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**Real Analysis Exchange**
Vol. 34(1), 2008/2009, pp. 49–60

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# ON FUNCTIONS OF $(p, \alpha)$–BOUNDED VARIATION

**Abstract**

In this paper we introduce the concept of $(p, \alpha)$-bounded variation which generalizes the Riesz p-variation. The following result is proved: a function $f : [a, b] \to \mathbb{R}$ is of $(p, \alpha)$-bounded variation $(1 < p < \infty)$ if and only if $f$ is $\alpha$-absolutely continuous on $[a, b]$ and $f'_\alpha \in L_{(p,\alpha)}[a, b]$. Moreover it is shown that the $(p, \alpha)$-bounded variation of a function $f$ on $[a, b]$ is given by

$$V_{(p,\alpha)}(f) = \|f'_\alpha\|^p_{L_{(p,\alpha)}[a,b]}.$$

## 1 Introduction.

Two centuries ago, around 1880, C. Jordan (see [1]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones. Later, the concept of bounded variation was generalized in various directions.
In his 1910 paper F. Riesz (see [3]) defined the concept of bounded p-variation $(1 \le p < \infty)$ and proved that, for $1 < p < \infty$, this class coincides with the class of functions $f$, absolutely continuous with derivative $f' \in L_p[a, b]$. Moreover the p-variation of a function $f$ on $[a, b]$ is given by

$$V_p(f, [a, b]) = V_p(f) = \|f'\|^p_{Lp[a,b]}.$$

Mathematical Reviews subject classification: Primary: 26A45, 26B30; Secondary: 26A16, 26A24
Key words: Riesz p-variation, $(p, \alpha)$-bounded variation
Received by the editors July 19, 2007
Communicated by: Brian S. Thomson

In this paper we generalize the concept of bounded p-variation. In order to do that, we introduce a strictly increasing continuous function $\alpha : [a,b] \to \mathbb{R}$. Then, we consider the bounded p-variation with respect to $\alpha$. We called this new concept $(p,\alpha)$-bounded variation and denote it by $BV_{(p,\alpha)}[a,b]$.
This space is described as follows: let $f$ be a real function on $[a,b]$. For a given partition of the form $\pi : a = x_0 < x_1 < \ldots < x_n = b$ of $[a,b]$, we set

$$\sigma_{(p,\alpha)}(f,\pi) = \sum_{i=1}^{n} \frac{|f(x_i) - f(x_{i-1})|^p}{|\alpha(x_i) - \alpha(x_{i-1})|^{p-1}}$$

and

$$V_{(p,\alpha)}(f;[a,b]) = V_{(p,\alpha)}(f) = \sup_{\pi} \sigma_{(p,\alpha)}(f,\pi),$$

where the supremum is taken over all partitions $\pi$ of $[a,b]$. If $V_{(p,\alpha)}(f) < +\infty$, we say that $f$ is a function of $(p,\alpha)$-bounded variation. The class $BV_{(p,\alpha)}[a,b]$ is a normed space equipped with the norm

$$\|f\|_{(p,\alpha)} = |f(a)| + (V_{(p,\alpha)}(f))^{1/p}.$$

It is not to hard to show that $\left(BV_{(p,\alpha)}[a,b], \|\cdot\|_{(p,\alpha)}\right)$ is a Banach space.
Also, we generalize Riesz's Lemma, that is $f \in BV_{(p,\alpha)}[a,b]$ if and only if $f \in \alpha\text{-AC}[a,b]$ and $f^{'} \in L_{(p,\alpha)}[a,b]$. Moreover $V_{(p,\alpha)}(f) = \|f^{'}_{\alpha}\|_{L(p,\alpha)}[a,b]$ with $1 < p < \infty$.

## 2 Definitions and Notations.

In this section, we gather definitions and notations that will be used throughout the paper. Let $\alpha$ be any strictly increasing, continuous function defined on $[a,b]$.

