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A GENERALIZATION OF STEINHAUS' THEOREM AND SOME NONMEASURABLE SETS

Abstract

We generalize a classical Steinhaus theorem replacing addition by any two variable function which is differentiable. We deal with category and measure case. Measure case was done before in papers [3], [9]. We give a proof which can be easily generalized to category case. We also obtain a variety of examples of nonmeasurable sets which have interesting algebraic properties.

1 Definitions and Notations.

We use the standard set-theoretic notation. The set of all natural numbers we denote by ω; i.e. $\omega = \{0, 1, 2, \ldots\}$. The interior of a set A we denote by $int(A)$. The family of all Borel subsets of the real line we denote by Bor. The σ-ideals of null and meagre subsets of $\mathbb{R}$ we denote by $\mathbb{L}$ and $\mathbb{K}$ respectively. For any binary operation $\circ$ and sets $A, B \subseteq \mathbb{R}$ we denote by $A \circ B$ the algebraic operation; i.e. $\{a \circ b : a \in A \wedge b \in B\}$. Moreover, $A \circ b = A \circ \{b\}$ for a real b.

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Let $\mathbb{I} = \mathbb{L}$ or $\mathbb{I} = \mathbb{K}$. We say that a set A is a completely $\mathbb{I}$ nonmeasurable set if

$$(\forall B \in Bor \setminus \mathbb{I})(A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset).$$

2 Generalization of Steinhaus Theorem.

Let us recall the famous Steinhaus theorem: *for any two sets A, B of positive Lebesgue measure the interior of the algebraic sum $A + B$ is nonempty.*

In [3] and [9] there are generalizations of the Steinhaus theorem to first class functions with non-vanishing first derivatives. Further generalization in the case of measure was given by [7, 8]. Moreover the Steinhaus theorem in the category case was given in [10]: *if $A, B \subset V$ are two Baire measurable second category subsets of the topological vector space, then both $A + B$ and $A - B$ contain some open set.*

On the other hand, the ideal $\mathbb{L} \cap \mathbb{K}$ doesn't have the Steinhaus property that was given in [11].

We will prove a generalization of the Steinhaus theorem to first class functions with non-vanishing first derivatives. The measure case was proved in [3] and [9]. We give a proof which is similar in both cases: measure and category.

Theorem 2.1. *Let $\mathbb{I} = \mathbb{L}$ or $\mathbb{I} = \mathbb{K}$. Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that*

$$\left\{(x, y) : \frac{\partial f}{\partial x}(x, y) = 0 \vee \frac{\partial f}{\partial y}(x, y) = 0\right\} \in \mathbb{I}.$$

Let A, B be positive subsets of the real line; i.e. $A, B \in Bor(\mathbb{R}) \setminus \mathbb{I}$. Then the set

$$f(A, B) = \{f(a, b) : a \in A \wedge b \in B\}$$

contains an interval.

Before we start the proof of Theorem 2.1, we state two claims, one for measure and another for category.

Claim 2.1 (measure case)**.** *Let $A, B \in Bor \setminus \mathbb{L}$. Let a_0 be a point of density 1 in A and b_0 be a point of density 1 in B and $h_0 \in \mathbb{R}$ Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function such that*

$$g(a_0, h_0) = b_0 \quad \text{and} \quad \frac{\partial g}{\partial x}(a_0, h_0) \neq 0.$$

Then there exists an interval I such that $h_0 \in I$ and

$$(\forall h \in I)(g(A,h) \cap B \notin \mathbb{L}),$$

where $g(A,h) = \{g(a,h) : a \in A\}$.

Proof. For convenience we will write $g_h(a)$ instead of $g(a,h)$. By continuity of partial derivatives of g around (a_0, h_0) there exists positive reals α, β, δ and an interval I which contains a_0 such that

$$(\forall x, x' \in I)(\forall h)(|h - h_0| < \delta \longrightarrow \alpha|x - x'| \leq |g_h(x) - g_h(x')| \leq \beta|x - x'|).$$

Let us fix h such that $|h - h_0| < \delta$. There exist intervals $I_h \subseteq I$, J_h such that if $\tilde{A} = A \cap J_h$ and $\tilde{B} = B \cap J_h$, then:

1. $a_0 \in I_h$, $b_0 \in J_h$,
2. $g_h(I_h) \subseteq J_h$,
3. $\lambda(g_h(I_h) \cap J_h) \geq \frac{1}{2}\lambda(J_h)$,
4. $\lambda(\tilde{B}) - \lambda(J_h) < \frac{1}{2}\alpha\lambda(\tilde{A})$ (by density argument).

