the set E. Let q_1 and q_2 be two positive constants such that $q_1<q_2$. If the following inequalities hold:*

$$\frac{q_1}{\varepsilon+\vartheta_0(\zeta)+h(\zeta,\ x,q_1)+{}_0I_x^\gamma f(\zeta,\rho,q_1)+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,q_1)}<1<\frac{q_2}{\varepsilon+\vartheta_0(\zeta)+h(\zeta,\ x,q_2)+{}_0I_x^\gamma f(\zeta,\rho,q_2)+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,q_2)}.$$

Then there exists a maximal and minimal solution of problem (1)-(2) on $I\times J$.

Proof. The fractional integral equation of the problem (1)-(2) is

$$\vartheta(\zeta,\ x)=\vartheta_0(\zeta)+h(\zeta,\ x,\vartheta(\zeta,\ x))+{}_0I_x^\gamma f(\zeta,\rho,\vartheta(\zeta,\rho))+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho)).$$

Consider the fractional order integral equation

$$\vartheta(\zeta,\ x)=\varepsilon+\vartheta_0(\zeta)+h(\zeta,\ x,\vartheta(\zeta,\ x))+{}_0I_x^\gamma f(\zeta,\rho,\vartheta(\zeta,\rho))+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho)), \tag{17}$$

where $\varepsilon>0$. Then $\vartheta(\zeta,\ x)$ given by equation (17) is solution of problem (1)-(2) in (q_1,q_2), $(\zeta,\ x)\in I\times J$, for some constants $q_1,\ q_2>0$ such that

$$\frac{q_1}{\varepsilon+\vartheta_0(\zeta)+h(\zeta,\ x,q_1)+{}_0I_x^\gamma f(\zeta,\rho,q_1)+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,q_1)}<1<\frac{q_2}{\varepsilon+\vartheta_0(\zeta)+h(\zeta,\ x,q_2)+{}_0I_x^\gamma f(\zeta,\rho,q_2)+{}_0I_x^\delta\,{}_0I_\zeta^\beta K(\zeta,\ x,\varsigma,\rho)g(\varsigma,\rho,q_2)}.$$

Now, let $0<\varepsilon_2<\varepsilon_1\leq\varepsilon$. Then we have $\vartheta_{\varepsilon_2}(0,0)<\vartheta_{\varepsilon_1}(0,0)$. Thus we have to show that

$$\vartheta_{\varepsilon_2}(\zeta,\, x)<\vartheta_{\varepsilon_1}(\zeta,\, x),\ \forall\,(\zeta,\, x)\in I\times J. \tag{18}$$

Consider it to be false. Then there exist a (ζ_1,x_1) such that $\vartheta_{\varepsilon_2}(\zeta_1,x_1)=\vartheta_{\varepsilon_1}(\zeta_1,x_1)$ and $\vartheta_{\varepsilon_2}(\zeta,\, x)<\vartheta_{\varepsilon_1}(\zeta,\, x)$, $\forall\,(\zeta,\, x)\in I\times J$. Since $h,\ f,\ g$ are monotonic non-decreasing function in ϑ, it follows that

$$h(\zeta,\, x,\vartheta_{\varepsilon_2}(\zeta,\, x))\leq h(\zeta,\, x,\vartheta_{\varepsilon_1}(\zeta,\, x)),$$

$$f(\zeta,\, x,\vartheta_{\varepsilon_2}(\zeta,\, x))\leq f(\zeta,\, x,\vartheta_{\varepsilon_1}(\zeta,\, x)),$$

and

$$g(\zeta,\, x,\vartheta_{\varepsilon_2}(\zeta,\, x))\leq g(\zeta,\, x,\vartheta_{\varepsilon_1}(\zeta,\, x)).$$

