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A CONSTRUCTION OF MULTIWAVELET SETS IN THE EUCLIDEAN PLANE

Abstract

For $A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$, where a is an integer such that $|a| > 1$ and a natural number d satisfying $L = (|a| - 1)d$, we obtain that the product $W \times Q$ of a measurable set W of the Lebesgue measure $2\pi L$, and a measurable set Q in $\mathbb{R}$ such that $Q \subset aQ$, is an MRA A-multiwavelet set of order Ld in $\mathbb{R}^2$ if and only if W is an a-multiwavelet set of order L and Q is an a-multiscaling set of order d associated with W.

1 Introduction.

The concept of wavelet sets has been introduced by observing that the Lebesgue measure of the support of the Fourier transform of an orthonormal wavelet is at least 2π. Considering the notion of multiwavelets [7, 8, 12], wavelet sets have been generalized into multiwavelet sets by Bownik, Rzeszotnik and Speegle in [4]. The study related to wavelet sets and also to multiwavelet sets has attracted the attention of several workers [1, 3, 4, 10, 17, 18, 19, 20].

In this paper, we assume that a is an integer such that $|a| > 1$, and that L is a natural number for which $L/(|a| - 1)$ is an integer, say, d.

Having described necessary notation and preliminaries in Section 2, we prove that for an expansive matrix A, an A-multiwavelet set W has an A-multiscaling set if and only if it is an MRA A-multiwavelet set. In Section 3, we provide our main result, according to which the product $W \times Q$ of a measurable set W of Lebesgue measure $2\pi L$, and a measurable set Q in $\mathbb{R}$ such that $Q \subset aQ$, is an MRA A-multiwavelet set of order Ld in $\mathbb{R}^2$ if and

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only if W is an a-multiwavelet set of order L and Q is an a-multiscaling set of order d associated with W, where $A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$.

2 Notation and Preliminaries.

Throughout the paper, the symbols $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote, respectively, the set of natural numbers, the set of integers and the real line. By A, we denote an $n \times n$ expansive matrix such that $A\mathbb{Z}^n \subseteq \mathbb{Z}^n$, where $n \in \mathbb{N}$. The transpose of A is denoted by A^*. The Lebesgue measure of a measurable set E in the Euclidean space $\mathbb{R}^n$ is denoted by $|E|$. The collection of all square integrable complex valued functions on $\mathbb{R}^n$, in which two functions are identified if they are equal almost everywhere (abbreviated, a.e.), is denoted by $L^2(\mathbb{R}^n)$. With the usual addition, scalar multiplication and the inner product $\langle f, g\rangle$ of $f, g \in L^2(\mathbb{R}^n)$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx,$$

$L^2(\mathbb{R}^n)$ becomes a Hilbert space. For a function $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(t)\, e^{-i<\xi,\, t>} dt,$$

and the inverse Fourier transform $\check{f}$ of f is defined by

$$\check{f}(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\xi)\, e^{i<\xi,\, t>} d\xi.$$

A finite set $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^n)$, is called an *orthonormal A-multiwavelet* of order L, if the system $\{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, ..., L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, where

$$\psi^l_{j,k}(x) = |detA|^{\frac{j}{2}} \psi^l\left(A^j x - k\right), \qquad x \in \mathbb{R}^n.$$

In the case that Ψ consists of a single element, say ψ, we say ψ is an *n-dimensional orthonormal A-wavelet*, or simply an *A-wavelet*. The following result characterizes an orthonormal A-multiwavelet.

Theorem 2.1.[8, 12] *A subset $\Psi = \{\psi^1, ..., \psi^L\}$ of $L^2(\mathbb{R}^n)$ is an orthonormal A-multiwavelet if and only if the following hold*:

(i) $\sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}^l(A^{*j}\xi)|^2 = 1, \quad a.e., \ \xi \in \mathbb{R}^n,$

(ii) $\sum_{l=1}^{L}\sum_{j=0}^{\infty}\hat{\psi}^l(A^{*j}\xi)\overline{\hat{\psi}^l(A^{*j}(\xi+2s\pi))}=0,\ a.e.,\ \xi\in\mathbb{R}^n, s\in\mathbb{Z}^n\backslash A^*\mathbb{Z}^n$,

(iii) $||\psi^l||=1,\quad for\ l=1,...,L.$

A method to obtain A-multiwavelets in $L^2(\mathbb{R}^n)$ arises from the notion known as the A-multiresolution analysis of multiplicity d [2, 5, 11, 16], which is described below:

Definition 2.2. An *A-multiresolution analysis* (A-MRA) *of multiplicity* d associated with the lattice $\mathbb{Z}^n$ is a sequence of closed subspaces V_j, $j\in\mathbb{Z}$, of $L^2(\mathbb{R}^n)$ satisfying

(a) $V_j\subset V_{j+1}$, for all $j\in\mathbb{Z}$;

(b) $f(\cdot)\in V_j$, if and only if $f(A\cdot)\in V_{j+1}$, for all $j\in\mathbb{Z}$;

(c) $\cap_{j\in\mathbb{Z}}V_j=\{0\}$;

(d) $\overline{\cup_{j\in\mathbb{Z}}V_j}=L^2(\mathbb{R}^n)$;

(e) There exist functions $\varphi_1,\varphi_2,...,\varphi_d\in L^2(\mathbb{R}^n)$ such that $\{\varphi_i(\cdot\ -k):k\in\mathbb{Z}^n,\ i=1,...,d\}$ forms an orthonormal basis for V_0.