Let us observe that

$$\begin{aligned} \lambda(g_h(\tilde{A}) \cap \tilde{B}) &= \lambda(g_h(\tilde{A})) + \lambda(\tilde{B}) - \lambda(g_h(\tilde{A}) \cup \tilde{B}) \\ &\geq \alpha\lambda(\tilde{A}) + \lambda(\tilde{B}) - \lambda(g_h(I_h) \cup J_h) \\ &= \alpha\lambda(\tilde{A}) + \lambda(\tilde{B}) - \lambda(J_h) \\ &\geq \alpha\lambda(\tilde{A}) - \frac{1}{2}\alpha\lambda(\tilde{A}) = \frac{1}{2}\alpha\lambda(\tilde{A}) > 0. \end{aligned}$$

□

Claim 2.2 (category case)**.** *Let $A, B \in Bor \setminus \mathbb{K}$. Let $a_0 \in A$, $b_0 \in B$ and $h_0 \in \mathbb{R}$ be such that A is comeager in some open interval containing a_0 and B is comeager in some open interval containing b_0. Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a C^1 function such that*

$$g(a_0, h_0) = b_0 \quad \text{and} \quad \frac{\partial g}{\partial x}(a_0, h_0) \neq 0.$$

Then there exists an interval I such that $h_0 \in I$ and

$$(\forall h \in I)(g(A,h) \cap B \notin \mathbb{K}).$$

Proof. Let I_{a_0}, I_{b_0} be open intervals containing a_0 and b_0 respectively such that A, B is comeagre in I_{a_0}, I_{b_0} respectively. Then there exists a nonempty open interval I such that $g_h(I_{a_0}) \cap I_{b_0} \neq \emptyset$ for any $h \in I$. Thus $g_h(A) \cap B \notin \mathbb{K}$ for any $h \in I$. □

Proof of Theorem 2.1. We consider the measure case only. The category case is similar. From our assumption we see that there exists a point (a, b) such that the partial derivatives are non-zero in some neighborhood of it and $h_0 = f(a, b)$. Moreover, we can choose (a, b) such that a, b are density points of A and B respectively by Fubini's theorem.

By the implicit function theorem there exists an open interval I containing h_0 and family C^1 functions $\{g_h : h \in I\}$ such that $f(x, g_h(x)) = h$ where x belongs to some interval containing a. The assumptions of Claim 2.1 is fulfilled. Then there exists an interval I such that

$$(\forall h \in I)(g_h(A) \cap B \neq \emptyset),$$

which gives the required assertion. □

3 Nonmeasurable Sets.

In this section, we say that a set $A \subseteq \mathbb{R}$ is completely nonmeasurable iff A is completely $\mathbb{L}$-nonmeasurable and A is completely $\mathbb{K}$-nonmeasurable. We are interested in examples of completely nonmeasurable sets with some algebraic properties. This topic was investigated in papers [4, 5, 6]. Let us recall the following result from [1].

Theorem 3.1 (Cichoń, Szczepaniak)**.** *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism of $\mathbb{R}^2$ and $\mathbb{R}$ treated as linear spaces over $\mathbb{Q}$. Let $K \subseteq \mathbb{R}^2$ be bounded and int$(K) \neq \emptyset$. Then $\Phi[K]$ is completely nonmeasurable.*

We get the immediate corollary (see [1]):

Corollary 3.1 (Cichoń, Szczepaniak)**.** *Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$. Let $K \subseteq \mathbb{R}^2$ be such that int$(K) \neq \emptyset$ and int$(K^c) \neq \emptyset$. Then $\Phi[K]$ is completely nonmeasurable.*

We use the above corollary to obtain a few examples of completely nonmeasurable sets.

Theorem 3.2. *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that $A + A = A$ and $A - A = \mathbb{R}$.*

Proof. Consider $K = \mathbb{R} \times [0, \infty) \subseteq \mathbb{R}^2$. Here

$$K + K = K,\ int(K) \neq \emptyset,\ int(K^c) \neq \emptyset.$$

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$. Put $A = \Phi[K]$. By Corollary 3.1 A is completely nonmeasurable and since Φ preserves addition $A + A = A$. Moreover $K - K = \mathbb{R}^2$ so Φ preserves addition so that we get $A - A = \mathbb{R}$. □

Theorem 3.3. *There exists a partition* $\{A_n\}_{n\in\omega}$ *of the real line into countably many completely nonmeasurable sets such that* $A_n + A_n = A_n$ *for every* $n \in \omega$.