Consequently, using equation (17), we get

$$\begin{aligned}\vartheta_{\varepsilon_2}(\zeta_1,x_1)&=\varepsilon_2+\vartheta_0(\zeta_1)+h(\zeta_1,x_1,\vartheta_{\varepsilon_2}(\zeta_1,x_1))+{}_0I^{\gamma}_{x_1}f(\zeta_1,\rho,\vartheta_{\varepsilon_2}(\zeta_1,\rho))\\&\quad+{}_0I^{\delta}_{x_1}{}_0I^{\beta}_{\zeta_1}K(\zeta_1,x_1,\varsigma,\rho)g(\varsigma,\rho,\vartheta_{\varepsilon_2}(\varsigma,\rho))\\&<\varepsilon_1+\vartheta_0(\zeta_1)+h(\zeta_1,x_1,\vartheta_{\varepsilon_1}(\zeta_1,x_1))+{}_0I^{\gamma}_{x_1}f(\zeta_1,\rho,\vartheta_{\varepsilon_1}(\zeta_1,\rho))\\&\quad+{}_0I^{\delta}_{x_1}{}_0I^{\beta}_{\zeta_1}K(\zeta_1,x_1,\varsigma,\rho)g(\varsigma,\rho,\vartheta_{\varepsilon_1}(\varsigma,\rho))\\&=\vartheta_{\varepsilon_1}(\zeta_1,x_1),\end{aligned}$$

which defies the fact that $\vartheta_{\varepsilon_2}(\zeta_1,x_1)=\vartheta_{\varepsilon_1}(\zeta_1,x_1)$. As a result inequality (18) is true. That is, there exists a decreasing sequence ε_n such that $\varepsilon_n\to0$ as $n\to\infty$ and $\lim\limits_{n\to\infty}\vartheta_{\varepsilon_n}(\zeta,\, x)$ exists uniformly in $(\zeta,\, x)\in I\times J$. We write this limiting value by $l(\zeta,\, x)$. Evidently, by the uniform continuity of $h,\ f$, and g, the equation

$$\begin{aligned}\vartheta_{\varepsilon_n}(\zeta_1,x_1)&=\varepsilon_n+\vartheta_0(\zeta_1)+h(\zeta_1,x_1,\vartheta_{\varepsilon_n}(\zeta_1,x_1))+{}_0I^{\gamma}_{x_1}f(\zeta_1,\rho,\vartheta_{\varepsilon_n}(\zeta_1,\rho))\\&\quad+{}_0I^{\delta}_{x_1}{}_0I^{\beta}_{\zeta_1}K(\zeta_1,x_1,\varsigma,\rho)g(\varsigma,\rho,\vartheta_{\varepsilon_n}(\varsigma,\rho)),\end{aligned}$$

yields that $l(\zeta,\, x)$ is a solution of problem (1)-(2), let $\vartheta(\zeta,\, x)$ be any solution of problem (1)-(2) in $(\zeta,\, x)\in I\times J$. Then

$$\begin{aligned}\vartheta(\zeta,\, x)&<\varepsilon+\vartheta_0(\zeta)+h(\zeta,\, x,\vartheta(\zeta,\, x))+{}_0I^{\gamma}_{x}f(\zeta,\rho,\vartheta(\zeta,\rho))\\&\quad+{}_0I^{\delta}_{x}{}_0I^{\beta}_{\zeta}K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho)),\\&=\vartheta_{\varepsilon}(\zeta,\, x).\end{aligned}$$

Since the maximal solution is unique, it is clear that $\vartheta_\varepsilon(\zeta,\, x)$ tends to $l(\zeta,\, x)$ uniformly in $(\zeta,\, x)\in I\times J$ as $\varepsilon\to0$, which indicates the existence of maximal solution for the problem (1)-(2). Similarly, we can demonstrate the existence of the minimum solution.