**Definition 2.1.** Let $([a,b], A, \mu_\alpha)$ be a measure space equipped with the Lebesgue-Stieltjes measure. A measurable function $f : [a,b] \to \mathbb{R}$ is said to be in $L_{(p,\alpha)}[a,b]$ for $1 \le p < \infty$ if

$$\int_a^b |f|^p d\alpha < +\infty.$$

**Definition 2.2.** A function $f : [a,b] \to \mathbb{R}$ is said to be absolutely continuous with respect to $\alpha$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that if

$\{(a_j, b_j)\}_{j=1}^{n}$ are disjoint open subintervals of $[a, b]$, then

$$\sum_{j=1}^{n} |\alpha(b_j) - \alpha(a_j)| < \delta \quad \text{implies} \quad \sum_{j=1}^{n} |f(b_j) - f(a_j)| < \varepsilon.$$

Thus, the collection $\alpha$-AC$[a, b]$ of all $\alpha-$absolutely continuous functions on $[a, b]$ is a function space and an algebra of functions.

**Definition 2.3.** A function $f : [a, b] \to \mathbb{R}$ is said to be $\alpha$-Lipschitz if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|\alpha(x) - \alpha(y)|,$$

for all $x, y \in [a, b], x \neq y$ .

By $\alpha$-Lip $[a, b]$ we will denote the space of functions which are $\alpha$-Lipschitz. If $f \in \alpha$-Lip$[a, b]$ we define

$$Lip_{\alpha}(f) = inf\big\{M > 0 : |f(x) - f(y)| \leq M|\alpha(x) - \alpha(y)|, x \neq y \in [a, b]\big\}$$

and

$$Lip_{\alpha}^{'}(f) = \sup\left\{\frac{|f(x) - f(y)|}{|\alpha(x) - \alpha(y)} : x \neq y \in [a, b]\right\}.$$

It is not hard to prove that

$$Lip_{\alpha}(f) = Lip_{\alpha}^{'}(f).$$

$\alpha$-Lip$[a, b]$ equipped with the norm

$$\|f\|_{Lip_{\alpha}} = |f(a)| + Lip_{\alpha}(f)$$

is a Banach space.

**Definition 2.4.** Suppose $f$ and $\alpha$ are real-valued functions defined on the same open interval (bounded or unbounded). Suppose $x_0$ is a point in this interval. We say $f$ is $\alpha$-derivable at $x_0$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{\alpha(x) - \alpha(x_0)} \quad \text{exists.}$$

If the limit exists we denote its value by $f_{\alpha}^{'}(x_0)$, which we call the $\alpha$-derivative of $f$ at $x_0$.

## 3 Embeddings.

In this section we include several simple lemmas.

**Lemma 3.1.** *If $p > 1$, then*

$$\alpha\text{-}Lip[a,b] \hookrightarrow BV_{(p,\alpha)}[a,b].$$

Proof. Let $f \in \alpha$-Lip$[a,b]$ and $\pi : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. Then

$$\begin{aligned}\sigma_{(p,\alpha)}(f,\pi) &= \sum_{j=1}^{n} \frac{|f(x_j)-f(x_{j-1})|^p}{|\alpha(x_j)-\alpha(x_{j-1})|^{p-1}} \\ &= \sum_{j=1}^{n} \left|\frac{f(x_j)-f(x_{j-1})}{\alpha(x_j)-\alpha(x_{j-1})}\right|^p |\alpha(x_j)-\alpha(x_{j-1})| \\ &\le (Lip_\alpha(f))^p(\alpha(b)-\alpha(a)) < +\infty.\end{aligned}$$

Thus

$$V_{(p,\alpha)}(f) \le (Lip_\alpha(f))^p(\alpha(b)-\alpha(a)).$$

This completes the proof of Lemma 3.1. □

Take into account that

$$\|f\|_{(p,\alpha)} \le max\left\{1, (\alpha(b)-\alpha(a))^{\frac{1}{p}}\right\} \|f\|_{Lip_\alpha}.$$

**Lemma 3.2.** If $1 < q < p < +\infty$, then

$$BV_{(p,\alpha)}[a,b] \hookrightarrow BV_{(q,\alpha)}[a,b].$$

Proof. Let $f \in BV_{(p,\alpha)}[a,b]$ and $\pi : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. Next,

$$\begin{aligned}\sigma_{(q,\alpha)}(f,\pi) &= \sum_{j=1}^{n} \frac{|f(x_j)-f(x_{j-1})|^q}{|\alpha(x_j)-\alpha(x_{j-1})|^{q-1}} \\ &= \sum_{j=1}^{n} \frac{|f(x_j)-\alpha(x_{j-1})|^q}{\alpha(x_j)-\alpha(x_{j-1})|^{(p-1)\frac{q}{1}}} |\alpha(x_j)-\alpha(x_{j-1})|^{\frac{p-q}{p}}\end{aligned}$$

