Real Analysis Exchange
Vol. 36(2), 2010/2011, pp. 353–360

E. Liflyand, Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel. email: liflyand@math.biu.ac.il

ON ABSOLUTE CONVERGENCE OF FOURIER INTEGRALS

Abstract

New sufficient conditions for representation of a function as an absolutely convergent Fourier integral are obtained in the paper.

1 Introduction

If

$$f(x)=\int\limits_{\mathbb{R}} g(t)e^{ixt}dt, \qquad g\in L_1(\mathbb{R}),$$

we write $f\in A(\mathbb{R})$, with $||f||_A=||g||_{L_1(\mathbb{R})}$. The possibility to represent a function as an absolutely convergent Fourier integral was studied by many mathematicians and is of importance in various problems of analysis. For example, in [2] (see also [8, Ch.4, 7.4]) this problem was studied, in connection with multipliers, for the following function:

$$m(x)=\theta(x)\frac{e^{i|x|^\alpha}}{|x|^\beta}, \tag{1}$$

where θ is a C^∞ function on $\mathbb{R}$, which vanishes near zero, and equals 1 outside a bounded set, and $\alpha,\beta>0$. In is known that

I) If $\frac{\beta}{\alpha}>\frac12$, then $m\in A(\mathbb{R})$;

II) If $\frac{\beta}{\alpha}<\frac12$ and $\alpha\ne 1$, then $m\notin A(\mathbb{R})$.

Various sufficient conditions for absolute convergence of Fourier integrals were obtained by Titchmarsh, Beurling, Karleman, Sz.-Nagy, and many others. One can find more or less comprehensive and very useful survey on this

Mathematical Reviews subject classification: Primary: 42A38; Secondary: 42A25
Key words: Fourier integral, absolute convergence, Hardy inequality
Received by the editors June 2, 2010
Communicated by: Alexander Olevskii

problem in [7]. It is worth mentioning that the problem descends from the study of absolute convergence of Fourier series, see, e.g., [3].

New sufficient conditions of belonging to $A(\mathbb{R})$, are obtained in this paper. Multidimensional generalizations will appear elsewhere.

Let us unite certain of known results closely related to our study in the following theorem. First, it is natural to consider functions $f \in A(\mathbb{R})$ that satisfy the condition

(N) *Let* $f \in C_0(\mathbb{R})$, *that is,* $f \in C(\mathbb{R})$ *and* $\lim f(t) = 0$ *as* $|t| \to \infty$, *and let* f *be locally absolutely continuous on* $\mathbb{R}$.

Theorem A. *Let* f *satisfy condition* **(N)**, $f \in L^p(\mathbb{R})$ *with* $1 \le p \le 2$, *and* $f' \in L^q(\mathbb{R})$ *with* $1 < q \le 2$. *Then* $f \in A(\mathbb{R})$.

After fixing certain notation and conventions we formulate the results. The main one extends the range of p and q in Theorem A. In the next section we give the proofs.

We shall denote absolute constants by c or maybe by c with various subscripts, like c_1, c_2, etc., while $\gamma(...)$ will denote positive quantities depending only on the arguments indicated in the parentheses.

Our main result reads as follows.

Theorem 1.1. *Suppose a function* f *satisfies the condition* **(N)**.

a) *Let* $f(t) \in L_p(\mathbb{R})$, $1 \le p < \infty$, *and* $f'(t) \in L_q(\mathbb{R})$, $1 < q < \infty$. *If* $\frac{1}{p} + \frac{1}{q} > 1$, *then* $f \in A(\mathbb{R})$.

b) *If* $\frac{1}{p} + \frac{1}{q} < 1$, *then there exists a function* f *satisfying* **(N)** *such that* $f(t) \in L_p(\mathbb{R})$ *and* $f'(t) \in L_q(\mathbb{R})$ *but* $f \notin A(\mathbb{R})$.

As a corollary, we present a different proof of **I)** for the main range of the parameters α and β.

