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ON THE LATTICE GENERATED BY HAMEL FUNCTIONS

Abstract

We say that $f : \mathbb{R} \to \mathbb{R}$ is LIF if it is linearly independent over $\mathbb{Q}$ as a subset of $\mathbb{R}^2$ and that it is a Hamel function (HF) if it is a Hamel basis of $\mathbb{R}^2$. In this paper we present a discussion on the lattices generated by the classes HF and LIF. We also investigate extensions of partial LIF functions to HF and LIF functions defined on whole $\mathbb{R}$.

1 Introduction.

Let us establish some of the terminology to be used. Symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the set of real and rational numbers, respectively. Ordinal numbers will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. The symbol $|X|$ denotes the cardinality of a set X. In particular, the symbol $\mathfrak{c}$ stands for $|\mathbb{R}|$ and $|\mathbb{Q}| = \omega$. For a set X and a cardinal κ, $[X]^\kappa$ is the family of all subsets of X with cardinality κ. Similarly we define the family $[X]^{<\kappa}$. No distinction is made between a function and its graph. For any function $f : X \to Y$ symbols $\operatorname{rng}(f)$ and $\operatorname{dom}(f)$ denote the range and the domain of f respectively. The symbol $f_{|A}$ denotes the restriction of f to A. Suppose V is linear space over some field K and $W \subset V$ is a linear subspace. Then the symbol $\operatorname{codim_V}(\mathrm{W}) = \dim(V/W)$ will stand for the codimension of the subspace W.

We will consider $\mathbb{R}$ ($\mathbb{R}^2$) as a linear space over the field $\mathbb{Q}$. For $A \subset \mathbb{R}^k$ ($k = 1, 2, \ldots$), $\mathrm{LIN}_{\mathbb{Q}}(A)$ denotes the linear subspace of $\mathbb{R}^k$ over $\mathbb{Q}$ generated by A. Any basis of $\mathbb{R}^k$ over $\mathbb{Q}$ will be referred to as a Hamel basis. A function $f : \mathbb{R} \to \mathbb{R}$ is

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- *additive* ($f \in \mathrm{Add}$) if $f(x) + f(y) = f(x + y)$ for all $x, y \in \mathbb{R}$;
- *linearly independent* ($f \in \mathrm{LIF}$) if f is a linearly independent subset of $\mathbb{R}^2$;
- *Hamel function* ($f \in \mathrm{HF}$) if f is a Hamel basis of $\mathbb{R}^2$.

Let $X \subset \mathbb{R}$. We will say that $f : X \to \mathbb{R}$ is PHF if it is a Hamel basis of $\mathbb{R}^2$ and that it is PLIF if it is a linearly independent subset of $\mathbb{R}^2$. For $f : X \to \mathbb{R}$ and $x \in \mathbb{R}$ let

$$\mathrm{LC}(f, x) = \left\{ \sum_{i=0}^{k} p_i f(x_i) : k < \omega, p_i \in \mathbb{Q}, x_i \in X, \sum_{i=0}^{k} p_i x_i = x \right\}.$$

When $x = 0$ we will write $\mathrm{LC}(f)$.

A family $\mathcal{F}$ of real functions $f : X \to \mathbb{R}$ is a lattice iff $\min(f, g) \in \mathcal{F}$ and $\max(f, g) \in \mathcal{F}$ for $f, g \in \mathcal{F}$. If $\mathcal{F}$ is a family of real functions, then the symbol $\mathcal{L}(\mathcal{F})$ stands for the lattice generated by $\mathcal{F}$, i.e. the smallest lattice of functions containing $\mathcal{F}$. Evidently, we have $\mathcal{L}(\mathcal{A}) \subset \mathcal{L}(\mathcal{B})$ if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{L}(\mathcal{L}(\mathcal{A})) = \mathcal{L}(\mathcal{A})$.

Let $\mathcal{F}$ be a family of real functions. Then the symbols $\mathrm{Max}(\mathcal{F}) = \{\max(g, h) : g, h \in \mathcal{F}\}$ and $\mathrm{Min}(\mathcal{F}) = \{\min(g, h) : g, h \in \mathcal{F}\}$ will stand for the maxima and minima sets for family $\mathcal{F}$, respectively. Obviously if $\mathcal{A} \subset \mathcal{B}$ are families of real functions, then $\mathrm{Max}(\mathcal{A}) \subset \mathrm{Max}(\mathcal{B})$ and $\mathrm{Min}(\mathcal{A}) \subset \mathrm{Min}(\mathcal{B})$.

The class of Hamel function was first introduced and researched by Płotka in papers [7, 9, 8, 10]. The aim of this paper is to answer some questions concerning lattices generated by HF and LIF functions. Finding lattice generated by a family of real functions is a typical problem in real analysis (see e.g. [6]). Our main result in this topic consist of showing that $\mathcal{L}(\mathrm{HF}) = \mathcal{L}(\mathrm{LIF})$. Another important problem is extendability of partial functions (see e.g. [3]). We prove in Theorem 8 a sufficient condition for a PLIF function to be extendable to a HF function. Next we use Theorem 8 to prove that $\mathcal{L}(\mathrm{LIF}) = \mathcal{L}(\mathrm{HF})$.

2 The lattices of Hamel and linearly independent functions.

Lemma 1 ([5, Fact 2.3])**.** *Suppose $f \in \mathrm{HF}$, then $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = f(x) + q_x f(0)$ where $q_x \in \mathbb{Q} \setminus \{-1\}$ for $x \in \mathbb{R}$ is a Hamel function.*

Lemma 2 ([8, Fact 6])**.** *Suppose $X \in [\mathbb{R}]^{<\mathfrak{c}}$, $f : X \to \mathbb{R}$, $f \in \mathrm{LIF}$. Then f can be extended to* HF *function $\tilde{f} : \mathbb{R} \to \mathbb{R}$.*

Definition 1. *For $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ the class $\mathcal{M}_{\max}(\mathcal{F}) = \{f \in \mathbb{R}^{\mathbb{R}} : \max\{f,g\} \in \mathcal{F}$ for every $g \in \mathcal{F}\}$ is called the maximal upper family for $\mathcal{F}$ and $\mathcal{M}_{\min}(\mathcal{F}) = \{f \in \mathbb{R}^{\mathbb{R}} : \min\{g,f\} \in \mathcal{F}$ for every $g \in \mathcal{F}\}$ is called the maximal lower family for $\mathcal{F}$.*

Fact 1. $\mathcal{M}_{\max}(\mathrm{HF}) = \mathcal{M}_{\min}(\mathrm{HF}) = \mathcal{M}_{\max}(\mathrm{LIF}) = \mathcal{M}_{\min}(\mathrm{LIF}) = \emptyset$