5.3. *Continuation theorem.* This section examines the continuation of the solution to the problem (1)-(2) for the particular case $0<\gamma\leq1$, $\alpha=0$, and $\beta=1$, then the corresponding integral equation of problem (1)-(2) reduces to

$$\begin{aligned}\vartheta(\zeta,\, x)&=\vartheta_0(\zeta)+h(\zeta,\, x,\vartheta(\zeta,\, x))+{}_0I^{\gamma}_{x}f(\zeta,\rho,\vartheta(\zeta,\rho))\\&\quad+{}_0I^{\gamma}_{x}\int_0^{\zeta}K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho))d\varsigma.\end{aligned}\tag{19}$$

Theorem 5.5. *Let $h(\zeta,\, x,\vartheta(\zeta,\, x))$, $f(\zeta,\, x,\vartheta(\zeta,\, x))$, $g(\zeta,\, x,\vartheta(\zeta,\, x))$ and $K(\zeta,\, x,\varsigma,\rho)$ are continuous functions on E, then*
$\lim_{\gamma\to z}{}_0I^{\gamma}_{x}\left\{f(\zeta,\rho,\vartheta(\zeta,\rho))+\int_0^{\zeta}K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho))d\varsigma\right\}$
$={}_0I^{z}_{x}\left\{f(\zeta,\rho,\vartheta(\zeta,\rho))+\int_0^{\zeta}K(\zeta,\, x,\varsigma,\rho)g(\varsigma,\rho,\vartheta(\varsigma,\rho))d\rho\right\}.$

Proof. We have

$$\begin{aligned}
&\Big| {}_0I_x^{\gamma}\Big\{ f(\zeta,\rho,\vartheta(\zeta,\rho)) + \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)) d\varsigma \Big\} \\
&\qquad\qquad - {}_0I_x^{z}\Big\{ f(\zeta,\rho,\vartheta(\zeta,\rho)) + \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)) d\varsigma \Big\} \Big| \\
&\le \Big| \frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} f(\zeta,\rho,\vartheta(\zeta,\rho)) d\rho - \frac{1}{\Gamma(z)} \int_0^x (x-\rho)^{z-1} f(\zeta,\rho,\vartheta(\zeta,\rho)) d\rho \Big| \\
&\quad + \Big| \frac{1}{\Gamma(\gamma)} \int_0^x \int_0^{\zeta} (x-\rho)^{\gamma-1} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)) d\varsigma d\rho \\
&\quad - \frac{1}{\Gamma(z)} \int_0^x \int_0^{\zeta} (x-\rho)^{z-1} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)) d\varsigma d\rho \Big| \\
&= \Big| \Big(\frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} - \frac{1}{\Gamma(z)} \int_0^x (x-\rho)^{z-1} \Big) f(\zeta,\rho,\vartheta(\zeta,\rho)) d\rho \Big| \\
&\quad + \Big| \Big(\frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} - \frac{1}{\Gamma(z)} \int_0^x (x-\rho)^{z-1} \Big) \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta(\varsigma,\rho)) d\varsigma d\rho \Big| .
\end{aligned}$$

Since $\frac{1}{\Gamma(\gamma)} \int_0^x (x-\rho)^{\gamma-1} \to \frac{1}{\Gamma(z)} \int_0^x (x-\rho)^{z-1}$, as $\gamma \to z$, $z = 1,2,3,\ldots\ldots$
we get the result.

Theorem 5.6. *If the solution $\vartheta_1(\zeta,\, x)$ of eq. (19) exists, and if $\vartheta_\gamma(\zeta,\, x)$ is the solution of problem (1)-(2), then* $\lim_{\gamma\to 1} \vartheta_\gamma(\zeta,\, x) = \vartheta_1(\zeta,\, x)$

Proof. We have

$$\begin{aligned}
\vartheta_\gamma(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta_\gamma(\zeta,\, x)) + {}_0I_x^{\gamma} f(\zeta,\rho,\vartheta_\gamma(\zeta,\rho)) \\
&\quad + {}_0I_x^{\gamma} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_\gamma(\varsigma,\rho)) d\varsigma
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\vartheta_1(\zeta,\, x) &= \vartheta_0(\zeta) + h(\zeta,\, x, \vartheta_1(\zeta,\, x)) + {}_0I_x^{1} f(\zeta,\rho,\vartheta_1(\zeta,\rho)) \\
&\quad + {}_0I_x^{1} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) d\varsigma.
\end{aligned} \tag{21}$$