(and by Hölder's inequality)

$$\le \left[\sum_{j=1}^{n}\left(\frac{|f(x_j)-f(x_{j-1})|^q}{|\alpha(x_j)-\alpha(x_j)|^{(p-1)\cdot\frac{q}{p}}}\right)^{p/q}\right]^{q/p}\left[\sum_{j=1}^{n}\left(|\alpha(x_j)-\alpha(x_{j-1})|^{\frac{p-q}{p}}\right)^{\frac{p}{p-q}}\right]^{\frac{p-q}{p}}$$

$$=\left[\sum_{j=1}^{n}\frac{|f(x_j)-f(x_{j-1})|^p}{|\alpha(x_j)-\alpha(x_{j-1})|^{p-1}}\right]^{q/p}\cdot\left[\sum_{j=1}^{n}|\alpha(x_j)-\alpha(x_{j-1}|)\right]^{\frac{p-q}{p}}$$

$$=\left(\sigma_{(p,\alpha)}(f,\pi)\right)^{\frac{1}{p}}\cdot\left(\alpha(b)-\alpha(a)\right)^{\frac{p-q}{p}}.$$

Therefore

$$\left(\sigma_{(q,\alpha)}(f,\pi)\right)^{\frac{1}{q}}\leq\left(\alpha(b)-\alpha(a)\right)^{\frac{p-q}{pq}}\cdot\left(\sigma_{(p,\alpha)}(f,\pi)\right)^{\frac{1}{p}},$$

since $f\in BV_{(p,\alpha)}[a,b]$ and

$$\left(V_{(q,\alpha)}(f)\right)^{\frac{1}{q}}\leq[V_{(p,\alpha)}(f)]^{1/p}(\alpha(b)-\alpha(a))^{\frac{1}{q}-\frac{1}{p}}<\infty.$$

Therefore $f\in BV_{(q,\alpha)}[a,b]$

$$\|f\|_{(q,\alpha)}\leq\max\{1,(\alpha(b)-\alpha(a))^{\frac{1}{q}-\frac{1}{p}}\}\|f\|_{(p,\alpha)}.$$

This completes the proof of Lemma 3.2. □

**Corollary 3.1.** If $p\geq 1$, then

$$BV_{(p,\alpha)}[a,b]\hookrightarrow BV[a,b],$$

and

$$\|f\|_{BV}\leq\max\{1,(\alpha(b)-\alpha(a))^{\frac{p-1}{r}}\}\|f\|_{(p,\alpha)}.$$

**Lemma 3.3.** If $p\geq 1$, then

$$BV_{(p,\alpha)}[a,b]\subset\alpha\text{-AC}[a,b].$$

Proof. Let $f\in BV_{(p,\alpha)}[a,b]$ and let $(a_j,b_j), j=1,\ldots,n$ be pairwise disjoint subintervals of $[a,b]$. By Hölder's inequality we have

$$\begin{aligned}\sum_{j=1}^{n}|f(b_j)-f(a_j)|&\leq\left[\sum_{j=1}^{n}\left(\frac{|f(b_j)-f(a_j)|}{|\alpha(b_j)-\alpha(a_j)|^{\frac{p-1}{p}}}\right)^p\right]^{\frac{1}{p}}\left[\sum_{j=1}^{n}|\alpha(b_j)-\alpha(a_j)|\right]^{\frac{p-1}{p}}\\&=\left[\sum_{j=1}^{n}\frac{|f(b_j)-f(a_j)|^p}{|\alpha(b_j)-\alpha(a_j)|^{p-1}}\right]^{\frac{1}{p}}\left[\sum_{j=1}^{n}|\alpha(b_j)-\alpha(a_j)|\right]^{\frac{p-1}{p}}\\&\leq V_{(p,\alpha)}(f)\left[\sum_{j=1}^{n}|\alpha(b_j)-\alpha(a_j)|^p\right]^{\frac{p-1}{p}}.\end{aligned}$$

From this the proof follows easily. □

Finally, from Lemmas 3.1, 3.2 and 3.3 we have

$$\alpha\text{-Lip}[a,b] \subset BV_{(p,\alpha)}[a,b] \subset BV_{(q,\alpha)}[a,b] \subset \alpha\text{-AC}[a,b]$$

for $1 < q < p < \infty$.