Proof. We will show only the case $\mathcal{M}_{\max}(\mathrm{HF}) = \emptyset$, the other equalities can be proven in a similar fashion. Let $g : \mathbb{R} \to \mathbb{R}$ be a function, we will show that there exists $f : \mathbb{R} \to \mathbb{R}$ such that $f \in \mathrm{HF}$ and $h = \max\{f,g\} \notin \mathrm{HF}$. Fix linearly independent vectors $x, y \in \mathbb{R}$. First suppose that $g(x)+g(y) = g(x+y)$. Fix a Hamel function f such that $f(x) < g(x)$, $f(y) < g(y)$ and $f(x+y) < g(x+y)$. Such a function exists in virtue of Lemma 1. Then $\max\{f,g\} \notin \mathrm{LIF}$. Now let us assume that $g(x)+g(y) \neq g(x+y)$. Hence we get two cases. First suppose that $g(x)+g(y) > g(x+y)$. Fix $z_1 \in (-\infty, g(x))$, $z_2 \in (-\infty, g(y))$ and put $f' = \{\langle x, z_1\rangle, \langle y, z_2\rangle, \langle x+y, g(x)+g(y)\rangle\}$. We will show that $f' \in \mathrm{PLIF}$. Indeed suppose that $q_0 \langle x, z_1\rangle + q_1 \langle y, z_2\rangle + q_2 \langle x+y, g(x)+g(y)\rangle = 0$ for some $q_0, q_1, q_2 \in \mathbb{Q}$. From $q_0 x + q_1 y + q_2(x+y) = 0$ we get that $q_0 = q_1 = -q_2$. Hence $q_0(z_1+z_2-g(x)-g(y)) = 0$. Hence $q_0 = 0$ or $z_1+z_2-g(x)-g(y) = 0$. Notice that $z_1+z_2-g(x)-g(y) < g(x)+g(y)-g(x)-g(y) = 0$, consequently $q_0 = q_1 = q_2 = 0$. Let $f \in \mathrm{HF}$ be an extension of function f'. Let $h = \max\{f,g\}$, then $h(x)+h(y)-h(x+y) = g(x)+g(y)-g(x)-g(y) = 0$, so $h \notin \mathrm{HF}$.

Now assume that $g(x)+g(y) < g(x+y)$, then $z = g(x+y)-g(y) > g(x)$. Choose $z_1 \in (-\infty, g(y))$ and $z_2 \in (-\infty, g(x+y))$ such that $z+z_1-z_2 \neq 0$ and put $f' = \{\langle x, z\rangle, \langle y, z_1\rangle, \langle x+y, z_2\rangle\}$. We will show that $f' \in \mathrm{PLIF}$. Suppose $q_0 \langle x, z\rangle + q_1 \langle y, z_1\rangle + q_2 \langle x+y, z_2\rangle = 0$ for some $q_0, q_1, q_2 \in \mathbb{Q}$. Then $q_0 = q_1 = -q_2$, so $q_0(z+z_1-z_2) = 0$. Hence $q_0 = q_1 = q_2 = 0$. Let $f \in \mathrm{HF}$ be an extension of function f' and $h = \max\{f,g\}$. Then $h(x)+h(y)-h(x+y) = z+g(y)-g(x+y) = 0$, so $h \notin \mathrm{HF}$. □

Definition 2. *We will say that a function $f : \mathbb{R} \to \mathbb{R}$ is n-Hamel function if there exist sets $A_0, \dots, A_{n-1} \subset \mathbb{R}$ and functions $f_0, \dots, f_{n-1} \in \mathrm{HF}$ such that $\bigcup_{i=0}^{n-1} A_i = \mathbb{R}$, $A_m \cap A_k = \emptyset$ for $m, k < n$, and $f_{i|A_i} = f_{|A_i}$ for $i < n$.*

Fact 2. $\mathrm{Max}(\mathrm{HF}) = \mathrm{Min}(\mathrm{HF})$.

Proof. First we will show that if $f \in \mathrm{Max}(\mathrm{HF})$ ($f \in \mathrm{Min}(\mathrm{HF})$) then $-f \in \mathrm{Max}(\mathrm{HF})$ ($-f \in \mathrm{Min}(\mathrm{HF})$). Pick $f \in \mathrm{Max}(\mathrm{HF})$, then there exist functions $g, h \in \mathrm{HF}$ such that $f = \max\{g,h\}$. Define $A_0 = \{x \in \mathbb{R} : g(x) = f(x)\}$ and $A_1 = \{x \in \mathbb{R} \setminus A_0 : h(x) = f(x)\}$. Put $\tilde{g}(x) = \begin{cases} -g(x) & \text{for } x \in A_0 \\ -g(x) - q_x g(0) & \text{for } x \in A_1 \end{cases}$

and $\tilde{h}(x) = \begin{cases} -h(x) & \text{for } x \in A_1 \\ -h(x) - p_x h(0) & \text{for } x \in A_0 \end{cases}$, where $q_x, p_x \in \mathbb{Q}$, $-g(x) - q_x g(0) < -h(x)$ for $x \in A_1$, $q_x \neq -1$ and $-h(x) - p_x h(0) < -g(x)$ for $x \in A_0$, $p_x \neq -1$. Such p_x, q_x exist because $g(0) \neq 0 \neq h(0)$. Since $g, h \in$ HF, so $-g, -h \in$ HF and consequently in virtue of Lemma 1 we get that $\tilde{g}, \tilde{h} \in$ HF. Then $-f = \max\{\tilde{g}, \tilde{h}\} \in$ Max(HF). The case $f \in$ Min(HF) can be proved analogously. To finish the proof notice that $f \in$ Max(HF) iff $-f \in$ Min(HF), hence Max(HF) = Min(HF). □

Lemma 3. *If f is n-Hamel function then f is a maximum of n Hamel functions.*

PROOF. Fix an n-Hamel function f. Hence there exists a partition $A_0, \ldots, A_{n-1}$ of $\mathbb{R}$ and Hamel functions $f_0, \ldots, f_{n-1}$ such that $f_{i|A_i} = f_{|A_i}$. Define $\tilde{f}_i(x) = \begin{cases} f_i(x) & \text{for } x \in A_i \\ f_i(x) + q_x^i f_i(0) & \text{for } x \notin A_i \end{cases}$, where $q_x^i \in \mathbb{Q} \setminus \{-1\}$ is such that $f_i(x) + q_x^i f_i(0) < f_j(x)$ for $x \in A_j$, $i \neq j$. Such q_x^i exists since $f_i(0) \neq 0$. By Lemma 1, $\tilde{f}_i \in$ HF. Fix $i < n$ and $x \in A_i$. Then $\tilde{f}_j(x) = f_j(x) + q_x^j f_j(0) < f_i(x) = \tilde{f}_i(x)$ for $j \neq i$, so $f(x) = \tilde{f}_i(x)$, and therefore $f = \max\{\tilde{f}_0, \ldots, \tilde{f}_{n-1}\}$. □

Theorem 1. *$f \in$ Max(HF) iff f is 2-Hamel function.*

PROOF. $\Rightarrow$ Fix $f \in$ Max(HF). Then there exist $g, h \in$ HF such that $f = \max\{g, h\}$. Put $A_0 = \{x \in \mathbb{R} : g(x) < h(x)\}$ and $A_1 = \{x \in \mathbb{R} : h(x) \leq g(x)\}$. Then $A_0 \cap A_1 = \emptyset$ and $A_0 \cup A_1 = \mathbb{R}$ and $f_{|A_0} = h_{|A_0}$, $f_{|A_1} = g_{|A_1}$. Hence f is 2-Hamel function.