The functions $\varphi_1,\varphi_2,...,\varphi_d$ are called *scaling functions* of the A-MRA, and the vector $\Phi=(\varphi_1,...,\varphi_d)^*$ is called a *multiscaling function with multiplicity* d [6, 15] for the A-MRA.

In [2], it is shown that an A-multiresolution analysis of multiplicity d gives rise to an A-multiwavelet Ψ of order L, where $L=(|detA|-1)d$.

It is well known that $|supp\ \hat{\psi}|$, where ψ is an n-dimensional orthonormal A-wavelet, is at least $(2\pi)^n$. An A-wavelet ψ for which $|supp\ \hat{\psi}|=(2\pi)^n$, is said to be a *minimally supported frequency* (MSF) *A-wavelet* [8, 9, 10]. It is also known that for an MSF A-wavelet ψ, there exists a measurable set W of measure $(2\pi)^n$ such that $|\hat{\psi}|=\chi_W$. We call the set W is an *A-wavelet set.*

The concept of an MSF *A-wavelet* has been generalized to that of an MSF *A-multiwavelet of order* L [4] as follows:

Definition 2.3.[4] An MSF *A-multiwavelet* of order L is an orthonormal A-multiwavelet $\Psi=\{\psi^1,...,\psi^L\}$ such that $|\hat{\psi}^l|=\chi_{W_l}$, for some measurable sets $W_l\subset\mathbb{R}^n$, $l=1,...,L$.

Stated below is a characterization of MSF A-multiwavelets:

Theorem 2.4.[4] *A set* $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^n)$ *such that* $|\hat{\psi}^l| = \chi_{W_l}$, *for* $l = 1, ..., L$, *is an orthonormal A-multiwavelet if and only if*

(i) $\sum_{k\in\mathbb{Z}^n} \chi_{W_l}(\xi+2\pi k)\, \chi_{W_m}(\xi+2\pi k) = \delta_{l,m}$, *a.e.*, $\xi \in \mathbb{R}^n$, $l, m = 1, ..., L$,

(ii) $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \chi_{W_l}(A^{*j}\xi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$.

Notice that equality is, in general, almost everywhere. Also, we shall say sets A and B to be disjoint if $|A \cap B| = 0$. An empty set, is symbol ϕ, will mean a set of measure zero.

Observing that Theorem 2.4 (i) implies that the disjoint union (modulo sets of measure zero) of translates of W_l by $2\pi\mathbb{Z}^n$ covers $\mathbb{R}^n$, a.e., for $l = 1, ..., L$, while (ii) implies that $\{(A^*)^{-j}(\dot{\bigcup}_{l=1}^{L} W_l) : j \in \mathbb{Z}\}$ partitions $\mathbb{R}^n$, a.e., the notion of an *A-multiwavelet set* has been introduced in [4]. Precisely,

Definition 2.5.[4] A measurable set $W \subset \mathbb{R}^n$ is an *A-multiwavelet set* of order L, if $W = \dot{\bigcup}_{l=1}^{L} W_l$, for some measurable sets $W_1, ..., W_L \subset \mathbb{R}^n$ satisfying

(i) $\sum_{k\in\mathbb{Z}^n} \chi_{W_l}(\xi + 2k\pi)\, \chi_{W_m}(\xi + 2k\pi) = \delta_{l,m}$, *a.e.*, $\xi \in \mathbb{R}^n$, $l, m = 1, ..., L$, and

(ii) $\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \chi_{W_l}(A^{*j}\xi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$.

The following characterization of A-multiwavelet sets of order L established in [4], will be used in the sequel.

Theorem 2.6.[4] *A measurable set* $W \subset \mathbb{R}^n$ *is an A-multiwavelet set of order* L *if and only if*

(i) $\sum_{k\in\mathbb{Z}^n} \chi_W(\xi + 2k\pi) = L$, *a.e.*, $\xi \in \mathbb{R}^n$, *and*

(ii) $\sum_{j\in\mathbb{Z}} \chi_W(A^{*j}\xi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$.

Two measurable sets E and F of $\mathbb{R}^n$ are said to be 2π-*translation congruent* modulo null sets if there is a measurable bijection τ_1 from E to F such that $\tau_1(t) - t \in 2\pi\mathbb{Z}^n$, for each $t \in E$. These sets are said to be A-*dilation congruent* modulo null sets if there is a measurable bijection δ from E to F such that $\delta(t) = A^m t$, for an $m \in \mathbb{Z}$, where $t \in E$.

Dai, Larson and Speegle in [9] proved the existence of wavelets for any expansive dilation matrix A. Gu and Han in [13] proved that if $|detA| = 2$, then there exists an MSF A-wavelet ψ in $L^2(\mathbb{R}^n)$, which arises from an A-MRA having φ as its scaling function. It is known that there is a measurable set

S in $\mathbb{R}^n$ such that $|\hat{\varphi}| = \chi_S$. Also, for the scaling function φ of an A-MRA satisfying $|\hat{\varphi}| = \chi_S$, for some measurable set S in $\mathbb{R}^n$, there exists an MSF A-wavelet ψ associated with the A-MRA. Such a set is called an *A-scaling set* [4, 13].

