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ON FINITELY CONTINUOUS DARBOUX FUNCTIONS AND STRONG FINITELY CONTINUOUS FUNCTIONS

Abstract

Properties of the families of finitely continuous and strong finitely continuous functions are investigated. We show that the Darboux property implies continuity of strong finitely continuous functions and that the family DB_1^{**} is superporous in the space of all finitely continuous functions with the Darboux property.

1 Notions.

We apply standard symbols and notions. By $\mathbb{R}$ we denote the set of real numbers. By $\mathbb{Q}$ ($\mathbb{N}$) we denote the set of rational numbers (positive integers). For a metric space Z, $z \in Z$ and $R > 0$ by $B_Z(z,R)$ (or simply $B(z,R)$) we denote an open ball with center z and radius R. Let $A \subset Z$, $z \in Z$, $R > 0$. If there does not exist $y \in Z$ and $r > 0$ such that $B(y,r) \subset B(z,R) \setminus A$, then let $\gamma(z,R,A) = 0$. Otherwise let

$$\gamma(z,R,A) = \sup\{r > 0 : \exists_{y\in Z} B(y,r) \subset B(z,R) \setminus A\}.$$

If

$$\limsup_{R\to 0^+} \frac{\gamma(z,R,A)}{R} > 0,$$

then we say that A is porous at z. We say that A is superporous in z, if for every set $B \subset Z$, porous at z, the set $A \cup B$ is porous at z. If f is a function,

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then by C_f (D_f) we denote the set of all continuity (discontinuity) points of the function f. No distinction is made between a function and its graph.

Let X be a metric space. The function $f \in \mathbb{R}^X$ is said to be Darboux if, for every connected set $A \subset X$, the image $f(A)$ is a connected subset of $\mathbb{R}$ (i.e., an interval). The class $\mathcal{F} \subset \mathbb{R}^X$ is an ordinary system (in the sense of Aumann) if $\mathcal{F}$ contains all constants and for $f, g \in \mathcal{F}$, $\max(f,g) \in \mathcal{F}$, $\min(f,g) \in \mathcal{F}$, $f+g \in \mathcal{F}$, $f \cdot g \in \mathcal{F}$, and (if $\{x : g(x) = 0\} = \emptyset$) $\frac{f}{g} \in \mathcal{F}$.

If $f \in \mathbb{R}^{\mathbb{R}}$, $x \in \mathbb{R}$, then let $L^+(f,x)$, $L^-(f,x)$, and $L(f,x)$ denote the following limit sets: $L^+(f,x)$ denotes the set of all $\alpha \in \widetilde{\mathbb{R}}$ such that there exists a sequence $(x_n)_{n=1}^{\infty}$ ($x_n > x$, $n = 1, 2 \ldots$) for which $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} f(x_n) = \alpha$. Similarly $L^-(f,x)$ denotes the set of all $\alpha \in \widetilde{\mathbb{R}}$ such that there exists a sequence $(x_n)_{n=1}^{\infty}$ ($x_n < x$, $n = 1, 2 \ldots$) for which $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} f(x_n) = \alpha$. Finally, $L(f,x) = L^+(f,x) \cup L^-(f,x)$.

Let $\mathcal{A}$ be a covering of a metric space X (i.e., $\bigcup \mathcal{A} = X$). The function $f \in \mathbb{R}^X$ is said to be *$\mathcal{A}$-continuous* if, for all $A \in \mathcal{A}$, the restriction $f \restriction A$ is continuous. The function $f \in \mathbb{R}^X$ is said to be *n-continuous* (*finitely continuous*), if there exists a covering $\mathcal{A}$ of X such that $card(\mathcal{A}) = n$ ($card(\mathcal{A}) < \omega$), and f is $\mathcal{A}$-continuous.

R. Pawlak in [3] introduced the notions of the class of functions B_1^{**}, intermediate between the family of continuous functions and the class of Baire*1 functions. We say that the function f belongs to the class B_1^{**}, if either $D_f = \emptyset$ or $f \restriction D_f$ is a continuous function.

If $f \in \mathbb{R}^{\mathbb{R}}$ is a Darboux function, then f is 2-continuous if and only if f belongs to the class B_1^{**} (see [1]).