$\Leftarrow$ This follows from Lemma 3 for $n = 2$. □

Lemma 4. *The set $L = \{f : \mathbb{R}^{\mathbb{R}} : f$ is n-Hamel function for some $n \in \mathbb{N}\}$ is a lattice.*

PROOF. Fix $g, h \in L$ and put $B_0 = \{x \in \mathbb{R} : h(x) \leq g(x)\}$, $B_1 = \{x \in \mathbb{R} : g(x) < h(x)\}$ and $f = \max\{g, h\}$. Since $g, h \in L$, then there exist numbers $m, n \in \mathbb{N}$, sets $A_{0,0}, \ldots A_{0,m}, A_{1,0}, \ldots, A_{1,n} \subset \mathbb{R}$ and functions $g_0, \ldots, g_m, h_0, \ldots, h_n \in$ HF such that $A_{0,0}, \ldots, A_{0,m}$ and $A_{1,0}, \ldots, A_{1,n}$ are pairwise disjoint and $g_{i|A_{0,i}} = g_{|A_{0,i}}$, $h_{j|A_{1,j}} = h_{|A_{1,j}}$. Define $A_i^0 = B_0 \cap A_{0,i}$ for $i \leq m$ and $A_i^1 = B_1 \cap A_{1,i}$ for $i \leq n$. Then the family $\{A_i^0, A_j^1 : i \leq m, j \leq n\}$ and functions $g_0, \ldots, g_m, h_0, \ldots, h_n$ witness that f is $(n+m)$-Hamel function, so $f \in L$. Similarly we show that $\min\{g, h\} \in L$. □

Theorem 2. $\mathcal{L}(\text{HF}) = \{f \in \mathbb{R}^{\mathbb{R}} : f$ *is n-Hamel function for some* $n \in \mathbb{N}\}$.

Proof. In virtue of Lemma 4 we get the inclusion $\subset$. To prove the inclusion $\supset$ use Lemma 3. □

Notice that similarly as in the above discussion we could show that a real function $f \in \mathrm{Max}(\mathrm{LIF})$ iff there exists a decomposition of $\mathbb{R}$ into sets A, B such that $f_{|A}$ and $f_{|B}$ are extendable to linearly independent functions. Moreover $f \in \mathcal{L}(\mathrm{LIF})$ iff there exists $n \in \mathbb{N}$ and a decomposition of $\mathbb{R}$ into n sets $A_0, \ldots, A_{n-1}$ such that $f_{|A_0}, \ldots, f_{|A_{n-1}}$ are extendable to linearly independent functions.

In the next theorem we will use the following notation. For $f \in \mathcal{L}(\mathrm{HF})$ let $n(f)$ be the minimal number such that f is $n(f)$-Hamel function, $L_1 = \mathrm{HF}$ and, generally, $L_n = \{f : n(f) \le n\}$ for $n \in \mathbb{N}$.

Theorem 3. *$L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_n \subsetneq \ldots$ and $\bigcup L_n = \mathcal{L}(\mathrm{HF})$.*

Proof. Fix $n \in \mathbb{N}$, $g \in \mathrm{HF}$, $q_0 = 1$ and $x_0 \in \mathbb{R} \setminus \{0\}$. Put $x_i = q_i x_0$ where $q_i \in \mathbb{Q} \setminus \{0, 1\}$ are pairwise different, $1 \le i \le n$. Define $h : \mathbb{R} \setminus \{x_1, \ldots, x_n\} \to \mathbb{R}$ as $h = g_{|\mathbb{R}\setminus\{x_1,\ldots,x_n\}}$. Notice that since $\mathrm{LIN}_{\mathbb{Q}}(\mathbb{R} \setminus \{x_1, \ldots, x_n\}) = \mathbb{R}$, so $\mathrm{LC}(h, x_0) \neq \emptyset$. Choose $y_0 \in \mathrm{LC}(h, x_0)$ and put $y_i = q_i y_0$ for $1 \le i \le n$. Define $f : \mathbb{R} \to \mathbb{R}$ as $f = h \cup \{\langle x_i, y_i\rangle : 1 \le i \le n\}$. We will show that $f \in L_{n+1} \setminus L_n$. First notice that since $f_{|\mathbb{R}\setminus\{x_1,\ldots,x_n\}}, \{\langle x_1, y_1\rangle\}, \ldots, \{\langle x_n, y_n\rangle\}$ are extendable to Hamel functions, so $f \in L_{n+1}$. Now suppose that $f \in L_n$, hence there exists a partition of $\mathbb{R}$ into sets $B_0, \ldots B_{n-1}$ such that $f_{|B_i}$ is extendable to a Hamel function for $0 \le i \le n-1$. Since $|\{x_0, \ldots, x_n\}| = n+1$, so there exists i such that $|B_i \cap \{x_0, \ldots, x_n\}| \ge 2$. Hence there exist $k \neq l$ such that $x_k, x_l \in B_i$ and consequently $\langle x_k, y_k\rangle - \frac{q_k}{q_l}\langle x_l, y_l\rangle = 0$, so $f_{|B_i} \notin \mathrm{PLIF}$, a contradiction. Hence $f \notin L_n$. □

Remark 1. *Note that $\mathcal{L}(\mathrm{HF}) \cap \mathrm{Add} = \emptyset$, because $f(0) \neq 0$ for each $f \in \mathcal{L}(\mathrm{HF})$. Similarly, if $f = \max\{f_0, f_1, \ldots\}$ for some infinite sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of Hamel functions, then $f(0) \neq 0$ and therefore $f \notin \mathrm{Add}$.*

The following Lemma is a simple modification of [1, Lemma 7.3.10]. (See also [4] and [2].)