In [13], it has been found that a measurable set S in $\mathbb{R}^n$ is an *A-scaling* set if it satisfies the following:

(i) $S \subset A^*S$,

(ii) $W = A^*S \backslash S$, is an A-wavelet set of $\mathbb{R}^n$, and

(iii) $\{S + 2k\pi : k \in \mathbb{Z}^n\}$ is a measurable partition of $\mathbb{R}^n$, a.e.

It is easy to see that (ii) and (iii) imply (i). The following is an equivalent condition to (i) and (ii) [3, 4]:

(iv) $S = \cup_{j<0} A^{*j} W$, for some A-wavelet set W.

A measurable set S in $\mathbb{R}^n$ satisfying (i) and (ii), or equivalently (iv), is called a *generalized A-scaling set* [4]. In a similar way a generalized A-scaling set associated with an A-multiwavelet set has been described in [4] as follows:

Definition 2.7. A measurable set S in $\mathbb{R}^n$ is called a *generalized A-scaling set* if $|S| = (2\pi)^n L/(|detA| - 1)$, and $A^*S \backslash S$ is an A-multiwavelet set of order L.

Equivalently, a measurable set S in $\mathbb{R}^n$ is a generalized A-scaling set if and only if $S = \bigcup_{j=1}^{\infty} (A^*)^{-j} W$, for some A-multiwavelet set W.

Employing Lemma 2.2 in [4], and following the steps of the proof of Theorem 2.6 in [4], we easily obtain the proof of Lemma 2.8. Lemma 2.2 in [4] states that for a measurable subset $\bar{E}$ of $\mathbb{R}^n$, there is a measurable set $E \subset \bar{E}$, such that $\tau(E) = \tau(\bar{E})$ and $\tau|E$ is injective, where τ is a map from $\mathbb{R}^n$ to $(-\pi, \pi]^n$ defined by $\tau(\xi) = \xi + 2k\pi$, for some $k \in \mathbb{Z}^n$.

Lemma 2.8. *Let E be a measurable subset in $\mathbb{R}^n$ such that $|E| = (2\pi)^n d$. Then the following are equivalent*:

(a) $\sum_{k\in\mathbb{Z}^n} \chi_E(\xi + 2k\pi) = d$, *a.e.*, $\xi \in \mathbb{R}^n$.

(b) *There exists a disjoint partition $E_1, E_2, ..., E_d$ of E satisfying*
$\sum_{k\in\mathbb{Z}^n} \chi_{E_l}(\xi + 2k\pi) = 1$, *a.e.*, $\xi \in \mathbb{R}^n$, $l = 1, ..., d$.

(c) *There exists a disjoint partition* $E_1, E_2, ..., E_d$ *of* E *satisfying*
$\sum_{k\in\mathbb{Z}^n} \chi_{E_l}(\xi+2k\pi)\, \chi_{E_m}(\xi+2k\pi) = \delta_{l,m}$, *a.e.,* $\xi \in \mathbb{R}^n$, $l,m = 1,...,d.$

The following Lemma and its conclusion as stated below give rise the notion of multiscaling set of multiplicity d which is a particular case of multiscaling function of multiplicity d. We call a multiscaling set of multiplicity d associated with a dilation matrix A to be an A-*multiscaling set of order* d.

Lemma 2.9.[2; Lemma 5] *The sequence* $\{\varphi_i(\cdot - k) : k \in \mathbb{Z}^n,\ i = 1,...,d\}$ *is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}^n} \hat{\Phi}(\xi+2k\pi)\, \overline{\hat{\Phi}(\xi+2k\pi)^*} \equiv I_d,$$

where I_d *is an identity matrix of order* d.

From the above Lemma, we derive the following:

Let $\Phi = \{\varphi_1, \varphi_2, ..., \varphi_d\} \subset L^2(\mathbb{R}^n)$ *be such that* $|\hat{\varphi}_i| = \chi_{Q_i}$, *for some measurable sets* $Q_i \subset \mathbb{R}^n$, $i = 1,...,d$. *Then* $\{\varphi_i(. - k) : k \in \mathbb{Z}^n, i = 1,...,d\}$ *is an orthonormal system if and only if*

$$\sum_{k\in\mathbb{Z}^n} \chi_{Q_i}(\xi+2k\pi)\, \chi_{Q_j}(\xi+2k\pi) = \delta_{i,j}, \quad a.e.,\ \xi \in \mathbb{R}^n,\ i,j = 1,...,d.$$

Thus the disjoint union of translates of Q_i by $2\pi\mathbb{Z}^n$ covers $\mathbb{R}^n$, a.e., where $i = 1,...,d$. Using Lemma 2.8, we obtain that

$$\sum_{k\in\mathbb{Z}^n} \chi_Q(\xi+2k\pi) = d, \quad a.e.,\ \xi \in \mathbb{R}^n.$$

Now, we have

Definition 2.10. A measurable set $Q \subset \mathbb{R}^n$ is called an A-*multiscaling set of order* d if

(i) $|Q| = (2\pi)^n d$,

(ii) $W \equiv A^*Q \setminus Q$ is an A-multiwavelet set of order L, where $L = (|detA| - 1)d$, and

(iii) $\sum_{k\in\mathbb{Z}^n} \chi_Q(\xi+2k\pi) = d$, $a.e.,\ \xi \in \mathbb{R}^n$.