Then

$$\begin{aligned}
&\big\| \vartheta_\gamma(\zeta,\, x) - \vartheta_1(\zeta,\, x) \big\| \\
&\le \big\| h(\zeta,\, x, \vartheta_\gamma(\zeta,\, x)) - h(\zeta,\, x, \vartheta_1(\zeta,\, x)) \big\| + \big\| {}_0I_x^{\gamma} f(\zeta,\rho,\vartheta_\gamma(\zeta,\rho)) - {}_0I_x^{1} f(\zeta,\rho,\vartheta_1(\zeta,\rho)) \big\| \\
&\quad + \Big\| {}_0I_x^{\gamma} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_\gamma(\varsigma,\rho)) d\varsigma - {}_0I_x^{1} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) d\varsigma \Big\| \\
&\le F_h \big\| \vartheta_\gamma(\zeta,\, x) - \vartheta_1(\zeta,\, x) \big\| + \frac{T^{\gamma}}{\Gamma(\gamma+1)} F_f \big\| \vartheta_\gamma(\zeta,\, x) - \vartheta_1(\zeta,\, x) \big\| \\
&\quad + \big\| {}_0I_x^{\gamma} f(\zeta,\rho,\vartheta_1(\zeta,\rho)) - {}_0I_x^{1} f(\zeta,\rho,\vartheta_1(\zeta,\rho)) \big\| + \frac{T^{\gamma}}{\Gamma(\gamma+1)} F_g K_0 \big\| \vartheta_\gamma(\zeta,\, x) - \vartheta_1(\zeta,\, x) \big\| \\
&\quad + \Big\| {}_0I_x^{\gamma} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) d\varsigma - {}_0I_x^{1} \int_0^{\zeta} K(\zeta,\, x,\varsigma,\rho) g(\varsigma,\rho,\vartheta_1(\varsigma,\rho)) d\varsigma \Big\| .
\end{aligned}$$

Thus,

$$\left\|\vartheta_\gamma-\vartheta_1\right\| \leq\left[1-F_h-\frac{T^\gamma}{\Gamma(\gamma+1)} F_f-\frac{T^\gamma}{\Gamma(\gamma+1)} F_g K_0\right]^{-1}\left\{\left\|{}_0I_x^\gamma f(\zeta, \rho, \vartheta_1(\zeta, \rho))-{}_0I_x^1 f(\zeta, \rho, \vartheta_1(\zeta, \rho))\right\|\right.$$
$$\left.+\left\|{}_0I_x^\gamma \int_0^\zeta K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta_1(\varsigma, \rho)) d\varsigma-{}_0I_x^1 \int_0^\zeta K(\zeta,\, x, \varsigma, \rho) g(\varsigma, \rho, \vartheta_1(\varsigma, \rho)) d\varsigma\right\|\right\},$$

where $F_h+\frac{T^\gamma}{\Gamma(\gamma+1)} F_f+\frac{T^\gamma}{\Gamma(\gamma+1)} F_g K_0<1$ (from the uniqueness theorem). The proof is complete as we have $\left\|\vartheta_\gamma-\vartheta_1\right\| \rightarrow 0$ as $\gamma \rightarrow 1$, which is in accordance with Theorem 5.5.

6. Numerical examples

Exmaple 1. Consider the problem

$$ {}_0^C D_x^\gamma\left[\vartheta(\zeta,\, x)-\cos \pi \zeta \frac{x^2}{4} \sin \vartheta(\zeta,\, x)\right]=\frac{e^{-\zeta^2} x}{10} \vartheta(\zeta,\, x)+{}_0I_x^\alpha\, {}_0I_\zeta^\beta \frac{x \varsigma^2 e^\zeta\left(1+\rho^2\right)}{8} \frac{\varsigma^4 \rho^2}{16} \vartheta(\varsigma, \rho) \tag{22}$$

with the initial condition $\vartheta(\zeta, 0)=\vartheta_0(\zeta)=\frac{\zeta}{16}$, and $(\zeta,\, x) \in[0,1] \times[0,1]$.