Lemma 5. *Fix a cardinal δ and infinite regular cardinals γ, κ such that $\gamma < \kappa$ and $\delta < \gamma$. Let $|A| = \kappa$, $|B| = \gamma$ and $f : A \times B \to \delta$, then for every cardinal $\alpha < \delta$ there exist $B_0 \in [B]^\alpha$ and $A_0 \in [A]^\kappa$ such that $f(a_0, b_0) = f(a_1, b_1)$ for every $a_0, a_1 \in A_0$ and $b_0, b_1 \in B_0$.*

Proof. Fix $\alpha < \delta$. First we will show that for every $a \in A$ there exist $B_a \in [B]^\alpha$ and $\beta_a < \delta$ such that $f(a, b) = \beta_a$ for every $b \in B_a$. To see this, notice that for every $a \in A$ the sets $S_a^\beta = \{b \in B : f(a, b) = \beta\}$, $\beta < \delta$, form

a partition of the set B. Since $|B| = \gamma$ and $\gamma > \delta$, so there exists an $\beta = \beta_a$ such that the set S_a^β is of power γ. Notice that $[S_a^\beta]^\alpha \neq \emptyset$. Fix $B_a \in [S_a^\beta]^\alpha$ and observe that $f(a,b) = \beta_a$ for every $b \in B_a$.
Now $F : A \to [B]^\alpha \times \delta$ be defined by $F(a) = \langle B_a, \beta_a\rangle$. Then $f(a,b) = \beta_a$ for every $a \in A$ and $b \in B_a$. The set $[B]^\alpha \times \delta$ has cardinality γ, so there exists $\langle B_0, \beta\rangle \in [B]^\alpha \times \delta$ such that $A_0 = F^{-1}(B_0, \beta)$ has cardinality κ. Hence for every $a \in A_0$ and $b \in B_0$ we have $f(a,b) = \beta$. □

Theorem 4. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is constant on some nonempty open interval $(a,b) \subset \mathbb{R}$, then $f \notin \mathcal{L}(\mathrm{LIF})$.*

PROOF. Fix a partition $S_0, \ldots, S_n$ of $\mathbb{R}$. Let $U \subset \mathbb{R}$ be an non-empty open interval such that $x + y \in (a,b)$ for every $x, y \in U$. Fix a Hamel basis $H \subset U$ and disjoint sets $A, B \subset H$ such that $|A| = \omega_1$ and $|B| = \omega$. Define $g : A \times B \to \{0, \ldots, n\}$ by $g(a,b) = m$ iff $a + b \in S_m$. Then in virtue of Lemma 5 for function g and $\alpha = 2$ there exist $B_0 = \{b_0, b_1\} \in [B]^2$ and $A_0 \in [A]^\omega$ such that $b + a \in S_k$ for some $k \leq n$ and every $a \in A_0$, $b \in B_0$. Fix $a_0, a_1 \in A_0$ and put $x = b_0 + a_0$, $y = b_0 + a_1$, $z = b_1 + a_0$ and $t = b_1 + a_1$. Notice that x, y, z, t are pairwise different, $x, y, z, t \in (a,b)$ and $x - y = z - t$. Since f is constant on (a,b), so $\langle x, f(x)\rangle - \langle y, f(y)\rangle + \langle t, f(t)\rangle - \langle z, f(z)\rangle = 0$. Hence $f \notin \mathcal{L}(\mathrm{LIF})$. □

Problem 1. *Does there exist a function $f \in \mathcal{L}(\mathrm{HF})$ such that f is continuous on some nonempty open interval?*

Now we will consider maxima of countable families of functions.

Theorem 5. *The Continuum Hypothesis holds iff $\mathbb{R}^\mathbb{R} \setminus \{f \in \mathbb{R}^\mathbb{R} : f(0) = 0\} = \{\max\{f_0, f_1, \ldots\} : f_0, f_1, \ldots \in \mathrm{HF}\}$.*

PROOF. $\Rightarrow$ By Remark 1 it is enough to show that if $f(0) \neq 0$ then $f = \max\{f_0, f_1, \ldots\}$ for some $f_0, f_1, \ldots \in \mathrm{HF}$. Fix $f \in \mathbb{R}^\mathbb{R}$ with $f(0) \neq 0$. It is well known that the continuum hypothesis holds iff the set of all non-zero reals is a union of countably many Hamel bases [2]. Hence there exist pairwise disjoint linearly independent sets $H_1, H_2, \ldots \subset \mathbb{R}$ such that $\mathbb{R} \setminus \{0\} = \bigcup_{n=1}^\infty H_n$. Put $H_0 = \{0\}$ and define $\tilde{f}_i = f_{|H_i}$ for $i = 0, 1, \ldots$. Then $\tilde{f}_0, \tilde{f}_1 \ldots$ can be extended to a Hamel functions [8, Fact 6], denote those extensions again by $\tilde{f}_0, \tilde{f}_1, \ldots$ respectively. For $i = 0, 1, \ldots$ define functions $f_i(x) = \begin{cases} f(x) & \text{for } x \in H_i \\ \tilde{f}_i(x) + q_x^i \tilde{f}_i(0) & \text{for } x \notin H_i \end{cases}$ where $q_x^i \in \mathbb{Q} \setminus \{-1\}$ is such that $\tilde{f}_i(x) + q_x^i \tilde{f}_i(0) < f(x)$. Then $f = \max\{f_0, f_1, \ldots\}$ and $f_0, f_1, \ldots \in \mathrm{HF}$.

$\Leftarrow$ Suppose $\mathfrak{c} > \omega_1$. Define $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = \begin{cases} x & \text{for } x \neq 0 \\ a & \text{for } x = 0 \end{cases}$ for some $a \in \mathbb{R} \setminus \{0\}$. Hence there exists $(f_n)_{n\in\mathbb{N}} \subset \mathrm{HF}$ such that $f = \max\{f_n : n \in \mathbb{N}\}$. Define $H_n = \{x \in \mathbb{R} : f_n(x) = f(x)\} \setminus \{0\}$. Then $\mathbb{R} \setminus \{0\} = \bigcup_{n\in\mathbb{N}} H_n$. We will show that H_n is linearly independent for $n \in \mathbb{N}$. Suppose the opposite that there exists $i \in \mathbb{N}$ such that H_i is linearly dependent. Hence there exist different $x_0, \dots, x_n \in H_i$ and $q_0, \dots, q_n \in \mathbb{Q} \setminus \{0\}$ such that $\sum_{k=0}^n q_k x_k = 0$, so $\sum_{k=0}^n q_k \langle x_k, f_i(x_k)\rangle = 0$, a contradiction. Fix a Hamel basis H and $A, B \subset H$, $A \cap B = \emptyset$, such that $\mathfrak{c} \geq |A| > |B| > \omega$. Define function $g : A \times B \to \omega$ by $g(a, b) = m$ iff $a + b \in H_m$. Hence there exist sets A_0 and B_0 as in Lemma 5 for function g and $\alpha = 2$. Choose different $a_0, a_1 \in A_0$, $b_0, b_1 \in B_0$ and put $x_{ij} = a_i + b_j$ for $i, j < 2$. Then x_{ij} are different numbers all belonging to the same H_n. On the other hand we have $x_{00} - x_{10} = x_{01} - x_{11}$, a contradiction with the fact that H_n is linearly independent. $\square$

Remark 2. *Notice that by similar reasoning as in Theorem 5 if we assume* $\mathrm{ZFC} + \neg\mathrm{CH}$ *then no constant function is a maxima of a countable family of Hamel functions.*

3 Extensions of Hamel functions.

Fact 3. *Suppose* $f : X \to \mathbb{R}$. *Then* $\mathrm{LIN}_\mathbb{Q}(f) = \mathrm{LIN}_\mathbb{Q}(X) \times \mathbb{R}$ *iff there exists* $a \in \mathrm{LIN}_\mathbb{Q}(X)$ *such that* $\mathrm{LC}(f, a) = \mathbb{R}$.