Denote by the symbol $\mathcal{C}$ ($\mathcal{DB}_1^{**}$ respectively) the set of all bounded continuous functions from $\mathbb{R}^{\mathbb{R}}$ (all bounded Darboux functions from $\mathbb{R}^{\mathbb{R}}$ belonging to the class $\mathcal{B}_1^{**}$), with the metric of the uniform convergence. H. Pawlak [4] proved that $\mathcal{C} \subset_s \mathcal{DB}_1^{**}$; i.e., $\mathcal{C}$ is a superporous subset of $\mathcal{DB}_1^{**}$

Denote by the symbol $\mathcal{DB}_1\mathcal{C}^*$ ($\mathcal{DC}^*$, respectively) the set of all bounded finitely continuous Darboux functions from $\mathbb{R}^{\mathbb{R}}$ belonging to the class $\mathcal{B}_1$ (the set of all bounded finitely continuous Darboux functions from $\mathbb{R}^{\mathbb{R}}$).

Theorem 1. $\mathcal{C} \subset_s \mathcal{DB}_1^{**} \subset_s \mathcal{DB}_1\mathcal{C}^* \subset_s \mathcal{DC}^*$

PROOF. We show first that

$$\mathcal{DB}_1^{**} \subset_s \mathcal{DB}_1\mathcal{C}^*. \tag{1}$$

Let $f \in \mathcal{DB}_1\mathcal{C}^*$ and $\mathcal{Z} \subset \mathcal{DB}_1\mathcal{C}^*$ be porous at f. We shall prove that $\mathcal{DB}_1^{**} \cup \mathcal{Z}$ is porous at f.

Because $\mathcal{Z}$ is porous at f,

$$\limsup_{S \to 0^+} \frac{\gamma(f, S, \mathcal{Z})}{S} > 0. \tag{2}$$

Let $R > 0$. We shall show that $\gamma(f, R, \mathcal{DB}_1^{**} \cup \mathcal{Z}) \geq \gamma(f, R, \mathcal{Z})/16$. According to (2), there exists $g \in \mathcal{DB}_1\mathcal{C}^*$ and $r_1 > \gamma(f, R, Z)/2 > 0$ such that

$$B(g, r_1) \subset B(f, R) \setminus \mathcal{Z}. \tag{3}$$

Let $r = \frac{r_1}{8}$. Of course $r > \gamma(f, R, B)/16$. We shall show that there exists $h \in \mathcal{DB}_1\mathcal{C}^*$ such that

$$B(h, r) \subset B(f, R) \setminus (\mathcal{DB}_1^{**} \cup \mathcal{Z}). \tag{4}$$

Because g is finitely continuous and Darboux, the set of discontinuities of g is nowhere dense [1], hence $\operatorname{int}(C_g)$ is dense in $\mathbb{R}$. Let $x_0 \in \operatorname{int}(C_g)$ and a, b be such that $a < x_0 < b$, $[a, b] \subset intC_g$, $g([a, b]) \subset (g(x_0) - r, g(x_0) + r)$. Let C be a classical Cantor set on $[a, b]$, C° be the set of bilateral condensation points of C, $c_0 \in C^\circ$, $\big((a_n, b_n)\big)_{n=1}^{\infty}$ be a sequence of components of $[a, b] \setminus C$, $c_n = \frac{a_n + b_n}{2}$ for $n = 1, 2, \ldots$. Let ℓ_n, κ_n, $n = 1, 2, \ldots$, be functions linear on $[a_n, c_n]$ and on $[c_n, b_n]$ such that $\ell_n(a_n) = \ell_n(b_n) = g(x_0) - 3r$, $\ell_n(c_n) = g(x_0) + 3r$ $\kappa_n(a_n) = \kappa_n(b_n) = g(x_0) + 3r$, $\kappa_n(c_n) = g(x_0) - 3r$. Let $N_1 = \{n \in \mathbb{N} : b_n < c_0\}$, $N_2 = \{n \in \mathbb{N} : a_n > c_0\}$. Define

$$h(x) = \begin{cases} g(x), & x \notin (a, b) \\ \ell_n(x), & x \in [a_n, b_n], n \in N_1 \\ g(x_0) - 3r, & x \in C^\circ \cap (a, c_0] \\ \kappa_n(x), & x \in [a_n, b_n], n \in N_2 \\ g(x_0) + 3r, & x \in C^\circ \cap (c_0, b) \end{cases}$$

Then $h \restriction ((a, b) \setminus C)$, $h \restriction (C \cap (a, c_0])$ and $h \restriction (C \cap (c_0, b))$ are continuous. Moreover $h \restriction (\mathbb{R} \setminus (a, b)) = g \restriction (\mathbb{R} \setminus (a, b))$ and g is finitely continuous. Therefore h is finitely continuous.