Proof. Let $(\alpha_n)_{n\in\omega}$ be any strictly increasing sequence of reals such that

1. $\alpha_0 = 0$,
2. $\lim_{n\to\infty} \alpha_n = 2\pi$,
3. $(\forall n \in \omega)(\alpha_{n+1} - \alpha_n < \pi)$.

Consider the following family of subsets of $\mathbb{R}^2$:

$$K_0 = \{(0,0)\} \cup \{(r\cos\alpha, r\sin\alpha) : r > 0 \wedge \alpha_0 \leq \alpha < \alpha_1\},$$

$$K_n = \{(r\cos\alpha, r\sin\alpha) : r > 0 \wedge \alpha_n \leq \alpha < \alpha_{n+1}\},\ \text{for } n \geq 1.$$

Here for every $n \in \omega$

$$K_n + K_n = K_n,\ int(K_n) \neq \emptyset.$$

Moreover $\{K_n\}_{n\in\omega}$ is a partition of $\mathbb{R}^2$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$. For each n put $A_n = \Phi[K_n]$. By Corollary 3.1 A_n is completely nonmeasurable and since Φ preserves addition $A_n + A_n = A_n$. Moreover Φ is a bijection, so $\{A_n\}_{n\in\omega}$ is a partition of $\mathbb{R}$. □

Theorem 3.4. *Let* N *be a positive natural number. There exists a partition* $\{A_n\}_{n<N}$ *of the real line into completely nonmeasurable sets such that* $A_n + A_n = A_n$ *for every* $n < N$.

Proof. Fix a strictly increasing sequence of reals $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ satisfying

1. $\alpha_0 = 0$,
2. $\alpha_N = 2\pi$,

3. $(\forall n < N)(\alpha_{n+1} - \alpha_n < \pi)$.

The rest of the proof is the same as in Theorem 3.3. □

Theorem 3.5. *There exist countable partitions $(A_n)_{n\in\omega}$ of $\mathbb{R}\setminus\{0\}$ into countable completely many nonmeasurable sets such that*

$$\forall m, n \in \omega \;\; n \neq m \to A_m + A_n = \mathbb{R} \setminus \{0\}.$$

Proof. Let $(\alpha_n)_{n\in\omega}$ be any strictly increasing sequence of reals such that

1. $\alpha_0 = 0$,
2. $\lim_{n\to\infty} \alpha_n = \pi$.

Consider the following family of subsets of $\mathbb{R}^2$:

$$K_n = \{(r\cos\alpha, r\sin\alpha) : r \in \mathbb{R} \setminus \{0\} \wedge \alpha \in [\alpha_n, \alpha_{n+1}) \cup [\alpha_n + \pi, \alpha_{n+1} + \pi)\},$$

for any $n \in \omega$.

Here for every $m, n \in \omega,\ n \neq m$

$$K_m + K_n = \mathbb{R}^2 \setminus \{(0,0)\} \wedge \; int(K_n) \neq \emptyset.$$

Moreover $\{K_n\}_{n\in\omega}$ is a partition of $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$. For each n put $A_n = \Phi[K_n]$. By Corollary 3.1 A_n is completely nonmeasurable and since Φ preserves addition $A_m + A_n = \mathbb{R} \setminus \{0\}$ for any $m, n \in \omega,\ n \neq m$. Moreover Φ is a bijection, so $\{A_n\}_{n\in\omega}$ is a partition of $\mathbb{R} \setminus \{0\}$. □

Theorem 3.6. *Let N be a positive natural number. There exists a partition $\{A_n\}_{n<N}$ of $\mathbb{R}\setminus\{0\}$ into completely nonmeasurable sets such that $A_n + A_m = \mathbb{R} \setminus \{0\}$ for every $n, m < N,\ n \neq m$.*

Proof. Fix a strictly increasing sequence of reals $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ satisfying

1. $\alpha_0 = 0$,
2. $\alpha_N = \pi$.