Here we have,

$$h(\zeta,\, x, \vartheta(\zeta,\, x))=\cos \pi \zeta \frac{x^2}{4} \sin \vartheta(\zeta,\, x),$$
$$f(\zeta,\, x, \vartheta(\zeta,\, x))=\frac{e^{-\zeta^2} x}{10} \vartheta(\zeta,\, x),$$
$$K(\zeta,\, x, \varsigma, \rho)=\frac{x \varsigma^2 e^\zeta\left(1+\rho^2\right)}{8}$$

and

$$g(\zeta,\, x, \vartheta(\zeta,\, x))=\frac{\zeta^4 x^2}{16} \vartheta(\zeta,\, x).$$

As all the assumptions are satisfying for the functions $h,\ f,\ K$, and g, therefore the values are given as $F_h=\frac{1}{4}$, $F_f=\frac{1}{10}$, $K_0=\frac{e}{4}$, and $F_g=\frac{1}{16}$.

This leads to

$$K^*=\frac{1}{4}+\left(\frac{1}{10}\right) \frac{1}{\Gamma(\gamma+1)}+\left(\frac{e}{4}\right)\left(\frac{1}{16}\right) \frac{1}{\Gamma(\delta+1) \Gamma(\beta+1)}$$

Case I: For a particular value of $\alpha=\frac{1}{8},\ \beta=\frac{1}{4}$, and $\gamma=\frac{1}{8}$, this gives

$$\begin{aligned} K^* & =\frac{1}{4}+\left(\frac{1}{10}\right) \frac{1}{\Gamma(9/8)}+\left(\frac{e}{4}\right)\left(\frac{1}{16}\right) \frac{1}{\Gamma(5/4) \Gamma(5/4)}. \\ & =0.45958<1. \end{aligned}$$

Consequently, the problem has a unique solution.

Case II: For a particular value of $\alpha=\frac{1}{4},\ \beta=\frac{1}{2}$, and $\gamma=\frac{1}{3}$, this gives

$$\begin{aligned} K^* & =\frac{1}{4}+\left(\frac{1}{10}\right) \frac{1}{\Gamma(4/3)}+\left(\frac{e}{4}\right)\left(\frac{1}{16}\right) \frac{1}{\Gamma(19/12) \Gamma(3/2)} \\ & =0.41573<1. \end{aligned}$$

Consequently, the problem has a unique solution.

The approximate solution to the problem (23) is given by

$$\vartheta_n(\zeta,\, x)=\frac{\zeta}{16}+\cos \pi \zeta \frac{x^2}{4} \sin \vartheta_{n-1}(\zeta,\, x)+\frac{1}{\Gamma(\gamma)} \int_0^x(x-\rho)^{\gamma-1} \frac{e^{-\zeta^2} \rho}{10} \vartheta_{n-1}(\zeta, \rho) d\rho$$
$$+\frac{1}{\Gamma(\delta)} \frac{1}{\Gamma(\beta)} \int_0^x \int_0^\zeta(x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1} \frac{x \varsigma^2 e^\zeta\left(1+\rho^2\right)}{8} \frac{\varsigma^4 \rho^2}{16} \vartheta_{n-1}(\varsigma, \rho) d\varsigma d\rho.$$

For $n=1$, we have the approximate solution ϑ_1 for Case I and II as shown in Figures 1 and 2, respectively.