Because $g \in \mathcal{D}$, it follows that h has the Darboux property on $(-\infty, a)$ and (b, ∞). If I is an interval, $I \subset (a, b)$, then $h(I) = \ell_n(I)$ or $h(I) = \kappa_n(I)$ for some $n \in \{1, 2, \ldots\}$ or $h(J) = [g(x_0) - 3r, g(x_0) + 3r]$, so $h(J)$ is an interval. Thus h has the Darboux property on (a, b). Moreover h is left continuous at the point a, right continuous at the point b and

$$h(a) \in L^+(h, a) = [g(x_0) - 3r, g(x_0) + 3r] = L^-(f, b) \ni h(b).$$

So, h is a Darboux function.

Observe that $h \in \mathcal{B}_1$. Indeed, let K be a perfect set. If there exists $x \in K \cap C_h$, then $h \restriction K$ is continuous at x. We can assume, that $K \subset D_h$. Because $h \in \mathcal{DC}^*$, we have that D_h is a nowhere dense set, therefore K is a boundary set. Thus there exists $k \in \{1,2,3,4\}$ and $u < v$ such that $F = K \cap [u,v] = K \cap (u,v) \subset J_k$, where $J_1 = (-\infty, a)$, $J_2 = (b, \infty)$, $J_3 = (a, c_0)$, $J_4 = (c_0, b)$. It enough to show that $h \restriction F$ has a point of continuity. If $k \in \{1,2\}$, it is a consequence of $h \restriction F = g \restriction F$ and $g \in \mathcal{B}_1$. If $k \in \{3,4\}$, then $h \restriction F$ is continuous.

If $\xi \in B(h,r)$, then, for $x \in \mathbb{R}$, $|\xi(x) - g(x)| \leq |\xi(x) - h(x)| + |h(x) - g(x)| < 6r$ thus $B(h,r) \subset B(g, 6r) = B\left(g, \frac{3r_1}{4}\right)$, so, by (3),

$$B(h,r) \subset B(f,R) \setminus \mathcal{Z}. \tag{5}$$

Let $\zeta \in B(h,r)$. Then $C \subset [a,b] \cap D_\zeta$ and $\zeta \restriction D_\zeta$ is discontinuous at c_0. Therefore $\zeta \notin B_1^{**}$ and we obtain that

$$B(h,r) \cap \mathcal{DB}_1^{**} = \emptyset \tag{6}$$

From (6) and (5) we have (4) and the proof of (1) is finished.

Now we similarly prove that $\mathcal{DB}_1\mathcal{C}^* \subset_s \mathcal{DC}^*$. Let $f \in \mathcal{DC}^*$ and $\mathcal{Z} \subset \mathcal{DC}^*$ be porous at f. Because $\mathcal{Z}$ is porous at f we have $\limsup_{S \to 0^+} \frac{\gamma(f,S,\mathcal{Z})}{S} > 0$. Let $R > 0$. Then there exist $g \in \mathcal{DC}^*$ and $r_1 > \frac{\gamma(f_1,R,Z)}{2} > 0$ such, that $B(g_1, r_1) \subset B(f_1, R) \setminus \mathcal{Z}$. Let $r = \frac{r_1}{8}$.

Because g is finitely continuous and Darboux so $\operatorname{int}(C_g)$ is dense in $\mathbb{R}$. Let $x_0 \in \operatorname{int}(C_g)$ and let (as above) a, b be such that $a < x_0 < b$, $[a,b] \subset \operatorname{int}(C_g)$, $g([a,b]) \subset (g(x_0) - r, g(x_0) + r)$. Let C be a classical Cantor set on $[a,b]$, $\big((a_n, b_n)\big)_{n=1}^{\infty}$ be a sequence of components of $[a,b] \setminus C$, and C° be the set of bilateral condensation points of C. Let ℓ_n, (where $n = 1, 2, \ldots$) be linear functions such that $\ell_n(a_n) = g(x_0) - 3r$, $\ell_n(b_n) = g(x_0) + 3r$.