The rest of the proof is the same as in Theorem 3.5. □

Theorem 3.7. *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A,\ A+A,\ A+A+A,\ A+A+A+A, \dots$$

is completely nonmeasurable and $\bigcup_{n\in\omega} \underbrace{A+A+\cdots+A}_{n} = \mathbb{R}$.

PROOF. Consider $K = \mathbb{R}\times(-\infty,1] \subseteq \mathbb{R}^2$. Here for $n \geq 1$

$$\underbrace{K+K+\cdots+K}_{n} = \mathbb{R}\times(-\infty,n].$$

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$. Put $A = \Phi[K]$. By Corollary 3.1 A is completely nonmeasurable and since Φ preserves addition $\underbrace{A+A+\cdots+A}_{n}$ is also completely nonmeasurable. The fact that $\bigcup_{n\in\omega} \underbrace{K+K+\cdots+K}_{n} = \mathbb{R}^2$ implies $\bigcup_{n\in\omega} \underbrace{A+A+\cdots+A}_{n} = \mathbb{R}$. □

Theorem 3.8. *Let N be a positive natural number. There exists completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A,\ A+A,\ A+A+A,\ A+A+A+A, \dots, \underbrace{A+A+\cdots+A}_{N}$$

is completely nonmeasurable and $\underbrace{A+A+\cdots+A}_{N+1} = \mathbb{R}$.

PROOF. Consider a set $K = \bigcup_{k\in\mathbb{Z}}[(N+1)k,(N+1)k+1)\times\mathbb{R}$. As before we put $A = \Phi[K]$. □

Theorem 3.9. *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A \subsetneq A+A \subsetneq A+A+A \subsetneq A+A+A+A \subsetneq \dots$$

is completely nonmeasurable and $\bigcup_{n\in\omega} \underbrace{A+A+\cdots+A}_{n}$ *is completely nonmeasurable.*

Proof. It is enough to consider the set $K = [0,1) \times [0,\infty)$. □

Theorem 3.10. *Let N be a positive natural number. There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A \subsetneq A + A \subsetneq \ldots \subsetneq \underbrace{A + A + \cdots + A}_{N} = \underbrace{A + A + \cdots + A}_{N+1}$$

is completely nonmeasurable.

Proof. Consider a set $K = \bigcup_{k\in\mathbb{Z}}[(N+1)k, (N+1)k+1) \times [0,\infty)$. As before we put $A = \Phi[K]$. □

Theorem 3.11. *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A \supsetneq A + A \supsetneq A + A + A \supsetneq A + A + A + A \supsetneq \ldots$$

is completely nonmeasurable.

Proof. Consider a set $K = [1,\infty) \times \mathbb{R}$. As before we put $A = \Phi[K]$. □

We can obtain similar results for multiplication as we have obtained for addition. Let us consider a function $\varphi(x) = e^x$. Having a completely nonmeasurable set $A \subseteq \mathbb{R}$ we get a set $\varphi[A]$ which is completely nonmeasurable in $\mathbb{R}^+$. Now, it is enough to put $B = \varphi[A] \cup -\varphi[A]$. All properties of a set A with respect to addition remain true for a set B with respect to multiplication.

So, we get a series of corollaries. Let us quote two examples.

Corollary 3.2 (of Theorem 3.2)**.** *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that $A \cdot A = A$.*

Corollary 3.3 (of Theorem 3.9)**.** *There exists a completely nonmeasurable set $A \subseteq \mathbb{R}$ such that each of the sets*

$$A \subsetneq A \cdot A \subsetneq A \cdot A \cdot A \subsetneq A \cdot A \cdot A \cdot A \subsetneq \ldots$$

is completely nonmeasurable and $\bigcup_{n\in\omega} \underbrace{A \cdot A \cdot \ldots \cdot A}_{n}$ is completely nonmeasurable.

4 Questions.

Let us start with the following question.

Question 4.1. *Is it possible to replace addition in Theorem 3.1 by any differentiable function with non-vanishing derivatives?*

We hope that Theorem 2.1 can help in proving the above proposition since the proof of Theorem 3.1 strongly uses the classical Steinhaus theorem.

From the other hand, we are interested in the following problem.

Question 4.2. *Is it possible to obtain the analogues of examples given in Theorems (Corollaries) from the previous section replacing addition (multiplication) by any differentiable function with non-vanishing derivatives?*

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