We say W is an A-multiwavelet set of order L associated with the A-multiscaling set Q of order d.

An immediate consequence of Theorem 3 in [7] is the following characterization of an orthonormal A-multiwavelet in $\mathbb{R}^n$ of order L arising from an A-multiresolution analysis of multiplicity d.

Theorem 2.11. *Let $\Psi = \{\psi^1, ..., \psi^L\}$ be an orthonormal A-multiwavelet in $L^2(\mathbb{R}^n)$ with $L = (|detA| - 1)d$, where d is a natural number. Then Ψ arises from an A-multiresolution analysis of multiplicity d if and only if*

$$\sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^l(A^{*j}(\xi+2\pi k))\right|^2 = d, \qquad a.e.,\ \xi\in\mathbb{R}^n.$$

We, now, assume that $|\hat{\psi}^l| = \chi_{W_l},\ l = 1, ..., L$. Then Ψ arises from an A-multiresolution analysis of multiplicity d if and only if

$$\sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\chi_{W_l}(A^{*j}(\xi+2\pi k)) = d, \qquad a.e.,\ \xi\in\mathbb{R}^n,$$

or, equivalently,

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\chi_{W}(A^{*j}(\xi+2\pi k)) = d, \qquad a.e.,\ \xi\in\mathbb{R}^n,$$

where $W = \bigcup_{l=1}^{L} W_l$.

The above can be rewritten as

$$\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\chi_{(A^*)^{-j}W}(\xi+2\pi k) = d, \qquad a.e.,\ \xi\in\mathbb{R}^n,$$

or,

$$\sum_{k\in\mathbb{Z}^n}\chi_{Q}(\xi+2\pi k) = d, \qquad a.e.,\ \xi\in\mathbb{R}^n,$$

where $Q = \bigcup_{j=1}^{\infty}(A^*)^{-j}W$.

A straightforward computation shows that $|Q| = (2\pi)^n d$, and $Q \subset A^*Q$.

Thus, we have the following characterization of MRA A-multiwavelet sets.

Theorem 2.12. *An A-multiwavelet set W in $\mathbb{R}^n$ of order L, arises from an A-multiresolution analysis of multiplicity d if and only if there is an A-multiscaling set Q in $\mathbb{R}^n$ of order d associated with W, where $L = (|detA|-1)d$.*

3 A construction of MRA A-multiwavelet sets in $\mathbb{R}^2$.

In this section, we obtain our main result, which provides a method to generate MRA A-multiwavelet sets in $\mathbb{R}^2$ from MRA a-multiwavelet sets in $\mathbb{R}$ as their product with their associated a-multiscaling sets.

Now, onwards, A denotes the matrix $\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$, where a is an integer such that $|a| > 1$. We begin with the following Lemma:

Lemma 3.1. *Let W be a measurable set of the Lebesgue measure $2\pi L$ in $\mathbb{R}$, and Q be a measurable set in $\mathbb{R}$ such that $Q \subset aQ$. If $W \times Q$ is an A-multiwavelet set of order Ld in $\mathbb{R}^2$, where $L = (a-1)d$ then*

(a) $a^k W \cap a^j W = \phi$, *for* $j, k \in \mathbb{Z},\ j \neq k$.

(b) *for every* $k \in \mathbb{Z}$, (i) $W \cap a^k Q = \phi$ *and* (ii) $a^{k-1} W \cap Q = \phi$, *cannot hold simultaneously.*

(c) $Q \cap a^{k-1} W = \phi$, *where k is a natural number.*

(d) $W = aQ \backslash Q$, *a.e.*

(e) $\dot{\bigcup}_{j\in\mathbb{Z}} a^j W = \mathbb{R}$, *a.e.*

(f) $Q = \bigcup_{k=1}^{\infty} a^{-k} W$, *a.e.*

PROOF. (a). Since $W \times Q$ is an A-multiwavelet set, by Theorem 2.6 (ii), we have

$$\begin{aligned}
\mathbb{R}^2 &= \dot{\bigcup}_{j\in\mathbb{Z}} (A^*)^{-j} (W \times Q) \\
&= \dot{\bigcup}_{j\in\mathbb{Z}} \left[\begin{pmatrix} 0 & a^j \\ a^{j-1} & 0 \end{pmatrix} (W \times Q) \cup \begin{pmatrix} a^j & 0 \\ 0 & a^j \end{pmatrix} (W \times Q) \right] \\
&= \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^j Q \times a^{j-1} W) \cup (a^j W \times a^j Q) \right], \quad a.e. \qquad (3.1)
\end{aligned}$$

Since the right hand side of (3.1) consists of disjoint sets $a^j Q \times a^{j-1} W$, $j \in \mathbb{Z}$, for $j, k \in \mathbb{Z},\ j \neq k$,

$$(a^{j+1} Q \times a^j W) \cap (a^{k+1} Q \times a^k W) = (a^{j+1} Q \cap a^{k+1} Q) \times (a^j W \cap a^k W) = \phi.$$