Proof. $\Rightarrow$ Fix $a \in \mathrm{LIN}_\mathbb{Q}(X)$. Then $\{a\} \times \mathrm{LC}(f, a) = \mathrm{LIN}_\mathbb{Q}(f) \cap (\{a\} \times \mathbb{R})$ and since $\mathrm{LIN}_\mathbb{Q}(f) = \mathrm{LIN}_\mathbb{Q}(X) \times \mathbb{R}$, so $\mathrm{LC}(f, a) = \mathbb{R}$.

$\Leftarrow$ The inclusion $\subset$ is clear. To see $\supset$, pick $\langle x, y\rangle \in \mathrm{LIN}_\mathbb{Q}(X) \times \mathbb{R}$ and $a \in \mathrm{LIN}_\mathbb{Q}(X)$ such that $\mathrm{LC}(f, a) = \mathbb{R}$. Since $x - a \in \mathrm{LIN}_\mathbb{Q}(X)$, so there exist $q_0, \dots, q_n \in \mathbb{Q} \setminus \{0\}$ and $x_0, \dots, x_n \in X$ such that $\sum_{i=0}^n q_i x_i = x - a$. Put $z = \sum_{i=0}^n q_i f(x_i)$. Since $\mathrm{LC}(f, a) = \mathbb{R}$, so $y - z \in \mathrm{LC}(f, a)$. Hence there exist $p_0, \dots, p_m \in \mathbb{Q} \setminus \{0\}$ and $y_0, \dots, y_m \in X$ such that $\sum_{i=0}^m p_i \langle y_i, f(y_i)\rangle = \langle a, y - z\rangle$. Hence we get

$$\langle x, y\rangle = \langle a, y - z\rangle + \langle x - a, z\rangle = \sum_{i=0}^m p_i \langle y_i, f(y_i)\rangle + \sum_{i=0}^n q_i \langle x_i, f(x_i)\rangle .$$

Consequently, $\langle x, y\rangle \in \mathrm{LIN}_\mathbb{Q}(f)$. $\square$

Remark 3. *Suppose* $f : X \to \mathbb{R}$ *is linearly independent,* $|\mathbb{R} \setminus X| < \omega$ *and* $\mathrm{codim}_{\mathbb{R}^2}(\mathrm{LIN}_\mathbb{Q}(f)) = |\mathbb{R} \setminus X|$. *Then* f *can be extended to a Hamel function* $\tilde{f}$.

PROOF. Order $\mathbb{R} \setminus X = \{x_k : k \leq n\}$. We will define a function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by induction. For $x \in X$ put $\tilde{f}(x) = f(x)$. Suppose, for $l < k$, a function $\tilde{f}$ is defined for points x_l such that $f_l = f \cup \bigcup_{l<k} \left\{ \left\langle x_l, \tilde{f}(x_l) \right\rangle \right\}$ is linearly independent. Since $l < k \leq n$, so $\operatorname{codim}_{\mathbb{R}^2} (\mathrm{LIN}_{\mathbb{Q}} (f_l)) = n - l > n - k \geq 0$. Note that $\mathrm{LIN}_{\mathbb{Q}} (X) = \mathbb{R}$, hence by Fact 3 there exists $y \notin \mathrm{LC}(f_l, x_k)$. Put $\tilde{f}(x_k) = y$. Notice that $\tilde{f} \in \mathrm{HF}$. □

Theorem 6. *Suppose* $X \subset \mathbb{R}$, $\mathrm{LIN}_{\mathbb{Q}} (X) = \mathbb{R}$ *and* $f : X \to \mathbb{R}$ *is* PLIF. *Then* f *is extendable to a* HF *function iff* $\operatorname{codim}_{\mathbb{R}^2} (\mathrm{LIN}_{\mathbb{Q}} (f)) = |\mathbb{R} \setminus X|$.

PROOF. $\Rightarrow$ Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be an HF extension of the function f. Then $\operatorname{codim}_{\mathbb{R}^2} (\mathrm{LIN}_{\mathbb{Q}} (f)) = |\tilde{f} \setminus f| = |\mathrm{dom}(\tilde{f}) \setminus \mathrm{dom}(f)| = |\mathbb{R} \setminus X|$.

$\Leftarrow$ Without loss of generality we can assume that $|\mathbb{R} \setminus X| = \kappa \geq \omega$. Pick a Hamel basis $H \subset \mathbb{R}^2$ such that $f \subset H$. Well order $\mathbb{R} \setminus X = \{x_\alpha : \alpha < \kappa\}$ and $H \setminus f = \{\langle a_\alpha, b_\alpha \rangle : \alpha < \kappa\}$. For $\alpha < \kappa$ we will construct partial functions f_α such that

(i) $f \subset f_\beta \subset f_\alpha$ for $\beta < \alpha$ and $|f_\alpha| \leq |f| + |\alpha|$;

(ii) $f_\alpha \in \mathrm{PLIF}$;

(iii) $x_\alpha \in \mathrm{dom}(f_{\alpha+1})$;

(iv) $\langle a_\alpha, b_\alpha \rangle \in \mathrm{LIN}_{\mathbb{Q}} (f_{\alpha+1})$.

Then $\tilde{f} = \bigcup_{\alpha<\kappa} f_\alpha$ is a HF extension of the function f. Suppose that for $\beta < \gamma$ functions f_β are constructed. If γ is a limit ordinal then $f_\gamma = \bigcup_{\beta<\gamma} f_\beta$. Otherwise there exists α such that $\gamma = \alpha + 1$.

Step 1. If $x_\alpha \in \mathrm{dom}(f_\alpha)$ then $f'_\alpha = f_\alpha$. Otherwise, since $\operatorname{codim}_{\mathbb{R}^2} (\mathrm{LIN}_{\mathbb{Q}} (f_\alpha)) = \kappa$, so $f_\alpha \notin \mathrm{PHF}$. Hence in virtue of Fact 3 there exists $y \in \mathbb{R} \setminus \mathrm{LC}(f_\alpha, x_\alpha)$. Put $f'_\alpha = f_\alpha \cup \{\langle x_\alpha, y \rangle\}$. Then obviously $f'_\alpha \in \mathrm{PLIF}$.