Corollary 1.2. *If* $\frac{\beta}{\alpha} > \frac{1}{2}$ *with* $\beta > 0$ *and* $\alpha < 2$, *then* $m \in A(\mathbb{R})$.

2 Proofs

We give, step by step, proofs of the results formulated in the introduction. We need certain auxiliary results.

2.1 Auxiliary results.

One of the basic tools is the following lemma (see Lemma 4 in [9] or Theorem 3 in [1], in any dimension).

Lemma B. *Let* $f \in C_0(\mathbb{R})$*. If*

$$\sum_{\nu=-\infty}^{\infty} 2^{\nu/2} \left(\int_{\mathbb{R}} |f(t+h(\nu)) - f(t-h(\nu))|^2 dt \right)^{1/2} < \infty,$$

where $h(\nu) = \pi 2^{-\nu}$, $\nu \in \mathbb{Z}$, *then* $f \in A(\mathbb{R})$*.*

This lemma is a natural extension of the celebrated Bernstein's test for the absolute convergence of Fourier series (see [3, Ch.II, §6]).

We also need the following Hardy type inequality (see [4, Cor.3.14]):

For $F \geq 0$ and $1 < q \leq Q < \infty$

$$\left(\int_{\mathbb{R}} \left[\int_{t-h}^{t+h} F(s)\, ds \right]^Q dt \right)^{1/Q} \leq c h^{1/Q+1/q'} \left(\int_{\mathbb{R}} F^q(t)\, dt \right)^{1/q}. \qquad (2)$$

Here $\frac{1}{q} + \frac{1}{q'} = 1$. Similarly $\frac{1}{p} + \frac{1}{p'} = 1$.

2.2 Proof of Theorem 1.1.

The proof will be divided into several steps.

Step 1. To prove **b)** of the theorem, let us consider the function m from the introduction. Suppose that $p\beta > 1$ and $q(\beta - \alpha + 1) > 1$, with $\alpha \neq 1$. Simple calculations show that $m \in L_p(\mathbb{R})$ and $m' \in L_q(\mathbb{R})$. If $\frac{\beta}{\alpha} < \frac{1}{2}$, then $m \notin A(\mathbb{R})$. The last inequality is equivalent to $2\beta - \alpha + 1 < 1$. Therefore,

$$\frac{1}{p} + \frac{1}{q} < 2\beta - \alpha + 1 < 1,$$

and the considered m delivers the required counterexample.

Step 2. Let $p, q \leq 2$, the known case from Theorem A. In order to make the presentation self-contained and show the difference in the methods, let us give a simple proof. In this case the Fourier transforms of f:

$$\widehat{f}(x) = \int_{\mathbb{R}} f(t) e^{-ixt}\, dt \qquad (3)$$

can be understood in the sense of mean convergence of

$$\widehat{f}_M(x) = \int_{-M}^{M} f(t) e^{-ixt}\, dt, \qquad (4)$$

and hence for almost every x there is a convergent subsequence $\widehat{f}_{M_k}(x)$ of (4). The same is true for the Fourier transform $\widehat{f'}$ of the derivative f'.

We will prove that $\widehat{f} \in L_1(\mathbb{R})$. Indeed, let us estimate

$$\int_{\mathbb{R}} |\widehat{f}(x)|\, dx.$$

We split this integral into the two, with $|x| \le 1$ and otherwise. Applying Hölder's inequality and then the Hausdorff-Young theorem, we get

$$\int_{|x|\le 1} |\widehat{f}(x)|\, dx \le 2^{1/p} \Big(\int_{\mathbb{R}} |\widehat{f}(x)|^{p'}\, dx \Big)^{1/p'} \le \gamma(p) \|f\|_p.$$

Further, let $|x| > 1$. Integrating by parts in the integral (3), we derive that for almost every x

$$|\widehat{f}(x)| \le \frac{|\widehat{f'}(x)|}{|x|},$$

the Fourier transform on the right exists when $f' \in L_q$, $q \le 2$.