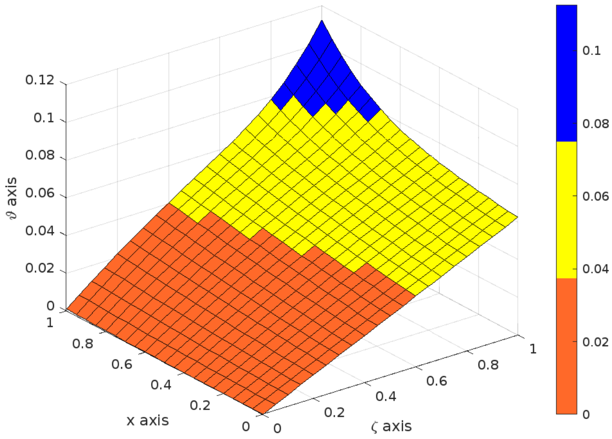

Figure 1: Graph for the approximate solution ϑ_1 of problem (22), where $\alpha = \frac{1}{8}$, $\beta = \frac{1}{4}$, $\gamma = \frac{1}{8}$.

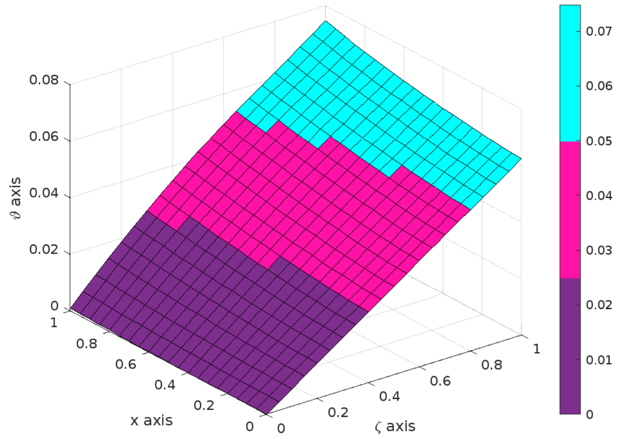

Figure 2: Graph for the approximate solution ϑ_1 of problem (22), where $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{3}$.

Figure 1 and Figure 2 represent the approximate solution ϑ_1 for the problem (22), but they correspond to different fractional order values. For the particular values of α, β, and γ, the first figure covers a broader range than the second figure.

Exmaple 2. Consider the problem

$$(23)\qquad {}_0^C D_x^\gamma \left[\vartheta(\zeta,\, x) - \sin 2x \frac{e^{-\zeta x}}{10}(2 + \vartheta(\zeta,\, x))\right] = \frac{x\zeta^3}{4}(1 + \vartheta(\zeta,\, x)) + {}_0I_x^\alpha\, {}_0I_\zeta^\beta \rho\varsigma^2 \frac{(\zeta^2 + +3)}{16}\left(x\zeta + \frac{e^{(\rho^2+\varsigma)}}{12}\vartheta(\varsigma,\rho)\right)$$

with the initial condition $\vartheta(\zeta, 0) = \vartheta_0(\zeta) = 0$. and $(\zeta,\, x) \in [0,1]\times[0,1]$.

Here we have,

$$h(\zeta,\, x, \vartheta(\zeta,\, x)) = \sin 2x \frac{e^{-\zeta x}}{10}(2 + \vartheta(\zeta,\, x)),$$

$$f(\zeta,\, x, \vartheta(\zeta,\, x)) = \frac{x\zeta^3}{4}(1 + \vartheta(\zeta,\, x)),$$

$$K(\zeta,\, x, \varsigma, \rho) = \rho\varsigma^2 \frac{(\zeta^2 + \zeta x + 3)}{16},$$

and

$$g(\zeta,\, x, \vartheta(\zeta,\, x)) = x\zeta + \frac{e^{(x^2+\zeta)}}{12}\vartheta(\zeta,\, x).$$

As all the assumptions are satisfying for the functions h, f, K, and g, therefore the values are given as $F_h = \frac{1}{10}$, $F_f = \frac{1}{4}$, $K_0 = \frac{5}{16}$, and $F_g = \frac{e^2}{12}$.