Define

$$h(x) = \begin{cases} g(x), & x \notin (a,b) \\ \ell_n(x), & x \in [a_n, b_n],\ n = 1, 2, \cdots \\ g(x_0) + 3r, & x \in C^\circ \end{cases}$$

It is easy to see that $h \in \mathcal{DC}^*$ and $h \in B(h,r) \subset B\left(g, \frac{3r_1}{4}\right) \subset B(f,R) \setminus \mathcal{Z}$. Observe that if $\zeta \in B(h,r)$, then $\zeta \restriction C$ has no continuity point. Therefore $\zeta \notin B_1$ and in consequence $B(h,r) \subset B(f,R) \setminus (\mathcal{DB}_1\mathcal{C}^* \cup \mathcal{Z})$. $\square$

2 Strong Finitely Continuous Functions.

The function $f \in \mathbb{R}^X$ is said to be *strong n-continuous* iff there exist continuous functions $f_1, \ldots, f_n \in \mathbb{R}^X$ such that $f \subset \bigcup_{k=1}^{n} f_k$. Of course, if f is strong

n-continuous, then it is n-continuous, because $X = \bigcup_{k=1}^{n} A_k$ where $A_k = \{x : f(x) = f_k(x)\}$, for $k = 1, 2, \ldots, n$. The function is said to be *strong finitely continuous*, if it is strong n-continuous for some n.

The family of finitely continuous, real-valued functions is an ordinary system in the sense of Aumann [1]. In [2] the authors proved that the family of functions with graph contained in the union of the graphs of a sequence of continuous functions is also closed with respect to algebraic and lattice operations. Similarly, as in [2], we obtain:

Proposition 1. *If f, g are strong finitely continuous, then $f + g$, $f \cdot g$, $\min(f, g)$, and $\max(f, g)$ are strong finitely continuous.*

We say that f is locally bounded iff for every $x \in \mathbb{R}$ there exists $M > 0$ and an open interval I such that $x \in I$ and $f(t) \in (-M, M)$ for $t \in I$. Of course:

Proposition 2. *If f is strong finitely continuous, then it is locally bounded.*

Proposition 3. *The family of strong finitely continuous functions (from $\mathbb{R}^{\mathbb{R}}$) is not an ordinary system in the sense of Aumann.*

PROOF. Let $f(x) = 1$ for $x \in \mathbb{R}$, $g(x) = x$, for $x \neq 0$ and $g(0) = 1$. Then f, g are strong finitely continuous and $\{x : g(x) = 0\} = \emptyset$. Let $h = f/g$. Then $h(x) = 1/x$ for $x \neq 0$ and $h(0) = 1$, so h is not strong finitely continuous by Proposition 2. □

3 Finitely Continuous Functions with a Nowhere Dense Set of Discontinuity Points.

The finitely continuous function (as well as the strong finitely continuous functions) can be everywhere discontinuous. For example $f \in \mathbb{R}^{\mathbb{R}}$, $f(x) = 1$ for $x \in \mathbb{Q}$ $f(x) = 0$ for $x \notin \mathbb{Q}$. On the other hand if f is finitely continuous and Darboux, then the set of discontinuity points is nowhere dense. For strong finitely continuous functions, the Darboux property is enough to imply continuity everywhere.

Theorem 2. *If f is strong finitely continuous and Darboux, then it is continuous.*

PROOF. Fix $a \in \mathbb{R}$. Let $g_1, \ldots, g_n : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f \subset \bigcup_{k=1}^{n} g_k$. Suppose that f is not continuous at a. Then there exists $p \in L(f, a)$, $p \neq f(a)$. We can assume that $p < f(a)$. (By Proposition 2 we have $-\infty < p < \infty$.) Let $\alpha \in (p, f(a)) \setminus \{g_i(a), i = 1, \ldots, n\}$. Denote $d_0 = |\alpha - p|$, $d_i = |\alpha - g_i(a)|$, $i = 1, \ldots, n$ and let $\epsilon = \min(d_0, d_1, \ldots, d_n)$.