In view of fact that $(a^{j+1} Q \cap a^{k+1} Q)$ is nonempty, we have (a). ⊠

(b). We establish it by contradiction. Suppose that for some $k \in \mathbb{Z}$, (i) and (ii) hold. Since (3.1) is a disjoint union of sets and $a^kW \cap a^jW = \phi$, where $j \neq k$, we have

$$\begin{aligned}
&\left|W \times a^{k-1}W\right| \\
&= \left|\left(W \times a^{k-1}W\right) \cap \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^jQ \times a^{j-1}W) \cup (a^jW \times a^jQ)\right]\right| \\
&= \left|\dot{\bigcup}_{j\in\mathbb{Z}} \left[(W \cap a^jQ) \times (a^{k-1}W \cap a^{j-1}W) \cup (W \cap a^jW) \times (a^{k-1}W \cap a^jQ)\right]\right| \\
&= \sum_{j\in\mathbb{Z}} \left(\left|(W \cap a^jQ) \times (a^{k-1}W \cap a^{j-1}W)\right| + \left|(W \cap a^jW) \times (a^{k-1}W \cap a^jQ)\right|\right) \\
&= \left|(W \cap a^kQ)\right| \left|(a^{k-1}W)\right| + |W| \left|(a^{k-1}W \cap Q)\right| = 0,
\end{aligned}$$

which implies $|W| = 0$, a contradiction. ⊠

(c). Since $W \times Q$ is an A-multiwavelet set, (3.1) holds. As $W \times Q$ appears in the disjoint union on the right hand side of (3.1), for an integer n,

$$(W \times Q) \cap (a^nQ \times a^{n-1}W) = \phi. \tag{3.2}$$

From (3.2), it follows that

$$(W \cap a^kQ) \times (Q \cap a^{k-1}W) = \phi,$$

where $k \in \mathbb{Z}$. Therefore, either $W \cap a^kQ = \phi$, or $Q \cap a^{k-1}W = \phi$.

To prove the result, we need to show that $Q \cap a^{k-1}W = \phi$, for $k \geq 1$. We achieve this by establishing that for $k \geq 1$, $W \cap a^kQ \neq \phi$, and using facts proved in (b). Suppose, for the sake of contradiction that $W \cap a^lQ = \phi$, for some $l \geq 1$. Since $l \geq 1$, $|a| > 1$, and $|(a^lW \cap aQ)| < |a^lW|$, first note that the set $(a^lW \backslash aQ)$ has positive measure. Using (3.1), we have

$$\begin{aligned}
&\left|(a^lW \backslash aQ) \times W\right| \\
&= \left|\left(a^lW \backslash aQ) \times W\right) \cap \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^jQ \times a^{j-1}W) \cup (a^jW \times a^jQ)\right]\right| \\
&= \left|\dot{\bigcup}_{j\in\mathbb{Z}} \left[((a^lW \backslash aQ) \cap a^jQ) \times (W \cap a^{j-1}W) \cup ((a^lW \backslash aQ) \cap a^jW) \times (W \cap a^jQ)\right]\right| \\
&= \sum_{j\in\mathbb{Z}} \left(\left|((a^lW \backslash aQ) \cap a^jQ) \times (W \cap a^{j-1}W)\right| + \left|((a^lW \backslash aQ) \cap a^jW) \times (W \cap a^jQ)\right|\right) \\
&= \left|((a^lW \backslash aQ) \cap aQ)\right| |W| + \left|((a^lW \backslash aQ) \cap a^lW)\right| \left|(W \cap a^lQ)\right| = 0,
\end{aligned}$$

which contradicts $|(a^l W\backslash aQ)| > 0$. ⊠

(d). Since $W \times Q$ is an A-multiwavelet set of order Ld, its Lebesgue measure is $(2\pi)^2 Ld$. Also, the Lebesgue measure of W is $2\pi L$. These facts together imply that the Lebesgue measure of Q is $2\pi d$. Since $Q \subset aQ, Q \cap W = \phi$ and $aQ \cap W \neq \phi, (aQ\backslash Q) \cap W \neq \phi$. Further, since $|(aQ\backslash Q)\backslash W| = 0$, we have $W = aQ\backslash Q,\ a.e.$ ⊠

(e). Further, on simplifying the expressions in the right hand side of (3.1), by using (d), we obtain that

$$\begin{aligned}\mathbb{R}^2 &= \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^jQ \times a^{j-1}W) \cup (a^{j-1}W \times a^{j-1}Q)\right], \ \ a.e.\\ &= \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^jQ \times (a^jQ\backslash a^{j-1}Q) \cup (a^jQ\backslash a^{j-1}Q) \times a^{j-1}Q)\right], \ \ a.e.\\ &= \dot{\bigcup}_{j\in\mathbb{Z}} \left[(a^jQ \times a^jQ)\backslash(a^{j-1}Q \times a^{j-1}Q)\right], \ \ a.e.\end{aligned}$$

Equivalently,

$$\begin{aligned}\chi_{\mathbb{R}^2}(\xi,\eta) &= \sum_{j\in\mathbb{Z}} \left[\chi_{(a^jQ\times a^jQ)}(\xi,\eta) - \chi_{(a^{j-1}Q\times a^{j-1}Q)}(\xi,\eta)\right], a.e., (\xi,\eta) \in \mathbb{R}^2\\ 1 &= lim_{j\to\infty}\chi_{(a^jQ\times a^jQ)}(\xi,\eta), \ \ a.e. \ \ (\xi,\eta) \in \mathbb{R}^2\\ &= lim_{j\to\infty}\left(\chi_{a^jQ}(\xi)\ \chi_{a^jQ}(\eta)\right), \ \ a.e., \ \ \xi,\eta \in \mathbb{R}.\end{aligned}$$