This leads to

$$K^* = \frac{1}{10} + \left(\frac{1}{4}\right)\frac{1}{\Gamma(\gamma+1)} + \left(\frac{5}{16}\right)\left(\frac{e^2}{12}\right)\frac{1}{\Gamma(\delta+1)\Gamma(\beta+1)}$$

Case I: For a particular value of $\alpha = \frac{1}{10}$, $\beta = \frac{1}{4}$, and $\gamma = \frac{1}{5}$, it gives

$$K^* = \frac{1}{10} + \left(\frac{1}{4}\right)\frac{1}{\Gamma(6/5)} + \left(\frac{5}{16}\right)\left(\frac{e^2}{12}\right)\frac{1}{\Gamma(13/10)\Gamma(5/4)}$$
$$= 0.608828 < 1.$$

Consequently, the problem has a unique solution.

Case II: For a particular value of $\alpha = \frac{1}{2}$, $\beta = \frac{1}{7}$, and $\gamma = \frac{1}{3}$, it gives

$$K^* = \frac{1}{10} + \left(\frac{1}{4}\right)\frac{1}{\Gamma(1/3)} + \left(\frac{5}{16}\right)\left(\frac{e^2}{12}\right)\frac{1}{\Gamma(11/6)\Gamma(8/7)}$$

$$= 0.455738 < 1.$$

Consequently, the problem has a unique solution.

The approximate solution to the problem (23) is given by

$$\vartheta_n(\zeta,\, x) = \sin 2x \frac{e^{-\zeta x}}{10}(2+\vartheta_{n-1}(\zeta,\, x)) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-\rho)^{\gamma-1}\frac{\rho\zeta^3}{4}(1+\vartheta_{n-1}(\zeta,\rho))d\rho$$
$$+\frac{1}{\Gamma(\delta)}\frac{1}{\Gamma(\beta)}\int_0^x\int_0^\zeta (x-\rho)^{\delta-1}(\zeta-\varsigma)^{\beta-1}\rho\varsigma^2\frac{(\zeta^2++3)}{16}\left(\rho\varsigma+\frac{e^{(\rho^2+\varsigma)}}{12}\vartheta_{n-1}(\varsigma,\rho)\right)d\varsigma d\rho.$$

For $n=1$, we have the approximate solution ϑ_1 for Case I and II, as shown in Figures 3 and 4, respectively.

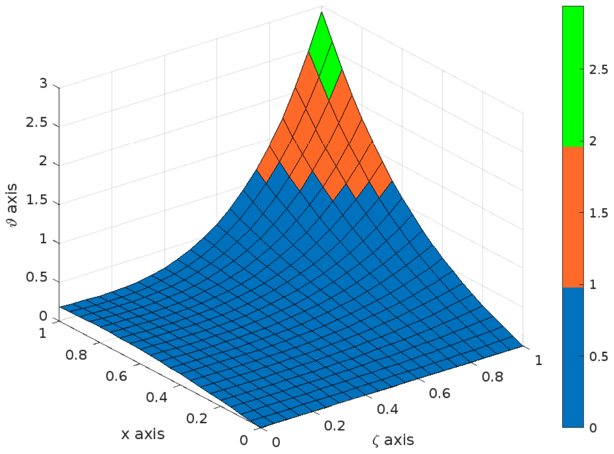

Figure 3: Graph for the approximate solution ϑ_1 of problem (23), where $\alpha=\frac{1}{10}$, $\beta=\frac{1}{4}$, $\gamma=\frac{1}{5}$.