For $i \in \{1, \ldots, n\}$, from the continuity g_i, we infer that there exists $\delta_i > 0$ such that $g_i((a - \delta_i, a + \delta_i)) \subset (g_i(a) - \epsilon, g_i(a) + \epsilon)$. Let $I = (a - \delta, a + \delta)$ where $\delta = \min\{\delta_1, \ldots, \delta_n\}$. There exists $y \in I$ such that $|p - f(y)| < \epsilon$. Then $f(y) < \alpha < f(a)$ and by the Darboux property there exists $z \in I$ such that $f(z) = \alpha$. Because $f \subset \bigcup_{k=1}^{n} g_k$, there exists $i \in \{1, \ldots, n\}$ such that $f(z) = g_i(z)$. Then $|g_i(z) - g_i(a)| = |\alpha - g_i(a)| = d_i \geq \epsilon$. This is a contradiction. $\square$

Corollary 1. *Let $f \in \mathbb{R}^{\mathbb{R}}$. If there exists $(a, b) \subset \mathbb{R}$ such that $f \restriction (a, b)$ is Darboux and $(a, b) \setminus C_f \neq \emptyset$, then f is not strong finitely continuous.*

Observe that f is a finitely continuous function with a nowhere dense set of discontinuity points iff there exists a covering $\{A_1, \ldots, A_n\}$ of X such that $\bigcup_{i=1}^{n} \operatorname{int}(A_i)$ is dense in X and $f \restriction A_i$ is continuous. Indeed, if $B_1, \ldots, B_k$ are such that $\bigcup_{k=1}^{k} B_i = X$ and $f \restriction B_i$ is continuous, for $i = 1, \ldots, k$, then it is enough to take $n = k + 1$, $A_1 = C_f$ and $A_i = B_{i-1}$, (for $i = 2, \ldots, n$). If there exists a covering $A_1, \ldots, A_n$ of X such that $\bigcup_{i=1}^{n} \operatorname{int}(A_i)$ is dense in X and $f \restriction A_i$ is continuous, then $\bigcup_{i=1}^{n} \operatorname{int}(A_i) \subset \operatorname{int}(C_f)$, hence $\operatorname{int}(C_f)$ is dense in X.

In [5] the authors introduced the notion of *almost semi open step functions.* A partition $\mathcal{P} = \{P\iota : \iota \in T\}$ of a topological space X is called almost semi-open if the union $\bigcup_{\iota \in T} \operatorname{int}(P_\iota)$ is dense in X. An almost semi-open step function is understood to be a real-valued function ϕ on X which is piecewise constant on the partition elements of an almost semi-open partition $\mathcal{P}$. We have:

Proposition 4. [5] *Let $f \in \mathbb{R}^{\mathbb{R}}$ be a real-valued, bounded, cliquish function on a compact metrizable space X. Then there exists a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of finite almost semi-open partitions of the space X, such that f can be attained as the uniform limit of a sequence of almost semi-open step functions defined on K.*

Denote by the symbol $\mathcal{N}_d$ ($\mathcal{N}_d\mathcal{C}^{**}$, $\mathcal{N}_d\mathcal{C}^{*}$ respectively) the set of all bounded functions from $\mathbb{R}^{\mathbb{R}}$ with a nowhere dense set of discontinuity points (all bounded strong finitely continuous functions belonging to the class $\mathcal{N}_d$, all bounded finitely continuous functions belonging to the class $\mathcal{N}_d$), with the metric of uniform convergence.

Semi-open step functions defined on a finite almost semi-open partition of the space X are strong finitely continuous and have a nowhere dense set of discontinuity points. Moreover, every bounded real-valued cliquish function is a uniform limit of a sequence of such functions. Thus the class $\mathcal{N}_d\mathcal{C}^{**}$ is a dense subset of $\mathcal{N}_d$.

Let $\mathcal{A} \subset_{db} \mathcal{B}$ mean that the class $\mathcal{A}$ is a boundary and a dense subset of $\mathcal{B}$.