## 4 Maligranda-Orlicz's Lemma.

The following lemma is due to L. Maligranda and W. Orlicz (see [2]). We will use this result to show that $BV_{(p,\alpha)}[a,b]$ becomes a normed Banach algebra with pointwise multiplication of function under an appropriate choice of norms.

**Lemma 4.1.** Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under multiplication of functions. Let us assume that $f \cdot g \in X$ and

$$\|fg\| \leq \|f\|_\infty \cdot \|g\| + \|f\| \cdot \|g\|_\infty$$

for any $f, g \in X$. Then the space $X$ equipped with the norm

$$\|f\|_1 = \|f\|_\infty + \|f\|$$

is a normed Banach algebra. Also, if $X \hookrightarrow B[a,b]$, Then the norms $\|\,.\,\|_1$ and $\|\,.\,\|$ are equivalent. Moreover, if $\|f\|_\infty \leq M\|f\|$ for $f \in X$, then $(X, \|\cdot\|_2)$ is a normed Banach algebra with $\|f\|_2 = 2M\|f\|$, $f \in X$ and the norms $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent.

**Lemma 4.2.** If $f, g \in BV_{(p,\alpha)}[a,b]$, then $f \cdot g \in BV_{(p,\alpha)}[a,b]$.

Proof. Let $\pi : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. Then

$$\sigma_{(p,\alpha)}(fg,\pi) = \sum_{j=1}^{n} \frac{|(fg)(x_j) - (fg)(x_{j-1})|^p}{|\alpha(x_j) - \alpha(x_{j-1})|^{p-1}}$$

$$\leq \sum_{j=1}^{n} \frac{\Delta_j \cdot |f(x_j)||g(x_j) - g(x_{j-1})|}{|\alpha(x_j) - \alpha(x_{j-1})|^{p-1}} + \sum_{j=1}^{n} \frac{\Delta_j \cdot |g(x_j)||f(x_j) - f(x_{j-1})|}{|\alpha(x_j) - \alpha(x_{j-1})|^{p-1}}$$

( where $\Delta_j = |f(x_j)(g(x_j) - g(x_{j-1})) + g(x_{j-1})(f(x_j) - f(x_{j-1}))|^{p-1}$ )

$$\leq \|f\|_\infty \sum_{j=1}^{n} \frac{|(fg)(x_j) - (fg)(x_{j-1})|^{p-1}}{|\alpha(x_j) - \alpha(x_{j-1})|^{\frac{(p-1)^2}{p}}} \cdot \frac{|g(x_j) - g(x_{j-1})|}{|\alpha(x_j) - \alpha(x_{j-1})|^{\frac{(p-1)}{p}}}$$
$$+ \|g\|_\infty \sum_{j=1}^{n} \frac{|(fg)(x_j) - (fg)(x_{j-1})|^{p-1}}{|\alpha(x_j) - \alpha(x_{j-1})|^{\frac{(p-1)^2}{p}}} \cdot \frac{|f(x_j) - f(x_{j-1})|}{|\alpha(x_j) - \alpha(x_{j-1})|^{\frac{(p-1)}{p}}}$$

(and by Hölder's inequality)

$$\leq \|f\|_\infty \left[\sum_{j=1}^{n}\left(\frac{|(fg)(x_j)-(fg)(x_{j-1})|^{p-1}}{|\alpha(x_j)-\alpha(x_{j-1})|^{\frac{(p-1)^2}{p}}}\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}$$

$$\times \left[\sum_{j=1}^{n}\left(\frac{|g(x_j)-g(x_{j-1})|}{|\alpha(x_j)-\alpha(x_{j-1})|^{\frac{(p-1)}{p}}}\right)^{p}\right]^{\frac{1}{p}}$$