As above, applying Hólder's inequality and then the Hausdorff-Young theorem, we obtain

$$\begin{aligned} \int_{|x|>1} |\widehat{f}(x)|\, dx &\le \int_{|x|>1} \frac{|\widehat{f'}(x)|}{|x|}\, dx \\ &\le \left(\frac{2}{q-1} \right)^{1/q} \Big(\int_{\mathbb{R}} |\widehat{f'}(x)|^{q'}\, dx \Big)^{1/q'} \le \gamma(q) \|f'\|_q. \end{aligned}$$

Contrary to *Step 2*, for $p > 2$ or $q > 2$ we are not able to make use of the Hausdorff-Young inequality. Thus, for more difficult cases of **a)**, we will apply the tools given in Subsection 2.1. Denoting

$$\Delta(h) = \left(\int_{\mathbb{R}} |f(t+h) - f(t-h)|^2 dt \right)^{1/2}, \tag{5}$$

we are going to prove that

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2} \Delta(h(-\nu)) + \sum_{\nu=0}^{\infty} 2^{\nu/2} \Delta(h(\nu)) < \infty. \tag{6}$$

It is obvious that for $h > 0$

$$|f(t+h) - f(t-h)| = |\int_{t-h}^{t+h} f'(s)\,ds|. \tag{7}$$

Step 3. Let start with the first sum in (6) which is

$$\sum_{\nu=1}^{\infty} 2^{-\nu/2}\Big(\int_{\mathbb{R}} |f(t+h(-\nu)) - f(t-h(-\nu)|^2 dt\Big)^{1/2}. \tag{8}$$

Using (7), we represent the integral as

$$\Big(\int_{\mathbb{R}} |f(t+h(-\nu)) - f(t-h(-\nu)| \left|\int_{t-h(-\nu)}^{t+h(-\nu)} f'(s)\,ds\right| dt\Big)^{1/2}.$$

By Hölder's inequality, it is estimated via

$$\Big(\int_{\mathbb{R}} |f(t+h(-\nu)) - f(t-h(-\nu)|^p dt\Big)^{\frac{1}{2p}} \Big(\left[\int_{t-h(-\nu)}^{t+h(-\nu)} |f'(s)|\,ds\right]^{p'} dt\Big)^{\frac{1}{2p'}}.$$

Since $p' > q$, we use (2) with $F(s) = |f'(s)|$ and $Q = p'$. Therefore, the first sum in (6) is controlled by

$$\|f\|_p^{1/2}\, \|f'\|_q^{1/2} \sum_{\nu=1}^{\infty} 2^{-\frac{\nu}{2}(1-\frac{1}{p'}-\frac{1}{q'})},$$

and is bounded since

$$1 - \frac{1}{p'} - \frac{1}{q'} = \frac{1}{p} + \frac{1}{q} - 1 > 0.$$

Step 4. In order to estimate the second sum in (6), equal to

$$\sum_{\nu=0}^{\infty} 2^{\nu/2}\Big(\int_{\mathbb{R}} |f(t+h(\nu)) - f(t-h(\nu)|^2 dt\Big)^{1/2}, \tag{9}$$

we observe that the integral can be represented as (see (7))

$$\Big(\int_{\mathbb{R}} |f(t+h(\nu)) - f(t-h(\nu)|^{1-\delta} \left|\int_{t-h(\nu)}^{t+h(\nu)} f'(s)\,ds\right|^{1+\delta} dt\Big)^{1/2} \tag{10}$$

for any $0<\delta<1$. We will specify δ later on. Applying Hölder's inequality with the exponents $\frac{p}{1-\delta}>1$ and $\frac{p}{p-1+\delta}$, we estimate (10) via

$$\left(\int_{\mathbb{R}} |f(t+h(\nu))-f(t-h(\nu)|^p dt\right)^{\frac{1-\delta}{2p}}$$

$$\times \quad \left(\int_{\mathbb{R}} \left[\int_{t-h(\nu)}^{t+h(\nu)} |f'(s)|\,ds\right]^{\frac{p(1+\delta)}{p-1+\delta}} dt\right)^{\frac{p-1+\delta}{2p}}. \tag{11}$$