Step 2. If $\langle a_\alpha, b_\alpha \rangle \in \mathrm{LIN}_{\mathbb{Q}} (f'_\alpha)$ then $f_{\alpha+1} = f'_\alpha$. Otherwise we get two cases. If $a_\alpha \notin \mathrm{dom}(f'_\alpha)$ then define $f_{\alpha+1} = f'_\alpha \cup \{\langle a_\alpha, b_\alpha \rangle\}$. Then, since $\langle a_\alpha, b_\alpha \rangle \notin \mathrm{LIN}_{\mathbb{Q}} (f'_\alpha)$, $f_{\alpha+1} \in \mathrm{PLIF}$. Now suppose that $a_\alpha \in \mathrm{dom}(f'_\alpha)$. Pick $x \notin \mathrm{dom}(f'_\alpha)$. Since $\mathrm{LIN}_{\mathbb{Q}} (\mathrm{dom}(f'_\alpha)) = \mathbb{R}$, so there exist $q_0, \ldots, q_n \in \mathbb{Q} \setminus \{0\}$ and $x_0, \ldots, x_n \in \mathrm{dom}(f'_\alpha)$ such that $-x = \sum_{i \leq n} q_i x_i$. Define $y = \sum_{i \leq n} q_i f'_\alpha(x_i)$ and $f_\gamma = f'_\alpha \cup \{\langle x, b_\alpha - f'_\alpha(a_\alpha) - y \rangle\}$. We will show that $f_\gamma \in \mathrm{PLIF}$. This fact follows easily from

$$\langle a_\alpha, b_\alpha \rangle = \langle a_\alpha, f'_\alpha(a_\alpha) \rangle + \langle 0, b_\alpha - f'_\alpha(a_\alpha) \rangle =$$

$$\langle a_\alpha, f'_\alpha(a_\alpha) \rangle + \langle -x, y \rangle + \langle x, b_\alpha - f'_\alpha(a_\alpha) - y \rangle \notin \mathrm{LIN}_{\mathbb{Q}} (f'_\alpha)$$

and since $\langle a_\alpha, f'_\alpha(a_\alpha) \rangle, \langle -x, y \rangle \in \mathrm{LIN}_{\mathbb{Q}} (f'_\alpha)$, so we get that $\langle x, b_\alpha - f'_\alpha(a_\alpha) - y \rangle \notin \mathrm{LIN}_{\mathbb{Q}} (f'_\alpha)$. Hence $f_{\alpha+1} \in \mathrm{PLIF}$ and $\langle a_\alpha, b_\alpha \rangle \in \mathrm{LIN}_{\mathbb{Q}} (f_{\alpha+1})$. □

Theorem 7.

1. *If* $|X| < \mathfrak{c}$, *then any* $f : X \to \mathbb{R}$, $f \in \mathrm{PLIF}$ *can be extended to a* HF *function.*
2. *Suppose* $|X| = \mathfrak{c}$. *Then there exists* $f_X : \mathrm{LIN}_{\mathbb{Q}}(X) \to \mathbb{R}$, $f_X \in \mathrm{PLIF}$ *such that* $\mathrm{LIN}_{\mathbb{Q}}(f_X) = \mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}$.
3. *If we additionally assume in* (2) *that* $\mathrm{LIN}_{\mathbb{Q}}(X) \neq \mathbb{R}$, *then* f_X *is not extendable to a* LIF *function.*

Proof. (1) follows from [5, Lemma 2.3].

(2) Fix $X \in [\mathbb{R}]^{\mathfrak{c}}$ and let $\varphi : \mathbb{R} \to \mathrm{LIN}_{\mathbb{Q}}(X)$ be a linear isomorphism. Define $\Phi : \mathbb{R}^2 \to \mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}$ by $\Phi(x,y) = \langle \varphi(x), y \rangle$. Then Φ is a linear isomorphism. Furthermore Φ preserves functions i.e. if $f : \mathbb{R} \to \mathbb{R}$ is a function, then $\Phi(f)$ is also a function. Hence for every $f \in \mathrm{HF}$ the function $\Phi(f) : \mathrm{LIN}_{\mathbb{Q}}(X) \to \mathbb{R}$ is a basis of the space $\mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}$. Set $f_X = \Phi(f)$.

(3) We will show that any extension of f_X on $\mathbb{R}$ is linearly dependent. Fix a function $f : \mathbb{R} \to \mathbb{R}$ which is an extension of function f_X. Let $Y \subset X$ be a basis of $\mathrm{LIN}_{\mathbb{Q}}(X)$ and $H \subset \mathbb{R}$ a Hamel basis such that $Y \subset H$. Define $\tilde{f} : \mathrm{LIN}_{\mathbb{Q}}(X) \cup H \to \mathbb{R}$ as the restriction $f_{|\mathrm{LIN}_{\mathbb{Q}}(X) \cup H}$. Notice that $f_X \subset \tilde{f}$ and since $\mathrm{LC}(f_X) = \mathbb{R}$, so $\mathrm{LC}\left(\tilde{f}\right) = \mathbb{R}$. Hence in virtue of Fact 3, $\mathrm{LIN}_{\mathbb{Q}}\left(\tilde{f}\right) = \mathrm{LIN}_{\mathbb{Q}}(Y \cup H) \times \mathbb{R} = \mathbb{R}^2$. On the other hand, $H \cup \mathrm{LIN}_{\mathbb{Q}}(X) \subsetneq \mathbb{R}$, so any extension of $\tilde{f}$ is linearly dependent and consequently f is linearly dependent. □

Lemma 6. *Suppose* $f : X \to \mathbb{R}$ *is* PLIF. *Then* $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}}(\mathrm{LIN}_{\mathbb{Q}}(f)) \geq \mathrm{codim}_{\mathbb{R}}(\mathrm{LC}(f))$.

Proof. Fix a basis $Y \subset \mathbb{R}$ of subspace $\mathrm{LC}(f)$ and a set $A \subset \mathbb{R}$ such that $Y \cup A$ is a Hamel basis of $\mathbb{R}$. Since $\mathrm{LC}(f)$ is linearly isomorphic to $\mathrm{LIN}_{\mathbb{Q}}(f) \cap (\{0\} \times \mathbb{R})$, so $f \cup (\{0\} \times A)$ is a linearly independent set. Finally $\mathrm{LIN}_{\mathbb{Q}}(f \cup (\{0\} \times A)) \subset \mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}$, so $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}}(\mathrm{LIN}_{\mathbb{Q}}(f)) \geq \mathrm{codim}_{\mathbb{R}}(\mathrm{LC}(f))$. □

Theorem 8. *Suppose* $X \subset \mathbb{R}$ *and* $f : X \to \mathbb{R}$ *is linearly independent. Then* f *is extendable to a* HF *function iff* $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X) \times \mathbb{R}}(\mathrm{LIN}_{\mathbb{Q}}(f)) = |\mathbb{R} \setminus X|$.