This implies that

$$lim_{j\to\infty}\chi_{a^jQ}(\xi) = 1, \quad a.e., \quad \xi \in \mathbb{R}.$$

Further, since $a^jQ = a^j(\cup_{k=1}^{\infty} a^{-k}W) = \cup_{t=-j+1}^{\infty} a^{-t}W,\ a.e.,$

$$\begin{aligned}lim_{j\to\infty}\chi_{a^jQ}(\xi) &= lim_{j\to\infty}\chi_{\cup_{t=-j+1}^{\infty} a^{-t}W}(\xi), \ \ a.e., \ \ \xi \in \mathbb{R}\\ 1 &= lim_{j\to\infty}\sum_{t=-j+1}^{\infty} \chi_{a^{-t}W}(\xi) \ \ a.e., \ \ \xi \in \mathbb{R}\\ &= \sum_{t\in\mathbb{Z}} \chi_{a^{-t}W}(\xi) \ \ a.e., \ \ \xi \in \mathbb{R}.\end{aligned}$$

Thus we obtain that $\dot{\bigcup}_{j\in\mathbb{Z}} a^jW = \mathbb{R},\ a.e.$ ⊠

(f). Since $Q \cap a^{k-1}W = \phi$, where k is any natural number, we have $Q \cap \bigcup_{k=1}^{\infty} a^{k-1}W = \phi$. This implies that $Q \subset \mathbb{R} - \left(\bigcup_{k=1}^{\infty} a^{k-1}W\right) = \bigcup_{k=1}^{\infty} a^{-k}W,\ a.e.$

Further, since the Lebesgue measure of $\bigcup_{k=1}^{\infty} a^{-k}W = \left|\bigcup_{k=1}^{\infty} a^{-k}W\right| = 2\pi d = |Q|$, *a.e.*, it follows that $Q = \bigcup_{k=1}^{\infty} a^{-k}W$, *a.e.* ⊠

Theorem 3.2. *Let W be a measurable set of Lebesgue measure $2\pi L$ in $\mathbb{R}$, and Q be a measurable set in $\mathbb{R}$ such that $Q \subset aQ$. If $W \times Q$ is an A-multiwavelet set of order Ld in $\mathbb{R}^2$, then W is an a-multiwavelet set of order L and Q is the a-multiscaling set of order d associated with W, where $L = (a-1)d$.*

Proof. In view of parts (a), (d), (e), and (f) of Lemma 3.1, to complete the proof, we need to show that

$$\sum_{m\in\mathbb{Z}} \chi_W(\xi + 2m\pi) = L, \qquad a.e., \;\; \xi \in \mathbb{R}, \tag{3.3}$$

and

$$\sum_{n\in\mathbb{Z}} \chi_Q(\xi + 2n\pi) = d, \qquad a.e., \;\; \xi \in \mathbb{R}. \tag{3.4}$$

From Lemma 2.8, there exists a disjoint partition $E_i, i = 1, ..., Ld$ of $W \times Q$, such that

$$\sum_{k\in\mathbb{Z}^2} \chi_{E_i}(\eta + 2k\pi) = 1, \qquad a.e., \;\; \eta \in \mathbb{R}^2.$$

Also, $|E_i| = (2\pi)^2, i = 1, ..., Ld$.

Let p_1 and p_2 be the first and second projection maps from $\mathbb{R}^2 \to \mathbb{R}$ defined by $p_1(x, y) = x$ and $p_2(x, y) = y$, for $(x, y) \in \mathbb{R}^2$. Since E_i is $2\pi\mathbb{Z}^2$-translation congruent to $(-\pi, \pi]^2$, a.e., $p_1(E_i)$ and $p_2(E_i)$ are $2\pi\mathbb{Z}$-translation congruent to $(-\pi, \pi]$, a.e., for $i = 1, ..., Ld$. Clearly, for $i = 1, ..., Ld$, $p_1(E_i)$ and $p_2(E_i)$ are subsets of W and Q respectively.

Since $W = \cup_{i=1}^{Ld} p_1(E_i)$, $\tau(W) = \tau(\cup_{i=1}^{Ld} p_1(E_i)) = (-\pi, \pi]$. Now, using Lemma 2.2 [4] and following the steps of the proof of Theorem 2.6 in [4], we easily obtain L disjoint sets $W_1, W_2, ..., W_L$ of W such that $|W_i| = 2\pi$, and $\sum_{k\in\mathbb{Z}} \chi_{W_i}(\xi + 2k\pi) = 1$, *a.e.*, $\xi \in \mathbb{R}$, $i = 1, ..., L$. An application of Lemma 2.8, yields (3.3).