$$+\ \|g\|_\infty \left[\sum_{j=1}^{n}\left(\frac{|(fg)(x_j)-(fg)(x_{j-1})|^{p-1}}{|\alpha(x_j)-\alpha(x_{j-1})|^{\frac{(p-1)^2}{p}}}\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}$$

$$\times \left[\sum_{j=1}^{n}\left(\frac{|f(x_j)-f(x_{j-1})|}{|\alpha(x_j)-\alpha(x_{j-1})|^{\frac{p-1}{p}}}\right)^{p}\right]^{\frac{1}{p}}$$

$$= \left[\sigma_{(p,\alpha)}(fg,\pi)\right]^{\frac{p-1}{p}} \left[\|f\|_\infty \left(\sigma_{(p,\alpha)}(g,\pi)\right)^{\frac{1}{p}} + \|g\|_\infty \left(\sigma_{(p,\alpha)}(f,\pi)\right)^{\frac{1}{p}}\right].$$

Thus,

$$\left(\sigma_{(p,\alpha)}(fg,\pi)\right)^{\frac{1}{p}} \leq \|f\|_\infty \left(\sigma_{(p,\alpha)}(g,\pi)\right)^{\frac{1}{p}} + \|g\|_\infty \left(\sigma_{(p,\alpha)}(f,\pi)\right)^{\frac{1}{p}}.$$

Therefore,

$$\left(V_{(p,\alpha)}(fg)\right)^{\frac{1}{p}} \leq \|f\|_\infty \left(V_{(p,\alpha)}(g)\right)^{\frac{1}{p}} + \|g\|_\infty \left(V_{(p,\alpha)}(f)\right)^{\frac{1}{p}}$$

and $fg \in BV_{(p,\alpha)}[a,b]$. This completes the proof of Lemma 4.2. □

**Theorem 4.1.**

*i)* $BV_{(p,\alpha)}[a,b]$ *equipped with the norm*

$$\|f\|_1 = \|f\|_\infty + \|f\|_{(p,\alpha)},\ f \in BV_{(p,\alpha)}[a,b]$$

*is a normed Banach algebra.*

*ii)* $BV_{(p,\alpha)}[a,b]$ *equipped with the norm*

$$\|f\|_2 = 2\max\{1,(\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\}\|f\|_{(p,\alpha)}, \quad f \in BV_{(p,\alpha)}[a,b]$$

*is a normed Banach algebra.*

*iii) The norms* $\|\cdot\|_{(p,\alpha)}$, $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent.*

Proof. i) We just need to check Maligranda-Orlicz's Lemma hypothesis. Indeed, from Lemma 4.2 we already know that if $f, g \in BV_{(p,\alpha)}[a,b]$, then $fg \in BV_{(p,\alpha)}[a,b]$ . Moreover

$$\begin{aligned}
\|fg\|_{(p,\alpha)} &= |(fg)(a)| + (V_{(p,\alpha)}(fg))^{\frac{1}{p}} \\
&\le 2|f(a)||g(a)| + \|f\|_\infty (V_{(p,\alpha)}(g))^{\frac{1}{p}} + \|g\|_\infty (V_{(p,\alpha)}(f))^{\frac{1}{p}} \\
&\le |f(a)|\|g\|_\infty + |g(a)|\|f\|_\infty + \|f\|_\infty (V_{(p,\alpha)}(g))^{\frac{1}{p}} \\
&\quad + \|g\|_\infty (V_{(p,\alpha)}(f))^{\frac{1}{p}} \\
&= \|f\|_\infty \|g\|_{(p,\alpha)} + \|g\|_\infty \|f\|_{(p,\alpha)}.
\end{aligned}$$

By Corollary 3.1 we have

$$\|f\|_{BV[a,b]} \le \max\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\}\|f\|_{(p,\alpha)}, \quad f \in BV_{(p,\alpha)}[a,b],$$

but $\|f\|_\infty \le \|f\|_{BV[a,b]}$, therefore

$$\|f\|_\infty \le \max\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\}\|f\|_{(p,\alpha)},\ f \in BV_{(p,\alpha)}[a,b].$$