Proof. First notice that in virtue of Theorem 6 and Lemma 2 without loss of generality we can assume that $\mathrm{LIN}_{\mathbb{Q}}(X) \neq \mathbb{R}$ and $|X| = \mathfrak{c}$.

$\Rightarrow$ Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a HF extension of a function f and $Y \subset X$ be a basis of the space $\mathrm{LIN}_{\mathbb{Q}}(X)$. Fix a Hamel basis $H \subset \mathbb{R}$ such that $Y \subset H$ and define

$F : X \cup H \to \mathbb{R}$, by $F = \tilde{f}_{|(X\cup H)}$.
Claim 1. $\operatorname{codim}_{\mathbb{R}} (\mathrm{LC}(F)) = \mathfrak{c}$.
Well order $\mathbb{R} \setminus (X \cup H) = \{x_\alpha : \alpha < \mathfrak{c}\}$. Notice that

$$\mathrm{LC}(F) \subsetneq \mathrm{LC}\left(F \cup \left\{\left\langle x_0, \tilde{f}(x_0)\right\rangle\right\}\right) \subsetneq \ldots \subsetneq \mathrm{LC}\left(\tilde{f}\right).$$

Indeed, fix $\alpha < \mathfrak{c}$. There exist different $x_0^\alpha, \ldots, x_n^\alpha \in X \cup H$ and $q_0^\alpha, \ldots, q_n^\alpha \in \mathbb{Q} \setminus \{0\}$ such that $\sum_{i=0}^n q_i^\alpha x_i^\alpha = -x_\alpha$. Let $y = \sum_{i=0}^n q_i^\alpha \tilde{f}(x_i^\alpha) + \tilde{f}(x_\alpha)$. Then

$$y \in \mathrm{LC}\left(F \cup \bigcup_{\gamma\le\alpha} \left\{\left\langle x_\gamma, \tilde{f}(x_\gamma)\right\rangle\right\}\right).$$

$\tilde{f}$ is a linearly independent set, so $y \notin \mathrm{LC}\left(F \cup \bigcup_{\gamma<\alpha} \left\{\left\langle x_\gamma, \tilde{f}(x_\gamma)\right\rangle\right\}\right)$. Recall that $\mathrm{LC}\left(F \cup \bigcup_{\gamma\le\alpha} \left\{\left\langle x_\gamma, \tilde{f}(x_\gamma)\right\rangle\right\}\right)$ is a linear subspace of $\mathbb{R}$ for every $\alpha < \mathfrak{c}$. Hence

$$\operatorname{codim}_{\mathbb{R}} (\mathrm{LC}(F)) = |\mathbb{R} \setminus (X \cup H)| = \mathfrak{c}.$$

Claim 2. $\mathrm{LC}(F) = \mathrm{LC}(f)$.
Since $f \subset F$, $\mathrm{LC}(f) \subset \mathrm{LC}(F)$. Fix $y \in \mathrm{LC}(F)$, so $\langle 0, y\rangle = \sum_{i=0}^n q_i \langle x_i, f(x_i)\rangle + \sum_{j=0}^m p_j \langle y_j, F(y_j)\rangle$, for some $p_j, q_i \in \mathbb{Q}$ and different $x_i \in X$, $y_j \in H \setminus Y$ for $i \le n$ and $j \le m$. Since $\mathrm{LIN}_{\mathbb{Q}}(\{x_i : i \le n\}) \cap \mathrm{LIN}_{\mathbb{Q}}(\{y_j : i \le m\}) = \{0\}$, so $\sum_{i=0}^n q_i x_i = 0$ and $\sum_{j=0}^m p_j y_j = 0$. Because $H \setminus X$ is a linearly independent set, so $p_j = 0$ for $j \le m$. Hence $y \in \mathrm{LC}(f)$.

Hence in virtue of Lemma 6 we get that

$$\operatorname{codim}_{\mathbb{R}^2} (\mathrm{LIN}_{\mathbb{Q}}(f)) \ge \operatorname{codim}_{\mathbb{R}} (\mathrm{LC}(f)) = \mathfrak{c}.$$

$\Leftarrow$ Again we start with showing that $\operatorname{codim}_{\mathbb{R}} (\mathrm{LC}(f)) = \mathfrak{c}$. To see this fix $x \in \mathbb{R} \setminus \mathrm{LIN}_{\mathbb{Q}}(X)$. Notice that since $|X| = \mathfrak{c}$, so $|\mathrm{LIN}_{\mathbb{Q}}(X \cup \{x\}) \setminus \mathrm{LIN}_{\mathbb{Q}}(X)| = \mathfrak{c}$. Put $Y = \mathrm{LIN}_{\mathbb{Q}}(X \cup \{x\})$ and well order $Y = \{x_\alpha : \alpha < \mathfrak{c}\}$. We define partial functions f_α, $\alpha < \mathfrak{c}$, such that

(i) $f \subset f_\beta \subset f_\alpha$ for $\beta < \alpha$;

(ii) $f_\alpha \in \mathrm{PLIF}$;

(iii) $x_\alpha \in \mathrm{dom}(f_{\alpha+1})$.

If α is a limit ordinal, then $f_\alpha = \bigcup_{\gamma<\alpha} f_\gamma$. Otherwise $\alpha = \beta + 1$. Since $\mathfrak{c} = \operatorname{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X)\times\mathbb{R}} (\mathrm{LIN}_{\mathbb{Q}}(f_\beta)) \le \operatorname{codim}_{Y\times\mathbb{R}} (\mathrm{LIN}_{\mathbb{Q}}(f_\beta))$ and $\alpha < \mathfrak{c}$, so in virtue of Fact 3 there exists $y \in \mathbb{R} \setminus \mathrm{LC}(f_\beta, x_\alpha)$. Put $f_\alpha = f_\beta \cup \{\langle x_\alpha, y\rangle\}$. It is easy to notice that

$$\mathrm{LC}\,(f) = \mathrm{LC}\,(f_0) \subsetneq \mathrm{LC}\,(f_1) \subsetneq \ldots \subsetneq \mathrm{LC}\,(f_{\mathfrak{c}}).$$