With the same arguments as above, we obtain disjoint partition $Q_1, Q_2, ..., Q_d$ of Q such that $|Q_j| = 2\pi$, and $\sum_{k\in\mathbb{Z}} \chi_{Q_j}(\xi + 2k\pi) = 1$, *a.e.*, $\xi \in \mathbb{R}$, $j = 1, ..., d$. We obtain (3.4) by aplying Lemma 2.8. ⊠

Theorem 3.3. *Let Q be an a-multiscaling set of order d of an a-multiwavelet set W of order L in $\mathbb{R}$. Then $W \times Q$ is an A-multiwavelet set of order Ld in $\mathbb{R}^2$, where $L = (a-1)d$.*

PROOF. For the proof, we show that $W \times Q$ satisfies:

$$\sum_{j\in Z} \chi_{W\times Q}\left(A^{*j}\xi\right) = 1, \qquad a.e., \ \ \xi \in \mathbb{R}^2, \tag{3.5}$$

$$\sum_{k\in\mathbb{Z}^2} \chi_{W\times Q}(\xi + 2k\pi) = Ld, \qquad a.e., \ \ \xi \in \mathbb{R}^2. \tag{3.6}$$

Let $(\xi_1, \xi_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}
I &\equiv \sum_{j\in Z} \chi_{W\times Q}(A^{*j}\xi) \\
&= \sum_{j\in\mathbb{Z}} \left\{ \chi_{W\times Q}\left(\begin{pmatrix} 0 & a^j \\ a^{j-1} & 0 \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right) + \chi_{W\times Q}\left(\begin{pmatrix} a^j & 0 \\ 0 & a^j \end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\right)\right\} \\
&= \sum_{j\in\mathbb{Z}} \{\chi_{W\times Q}\left(a^j\xi_2, a^{j-1}\xi_1\right) + \chi_{W\times Q}\left(a^j\xi_1, a^j\xi_2\right)\} \\
&= \sum_{j\in\mathbb{Z}} \chi_{W\times Q}\left(a^j\xi_2, a^{j-1}\xi_1\right) + \sum_{j\in\mathbb{Z}} \chi_{W\times Q}\left(a^j\xi_1, a^j\xi_2\right) \\
&= I_1 + I_2 \ \ (say).
\end{aligned}$$

Since Q is the a-multiscaling set of the a-multiwavelet set W, $W \subset aQ$ and $W \cap Q = \phi$. Let $\xi \in \mathbb{R}$. Then, for some $n \in \mathbb{Z}$, $\xi \in a^nW$. Before proceeding further, we observe the following:

(i) $\xi \notin a^mW$, where m is an integer different from n,

(ii) on account of the facts that $W \subset aQ$ and $Q \subset aQ$, $\xi \in a^lQ$, for any integer $l > n$, and

(iii) since $W \cap Q = \phi$, and $\xi \in a^nW$, $a^{-1}Q \subset Q$ implies that for an integer $p \leq n$, $\xi \notin a^pQ$.

Now, since W is an a-multiwavelet set and $(\xi_1, \xi_2) \in \mathbb{R}^2$, $\xi_1 \in a^kW$ and $\xi_2 \in a^lW$, for some $k, l \in \mathbb{Z}$. The following cases settle (3.5).

Case (a). Suppose $k \leq l$. Then from (ii), $\xi_1 \in a^{l+1}Q$. Therefore, $(a^{-l}\xi_2, a^{-l-1}\xi_1) \in W \times Q$. Using (i), we obtain that $I_1 = 1$. Next, from (iii), it follows that $\xi_2 \notin a^kQ$. Using (i) again, we get $I_2 = 0$. Hence, $I = 1$.

Case (b). Suppose $k > l$. Then, from (ii), $\xi_2 \in a^kQ$. Therefore, $(a^{-k}\xi_1, a^{-k}\xi_2) \in W \times Q$. From (i), we obtain that $I_2 = 1$. Using (iii), we have $\xi_1 \notin a^{l+1}Q$ which together with (i), gives $I_1 = 0$. Hence, $I = 1$.

Since W is an a-multiwavelet set of order L, it satisfies (3.3) and for $\xi \in \mathbb{R}$ there exist integers $m_1, m_2, ..., m_d$ such that $\xi + 2m_i\pi \in W$, $i = 1, ..., L$. Further, since Q is an a-multiscaling set of order d, it satisfies (3.4) and for $\xi \in \mathbb{R}$, there exist integers $n_1, n_2, ..., n_d$ such that $\xi + 2n_i\pi \in Q$, $i = 1, ..., d$. Now, for $\xi \in \mathbb{R}^2$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2} \chi_{W\times Q}(\xi + 2k\pi) &= \sum_{(m,n)\in\mathbb{Z}^2} \chi_W(\xi_1 + 2m\pi)\ \chi_Q(\xi_2 + 2n\pi),\ a.e., \xi_1, \xi_2 \in \mathbb{R} \\ &= L \sum_{n\in\mathbb{Z}} \chi_Q(\xi_2 + 2n\pi),\ a.e., \xi_2 \in \mathbb{R} \\ &= Ld.\end{aligned}$$

This completes the proof. ⊠

Combining Theorems 3.2 and 3.3, we have

Theorem 3.4. *Let W be a measurable set of the Lebesgue measure $2\pi L$ in $\mathbb{R}$, and Q be a measurable set in $\mathbb{R}$ such that $Q \subset aQ$. Then $W \times Q$ is an A-multiwavelet set of order Ld in $\mathbb{R}^2$ if and only if W is an a-multiwavelet set of order L and Q is an a-multiscaling set of order d associated with W, where $L = (a-1)d$.*