(This shows that $BV_{(p,\alpha)}[a,b] \hookrightarrow B[a,b]$.) By the Maligranda-Orlicz Lemma, we have thus shown that $(BV_{(p,\alpha)}[a,b], \|\cdot\|_1)$ is a normed Banach algebra with $\|f\|_1 = \|f\|_\infty + \|f\|_{(p,\alpha)}, f \in BV_{(p,\alpha)}[a,b]$ and the norms $\|\cdot\|_1$ and $\|\cdot\|_{(p,\alpha)}$ are equivalent; since $\|f\|_\infty \le \max\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\}\|f\|_{(p,\alpha)}$ for $f \in BV_{(p,\alpha)}[a,b]$, we obtain immediately that $(BV_{(p,\alpha)}[a,b], \|\cdot\|_2)$ is a normed Banach algebra with $\|f\|_2 = 2\max\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\}\|f\|_{(p,\alpha)}$, $f \in BV_{(p,\alpha)}[a,b]$ and the norms $\|\cdot\|_2$ and $\|\cdot\|_{(p,\alpha)}$ are equivalent.
Finally, from (i) and (ii) we have (iii). This completes the proof of Theorem 4.1. □

## 5 Characterization of the Space $BV_{(p,\alpha)}[a,b]$.

In this section we generalize the characterization due to F. Riesz (see [3]). Our characterization is as follows:

**Theorem 5.1.** *For* $1 < p < +\infty$, $f \in BV_{(p,\alpha)}[a,b]$ *if and only if* $f \in \alpha$-$AC[a,b]$ *and* $f'_\alpha \in L_{(p,\alpha)}[a,b]$. *Moreover*

$$V_{(p,\alpha)}(f) = \|f'_\alpha\|^p_{L(p,\alpha)}[a,b].$$

Proof. Let $\pi : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. Next, the fundamental theorem of calculus holds true precisely for the $\alpha$-absolutely continuous function, thus by Hölder's inequality we have

$$\begin{aligned}|f(x_j) - f(x_{j-1})|^p &= \left|\int_{x_{j-1}}^{x_j} f_\alpha^{'}(t)\, d\alpha(t)\right|^p \\ &\leq \left(\int_{x_{j-1}}^{x_j} |f_\alpha^{'}(t)|\, d\alpha(t)\right)^p \\ &\leq \left(\int_{x_{j-1}}^{x_j} |f_\alpha^{'}(t)|\, d\alpha(t)\right)^p \left(\left[\int_{x_{j-1}}^{x_j} d\alpha(t)\right]^{\frac{p-1}{p}}\right)^p \\ &= |\alpha(x_j) - \alpha(x_{j-1})|^{p-1} \int_{x_{j-1}}^{x_j} |f_\alpha^{'}(t)|^p\, d\alpha(t).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{|f(x_j) - f(x_{j-1})|^p}{|\alpha(x_j) - \alpha(x_{j-1})|^{p-1}} &\leq \int_{x_{j-1}}^{x_j} |f_\alpha^{'}(t)|^p\, d\alpha(t) \\ \sum_{j=1}^{n} \frac{|f(x_j) - f(x_{j-1})|^p}{|\alpha(x_j) - \alpha(x_{j-1})|^{p-1}} &\leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |f_\alpha^{'}(t)|^p\, d\alpha(t) \\ &= \int_a^b |f_\alpha^{'}(t)|^p\, d\alpha(t) \\ &= \|f_\alpha'\|_{L_{(p,\alpha)[a,b]}}^p.\end{aligned}$$

Since this holds for any partition of $[a,b]$ we obtain

$$V_{(p,\alpha)}(f) \leq \|f_\alpha'\|_{L_{(p,\alpha)[a,b]}}^p < +\infty. \tag{5.1}$$

Thus $f \in BV_{(p,\alpha)}[a,b]$.
If $f \in BV_{(p,\alpha)}[a,b]$, then $f$ is $\alpha-$absolutely continuous; by Lemma 3.3 $f_\alpha^{'}$ exists almost everywhere on $[a,b]$. For each $n \in \mathbb{N}$ we have to consider the partition of the interval $[a,b]$

$$\pi_n : a = x_{0,n} \leq x_{1,n} < \ldots < x_{n,n} = b$$

defined by

$$x_{i,n} = a + i\left(\frac{b-a}{n}\right), i = 0, 1, \ldots, n.$$

Next, let $(f_n)_{n\in\mathbb{N}}$ be the sequence of step functions given by

$$f_n(x) = \begin{cases} \dfrac{f(x_{i+1,n}) - f(x_{i,n})}{\alpha(x_{i+1,n}) - \alpha(x_{i,n})} & \text{if } x_{i,n} \leq x < x_{i+1,n} \quad i = 0, \cdots, n-1 \\ \text{any value} & \text{if } x = b. \end{cases} \tag{5.2}$$