Since $\mathrm{LC}\,(f_\alpha)$, $\alpha < \mathfrak{c}$, is a linear subspace of $\mathbb{R}$, so $\mathrm{codim}_{\mathbb{R}}\,(\mathrm{LC}\,(f)) \geq \mathfrak{c}$.
Fix a basis Y of the space $\mathrm{LIN}_{\mathbb{Q}}\,(X)$ such that $Y \subset X$ and a Hamel basis H such that $Y \subset H$. Define $\tilde{f} : X \cup H \to \mathbb{R}$ by $\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in X \\ 0 & \text{for } x \in H \setminus X \end{cases}$.
Obviously $\mathrm{LC}\left(\tilde{f}\right) = \mathrm{LC}\,(f)$. In virtue of Lemma 6

$$\mathrm{codim}_{\mathbb{R}^2}\left(\mathrm{LIN}_{\mathbb{Q}}\left(\tilde{f}\right)\right) \geq \mathrm{codim}_{\mathbb{R}}\left(\mathrm{LC}\left(\tilde{f}\right)\right) = \mathrm{codim}_{\mathbb{R}}\,(\mathrm{LC}\,(f)) = \mathfrak{c},$$

so in virtue of Theorem 6, $\tilde{f}$ can be extended to a Hamel function. □

Corollary 1. *Suppose $f : X \to \mathbb{R}$ is a* PLIF *function. Then f is extendable to a* LIF *function iff* $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X)\times\mathbb{R}}\,(\mathrm{LIN}_{\mathbb{Q}}\,(f)) \geq |\mathbb{R} \setminus X|$.

Proof. $\Rightarrow$ Assume the opposite that $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X)\times\mathbb{R}}\,(\mathrm{LIN}_{\mathbb{Q}}\,(f)) < |\mathbb{R} \setminus X|$. But then any function $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $f \subset \tilde{f}$, has to be linearly dependent, a contradiction.

$\Leftarrow$ First notice that if $|\mathbb{R} \setminus X| = \mathfrak{c}$ then $\mathrm{codim}_{\mathrm{LIN}_{\mathbb{Q}}(X)\times\mathbb{R}}\,(\mathrm{LIN}_{\mathbb{Q}}\,(f)) = \mathfrak{c}$. Hence from Theorem 8, f can be extended to a HF function. Now assume that $|\mathbb{R} \setminus X| = \kappa < \mathfrak{c}$. Well order $\mathbb{R} \setminus X = \{x_\alpha : \alpha < \kappa\}$ and define $\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in X \\ y_\alpha & \text{for } x = x_\alpha \end{cases}$, where $y_\alpha \in \mathbb{R} \setminus \mathrm{LC}(f \cup \bigcup_{\beta<\alpha}\{\langle x_\beta, y_\beta\rangle\}, x_\alpha)$. Such a choice is possible since $\alpha < \kappa$ and $\mathrm{codim}_{\mathbb{R}^2}\,(\mathrm{LIN}_{\mathbb{Q}}\,(f)) > \kappa$. Then $\tilde{f} \in \mathrm{LIF}$ and $f \subset \tilde{f}$. □

Next we apply the obtained extension theorem to prove a result concerning the lattice of Hamel functions.

Theorem 9. Max(HF)=Max(LIF).

Proof. Since HF $\subset$ LIF, so the inclusion $\subset$ is obvious.
$\supset$ Fix $f \in \mathrm{Max(LIF)}$. Hence there exist $g, h \in \mathrm{LIF}$ such that $f = \max\{g, h\}$. Let $A = \{x \in \mathbb{R} : g(x) = f(x)\}$ and $B = \{x \in \mathbb{R} \setminus A : h(x) = f(x)\}$. Since $A \cup B = \mathbb{R}$, so $|A| = \mathfrak{c}$ or $|B| = \mathfrak{c}$. Hence we get two cases.

First suppose that $|A| = \mathfrak{c}$ and $|B| < \mathfrak{c}$. Notice that $\mathrm{LIN}_{\mathbb{Q}}\,(A) = \mathbb{R}$. Fix disjoint Hamel bases $H_1, H_2 \subset \mathbb{R} \setminus B$ and a linearly independent set $X \subset H_1$ such that $\mathrm{LIN}_{\mathbb{Q}}\,(X) \cap \mathrm{LIN}_{\mathbb{Q}}\,(B) = \emptyset$ and $\mathrm{LIN}_{\mathbb{Q}}\,(B \cup X) = \mathbb{R}$. First notice that $f_{|(A\setminus X)}$ is extendable to a HF function. Indeed, since $f_{|A}$ is a linearly independent set, so $\mathrm{codim}_{\mathbb{R}^2}\left(\mathrm{LIN}_{\mathbb{Q}}\left(f_{|A\setminus X}\right)\right) \geq |X| = \mathfrak{c} = |\mathbb{R}\setminus(A\setminus X)|$. Furthermore $H_2 \subset A\setminus X$, so $\mathrm{LIN}_{\mathbb{Q}}(A\setminus X) = \mathbb{R}$. Hence in virtue of Theorem 6, $f_{|A\setminus X}$ is extendable to a HF function. It is easy to see that $f_{|(B\cup X)} \in$ PLIF and $\mathrm{LC}\left(f_{|(B\cup X)}\right) = \mathrm{LC}\left(f_{|B}\right)$. Since $|B| < \mathfrak{c}$, so $|\mathrm{LC}\left(f_{|B}\right)| < \mathfrak{c}$ and consequently $\mathrm{codim}_{\mathbb{R}}\left(\mathrm{LC}\left(f_{|B}\right)\right) = \mathfrak{c}$. Hence $\mathrm{codim}_{\mathbb{R}^2}\left(\mathrm{LIN}_{\mathbb{Q}}\left(f_{|B\cup X}\right)\right) \geq \mathrm{codim}_{\mathbb{R}}\left(\mathrm{LC}\left(f_{|B\cup X}\right)\right) = \mathfrak{c} = |\mathbb{R}\setminus(B\cup X)|$, so in virtue of Theorem 6, $f_{|B\cup X}$ can be extended to a HF function and by Theorem 1, $f \in \mathrm{Max(HF)}$.

Now assume that $|A| = |B| = \mathfrak{c}$. Notice that since $f_{|A}$ and $f_{|B}$ are extendable to LIF functions, so $\mathrm{codim}_{\mathbb{R}^2}\left(\mathrm{LIN}_{\mathbb{Q}}\left(f_{|A}\right)\right) \geq |B| = \mathfrak{c}$ and $\mathrm{codim}_{\mathbb{R}^2}\left(\mathrm{LIN}_{\mathbb{Q}}\left(f_{|B}\right)\right) \geq |A| = \mathfrak{c}$. Hence both $f_{|A}$ and $f_{|B}$ are extendable to Hamel functions and as above, $f \in \mathrm{Max(HF)}$. □

Corollary 2. $\mathcal{L}(\mathrm{HF}) = \mathcal{L}(\mathrm{LIF})$.

PROOF. The inclusion $\subset$ is obvious. To see the other inclusion notice that in virtue of Theorem 9, LIF $\subset \mathcal{L}(\mathrm{HF})$. Hence $\mathcal{L}(\mathrm{LIF}) \subset \mathcal{L}(\mathrm{HF})$. □

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