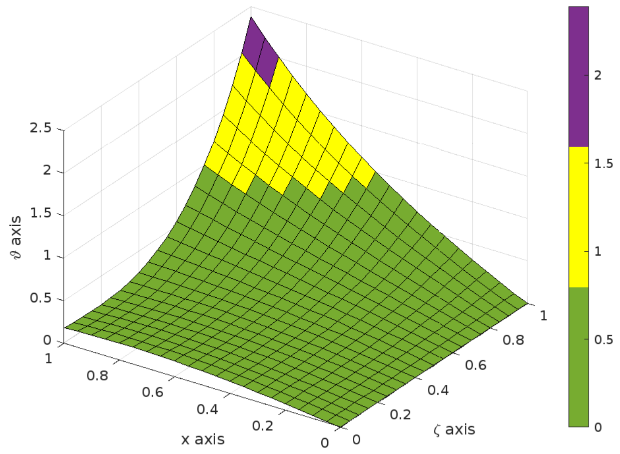

Figure 4: Graph for the approximate solution ϑ_1 of problem (23), where $\alpha=\frac{1}{2}$, $\beta=\frac{1}{7}$, $\gamma=\frac{1}{3}$.

Figure 3 and Figure 4 represent the approximate solution ϑ_1 for the problem (23), but they correspond to different fractional order values. For the particular values of α, β, and γ, the third figure covers a broader range compared to the fourth figure.

7. Conclusion

Two-dimensional IDEs have indeed attracted a lot of research interest in recent years due to their significance in various fields of science and engineering. Theoretical results and analytical and numerical solutions for these types of problems are of great interest to researchers. Several numerical methods have been developed for solving two-dimensional IDEs, including two-dimensional Triangular function, Haar wavelet, Tau method, and meshless methods.

In this paper, we have discussed the existence and uniqueness of the solution of the considered two-dimensional neutral integro-differential equation of fractional order by using Banach's and Krasnoselskii's fixed point theorems and then we have discussed Ulam-Hyers and Ulam-Hyers Rassias's stability of the considered problem. Additionally, we obtained a positive solution, maximal and minimal solution, and Continuation theorem. Validated our results with a few examples. In future work, we can find the numerical solution to the considered problem and establish qualitative properties for various types of two-dimensional integro-differential equations.

Appendix A. Coding for examples.

Figure 1. [X,Y] = meshgrid(0:0.05:1,0:0.05:1);
$Z = 0.0625.*X + cos(pi.*X).*0.015625.*Y.^2.*sin(0.0625.*Y) + 0.04444.*X.*exp(-X.^2).*Y.^1.125 + 0.017651.*X.^6.25.*exp(-X).*Y.^3.25.*(2.8444+Y.^2.*2.471191);$
surf(X,Y,Z)
colorbar
mycolors = [1 0 0; 1 1 0; 0 0 1];
colormap(mycolors);

Figure 2. [X,Y] = meshgrid(0:0.05:1,0:0.05:1);
$Z = 0.0625.*X + cos(pi.*X).*0.015625.*Y.^{2}.*sin(0.0625.*Y) + 0.028125.*X.*exp(-X.^{2}).*Y.^{1}.3333 + 0.005328.*X.^{6}.5.*exp(-X).*Y.^{3}.5833.*(0.83822 + Y.^{2}.*0.61245);$
surf(X,Y,Z)
colorbar
mycolors = [1 0 0; 1 1 0; 0 0 1];
colormap(mycolors);

Figure 3. [X,Y] = meshgrid(0:0.05:1,0:0.05:1);
$Z = 0.2.*exp(-X.*Y).*sin(2.*Y) + 1.04166667.*(X.^{3}).*Y.^{1}.14285714 + 0.36589.*(X.^{2} + X.*Y + 3).*X.^{3}.14285714.*Y.^{2}.833333;$
surf(X,Y,Z)
colorbar
mycolors = [1 0 0; 1 1 0; 0 0 1];
colormap(mycolors);

Figure 4. [X,Y] = meshgrid(0:0.05:1,0:0.05:1);
$Z = 0.2.*exp(-X.*Y).*sin(2.*Y) + 1.53125.*(X.^{3}).*Y.^{1}.14285714 + 0.157576569.*(X.^{2} + X.*Y + 3).*X.^{3}.14285714.*Y.^{2}.833333;$
surf(X,Y,Z)
colorbar
mycolors = [1 0 0; 1 1 0; 0 0 1];
colormap(mycolors);

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