Theorem 3.5. *Let Q be an a-multiscaling set of order d in $\mathbb{R}$. Then $Q \times Q$ is an A-multiscaling set of order d^2 in $\mathbb{R}^2$.*

PROOF. Since Q is an a-multiscaling set of order d, $|Q| = 2\pi d$ and $W \equiv aQ\backslash Q$ is an a-multiwavelet set of order $(|a|-1)d$, say, L. Therefore, $|Q \times Q| = |Q| \cdot |Q| = 4\pi^2 d^2$. That

$$A^*(Q \times Q)\backslash(Q \times Q) = (aQ \times Q)\backslash(Q \times Q) = (aQ\backslash Q) \times Q = W \times Q,$$

is an A-multiwavelet set of order $(|a|-1)d^2 = Ld$, follows from Theorem 3.3.

Furthermore, since Q is an a-multiscaling set of order d, it satisfies (3.4). Thus, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, there exist integers $m_1, m_2, ..., m_d$, and $l_1, l_2, ..., l_d$ such that $\xi_1 + 2m_i\pi \in Q_i$, and $\xi_2 + 2l_i\pi \in Q_i$, $i = 1, ..., d$. Now, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^2} \chi_{Q\times Q}(\xi + 2k\pi) &= \sum_{(k_1,\,k_2)\in\mathbb{Z}^2} \chi_{Q\times Q}(\xi_1 + 2k_1\pi, \xi_2 + 2k_2\pi) \\ &= \sum_{(k_1,\,k_2)\in\mathbb{Z}^2} \chi_Q(\xi_1 + 2k_1\pi)\ \chi_Q(\xi_2 + 2k_2\pi) = d^2.\end{aligned}$$

This completes the proof. ☒

Corollary 3.6. *If Q is an a-multiscaling set of order d in $\mathbb{R}$ associated with the a-multiwavelet set W of order L, then $Q \times Q$ is an A-multiscaling set of order d^2 associated with the A-multiwavelet set $W \times Q$ of order Ld in $\mathbb{R}^2$.*

Remark 3.7. Since a wavelet set W has a scaling set if and only if W is an MRA wavelet set, the product of a non-MRA wavelet set with any measurable set of $\mathbb{R}$ cannot provide an A-wavelet set of $\mathbb{R}^2$.

Below we provide some examples to illustrate certain A-wavelet sets of $\mathbb{R}^2$ obtained as the product of an MRA dyadic wavelet set with its scaling set, where A denotes the matrix $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$.

Example 3.8. For $a \in (0, 2\pi)$, $W_a = [2a - 4\pi, a - 2\pi) \cup [a, 2a)$ is known to be a 2-dilation MRA wavelet set [14]. Since its scaling set Q_a is $[a - 2\pi, a)$, by Theorem 3.4, it follows that $W_a \times Q_a$ is an A-wavelet set.

Example 3.9. Wavelet sets possessing three intervals have been characterized by Ha, Kang, Lee and Seo in [14]. These are precisely,

$$W(j,p) \equiv I_{j,p} \cup J_{j,p} \cup K_{j,p},$$

where

$$I_{j,p} \equiv \left[-2\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi \, , \, -\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\pi \right],$$

$$J_{j,p} \equiv \left[\frac{2(p+1)\pi}{2^{j+1}-1} \, , \, \frac{2(2p+1)\pi}{2^{j+1}-1} \right], \;\; K_{j,p} \equiv \left[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1} \, , \, \frac{2^{j+2}(p+1)\pi}{2^{j+1}-1} \right],$$

and j, p are natural numbers such that $j \geq 2$ and $1 \leq p \leq 2^j - 2$.

For $j \geq 2$, and an odd $p \in \mathbb{N}$, $W(j,p)$ is a non-MRA wavelet set [14; Theorem 4.7] while for $p = 2^j - 2$, $W(j,p)$ is an MRA wavelet set [19]. The scaling set of

$$W(j, 2^j - 2) = \left[\frac{-4\pi}{2^{j+1}-1}, \frac{-2\pi}{2^{j+1}-1} \right] \cup \left[\frac{(2^{j+1}-2)\pi}{2^{j+1}-1}, \frac{(2^{j+2}-6)\pi}{2^{j+1}-1} \right] \cup \left[\frac{2^{j+1}(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^{j+2}(2^j-1)\pi}{2^{j+1}-1} \right]$$

is given by

$$
\begin{aligned}
Q_j &= \bigcup_{k=1}^{\infty} 2^{-k} \;\; W(j, 2^j - 2) \\
&= \left[\frac{-2\pi}{2^{j+1}-1}, \frac{(2^{j+1}-2)\pi}{2^{j+1}-1}\right] \cup \left(\bigcup_{r=1}^{j} \left[\frac{2^r(2^{j+1}-3)\pi}{2^{j+1}-1}, \frac{2^{r+1}(2^j-1)\pi}{2^{j+1}-1}\right]\right).
\end{aligned}
$$

Thus from Theorem 3.4, $W(j, 2^j - 2) \times Q_j$ is an MRA A-wavelet set of $\mathbb{R}^2$, for $j \geq 2$. However, when p is odd, $W(j,p)$ does not provide an A-wavelet set of $\mathbb{R}^2$ as its product with any measurable set of $\mathbb{R}$.

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