The first integral in (11) is controlled by $\|f\|_p^{\frac{1-\delta}{2}}$. To estimate the second one, we use (2) with $F(s)=|f'(s)|$ and $Q=\frac{p(1+\delta)}{p-1+\delta}$. With these in hand, we estimate the second factor in (11) via

$$c_1[h(\nu)]^{(\frac{1}{Q}+\frac{1}{q'})\frac{1+\delta}{2}}\|f'\|_q^{\frac{1+\delta}{2}}.$$

To ensure the finiteness of (9), one must have

$$(\frac{1}{Q}+\frac{1}{q'})\,(1+\delta)>1.$$

This is equivalent to the inequality

$$\frac{p}{q'}>\frac{1-\delta}{1+\delta}. \tag{12}$$

Since $\frac{p}{q'}<1$, an appropriate choice of δ completes the proof.

2.3 Proof of Corollary 1.2.

We have $m\in L_p(\mathbb{R})$ for *any* p such that $p\beta>1$, and $m'\in L_q(\mathbb{R})$ for *any* q such that $q(\beta-\alpha+1)>1$ and

$$\beta-\alpha+1>0. \tag{13}$$

If $\beta<1$, then it follows from $\alpha<2\beta$ that

$$\beta-\alpha+1>\beta-2\beta+1>0.$$

If $\beta\ge 1$, then $\alpha<2$ ensures (13). Thus, there holds

$$2\beta-\alpha+1>\frac{1}{p}+\frac{1}{q}. \tag{14}$$

Further, $\frac{\beta}{\alpha} > \frac{1}{2}$ yields $2\beta - \alpha + 1 > 1$. Now, we can choose p and q such that the right-hand side of (14) be so close to the left-hand side of (14) that also $\frac{1}{p} + \frac{1}{q} > 1$. By Theorem 1.1, this gives $m \in A(\mathbb{R})$.

Acknowledgment. This paper is a part of the project initiated by R. M. Trigub (see [5], [6]). The problem solved in this paper was also posed by him. The author is indebted to his teacher Prof. R. M. Trigub for many years of mathematical collaboration and personal friendship.

References

[1] O. V. Besov, *Hörmander's theorem on Fourier multipliers*, Trudy Mat. Inst. Steklov **173** (1986), 164–180 (Russian). - English transl.: Proc. Steklov Inst. Math. **4** (1987), 4–14.

[2] I. I. Hirschman, Jr., *On multiplier transformations, I*, Duke Math. J. **26** (1959), 221–242; *II*, ibid **28** (1961), 45–56.

[3] J.-P. Kahane, *Séries de Fourier absolument convergentes*, Springer, Berlin, 1970.

[4] A. Kufner, L. E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, 2003.

[5] E. Liflyand and R. Trigub, *Conditions for the absolute convergence of Fourier integrals*, J. Approx. Theory, **163** (2011), 438–459.

[6] E. Liflyand and R. Trigub, *On the Representation of a Function as an Absolutely Convergent Fourier Integral*, Trudy Mat. Inst. Steklov, **269** (2010), 153–166 (Russian). - English transl.: Proc. Steklov Inst. Math., **269** (2010), 146–159.

[7] S. G. Samko and G. S. Kostetskaya, *Absolute integrability of Fourier integrals*, Vestnik RUDN (Russian Peoples Friendship Univ.), Math., **1** (1994), 138–168.

[8] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, N.J., 1970.

[9] R. M. Trigub, *Absolute convergence of Fourier integrals, summability of Fourier series, and polynomial approximation of functions on the torus*, Izv. Akad. Nauk SSSR, Ser.Mat., **44** (1980), 1378–1408 (Russian). - English transl.: Math. USSR Izv., **17** (1981), 567–593.

[10] R. M. Trigub and E. S. Belinsky, *Fourier Analysis and Appoximation of Functions*, Kluwer, 2004.