Now, we are going to show that $(f_n)_{n\in\mathbb{N}}$ converges to $f_\alpha^{'}$ almost everywhere on $[a,b]$. In order to do that, let us consider the following set:

$$A = \{x \in [a,b] : f_\alpha^{'}(x) \quad exists\} \setminus \{x_{i,n} : n \in \mathbb{N},\ i = 0, 1, \ldots, n\}.$$

Let $x \in A$; then, for each $n \in \mathbb{N}$ there exists $k \in \{0, \ldots, n-1\}$ such that $x_{k,n} \leq x < x_{k+1,n}$; thus

$$\begin{aligned} f_n(x) &= \frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \\ &= \frac{\alpha(x_{k+1,n}) - \alpha(x)}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \cdot \frac{f(x_{k+1,n}) - f(x)}{\alpha(x_{k+1}) - \alpha(x_k)} \\ &\quad + \frac{\alpha(x) - \alpha(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \cdot \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}. \end{aligned}$$

Next, one can see that $f_n(x)$ is a convex combination of the points

$$\frac{f(x_{k+1,n}) - f(x)}{\alpha(x_{k+1,n}) - \alpha(x)} \quad \text{and} \quad \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})},$$

since

$$\frac{\alpha(x_{k+1,n}) - \alpha(x)}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} + \frac{\alpha(x) - \alpha(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} = 1.$$

Now, $x_{k,n} \to x$ and $x_{k+1,n} \to x$ as $n \to \infty$. Then $\lim_{n\to\infty} f_n(x) = f_\alpha^{'}(x)$, $x \in A$.

By Fatou's Lemma we obtain

$$\begin{aligned}
\int_a^b \left|f_\alpha'(x)\right|^p \, d\alpha(x) &= \int_a^b |\liminf f_n(x)|^p \, d\alpha(x)\\
&\le \liminf \int_a^b |f_n(x)|^p \, d\alpha(x)\\
&= \liminf \sum_{i=0}^{n-1} \int_{x_{i,n}}^{x_{i+1,n}} |f_n(x)|^p \, d\alpha(x)\\
&= \liminf \sum_{i=0}^{n-1} \int_{x_{i,n}}^{x_{i+1,n}} \left|\frac{f(x_{i+1,n})-f(x_{i,n})}{\alpha(x_{i+1,n})-\alpha(x_{i,n})}\right|^p \, d\alpha(x)\\
&= \liminf \sum_{i=0}^{n-1} \frac{|f(x_{i+1,n})-f(x_{i,n})|}{|\alpha(x_{i+1,n})-\alpha(x_{i,n})|^p} \int_{x_{i,n}}^{x_{i+1,n}} \, d\alpha(x)\\
&= \liminf \sum_{i=0}^{n-1} \frac{|f(x_{i+1,n})-f(x_{i,n})|^p}{|\alpha(x_{i+1,n})-\alpha(x_{i,n})|^p} |\alpha(x_{i+1,n})-\alpha(x_{i,n})|\\
&= \liminf \sum_{i=0}^{n-1} \frac{|f(x_{i+1,n})-f(x_{i,n})|^p}{|\alpha(x_{i+1,n})-\alpha(x_{i,n})|^{p-1}}\\
&\le V_{(p,\alpha)}(f) < +\infty. \qquad (5.3)
\end{aligned}$$

Therefore $f_\alpha' \in L_{(p,\alpha)}[a,b]$.
Finally, by 5.1 and 5.3 we have

$$V_{(p,\alpha)}(f) = \|f_\alpha'\|_{L_{(p,\alpha)}}[a,b].$$

Hence, the proof of Theorem 5.1 is now complete. □

**Remark 5.1.** For $p = 1$, then the conclusion of Theorem 5.1 does not remain valid, because there is a bounded variation function which is neither continuous nor $\alpha-$absolutely continuous.

## References


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