Theorem 3. $\mathcal{N}_d\mathcal{C}^{**} \subset_{db} \mathcal{N}_d\mathcal{C}^* \subset_{db} \mathcal{N}_d$

PROOF. Let $f \in \mathcal{N}_d\mathcal{C}^*$ and $\epsilon > 0$. We show that there exists a finitely continuous function g with a nowhere dense set of discontinuity points such that $g \in B(f,\epsilon) \setminus \mathcal{C}^{**}$. Let $x_0 \in \operatorname{int}(C_f)$ and a,b,c,d be such that $c < a < x_0 < b < d$, $[c,d] \subset C_f$, $f([c,d]) \subset (f(x_0) - \frac{\epsilon}{2}, f(x_0) + \frac{\epsilon}{2})$. Let $w(x_0) = f(x_0)$, $w(x) = \frac{\epsilon}{2}\sin(\frac{1}{x-x_0}) + f(x_0)$, for $x \neq x_0$. Let s_1, s_2, be linear functions such that $s_1(c) = f(c), s_1(a) = w(a)$, $s_2(b) = w_0(b), s_2(d) = f(d)$. Define

$$g(x) = \begin{cases} f(x), & x \notin [c,d], \\ s_1(x), & x \in [c,a], \\ w(x), & x \in (a,b), \\ s_2(x), & x \in [b,d]. \end{cases}$$

Of course $g \in \mathcal{N}_d\mathcal{C}^*$ and $g \in B(f,\epsilon)$. By Corollary 1 g is not strong finitely continuous.

Now let $f \in \mathcal{N}_d$ and $\epsilon > 0$. We show that there exists a function g with a nowhere dense set of discontinuity points such that $g \in B(f,\epsilon) \setminus \mathcal{C}^*$. Let $x_0 \in \operatorname{int}(C_f)$ and a,b be such that $a < x_0 < b$, $[a,b] \subset C_f$, $f([a,b]) \subset (f(x_0) - \frac{\epsilon}{2}, f(x_0) + \frac{\epsilon}{2})$. Let C be a classical Cantor set on $[a,b]$, $\big((a_n,b_n)\big)_{n=1}^{\infty}$ be a sequence of components of $[a,b] \setminus C$ and C° be the set of bilateral condensation points of C. Then there exists a sequence $(C_n)_{n=1}^{\infty}$ such that C_n is dense in C°, $C_n \cap C_k = \emptyset$ for $n \neq k$, $(n,k = 1,2,\ldots)$ and $C^\circ = \bigcup_{n=1}^{\infty} C_n$. Let ℓ_n, (where $n = 1,2,\ldots$) be linear functions satisfying $\ell_n(a_n) = f(x_0) - \frac{\epsilon}{2}, \ell_n(b_n) = f(x_0) + \frac{\epsilon}{2}$ and let $(q_n)_{n=1}^{\infty}$ be a sequence of all rational numbers from $(f(x_0) - \frac{\epsilon}{2}, f(x_0) + \frac{\epsilon}{2})$.

Define

$$g(x) = \begin{cases} f(x), & x \notin (a,b), \\ \ell_n(x), & x \in [a_n,b_n], n = 1,2,\cdots, \\ q_n, & x \in C_n, n = 1,2,\cdots, \end{cases}$$

Of course, $g \in \mathcal{N}_d$ and $g \in B(f,\epsilon)$. Suppose that g is finitely continuous. Then it is n-continuous for some n. There exists $A_1,\ldots,A_n$ such that $C^\circ = \bigcup_{k=1}^{n} A_k$ and $g \restriction A_i$ is continuous. Let $r_1,\ldots,r_{n+1} \in K = (f(x_0) - \frac{\epsilon}{2}, f(x_0) + \frac{\epsilon}{2})$ be rational, $(r_i \neq r_j$ for $i \neq j)$. Let $L_i = f^{-1}(r_i) \cap C^\circ$ and $d > 0$ be such that $J_i = (r_i - d, r_i + d) \subset K$, $(i = 1,\ldots,n)$, are pairwise disjoint. Then L_i, $(i = 1,\ldots,n)$ are dense in C°. Therefore there exist open intervals $I_n \subset \ldots \subset I_1 \subset (a,b)$ and $k_1,\ldots,k_n \in \{1,\ldots,n\}$ such that $A_{k_i} \cap I_i \cap L_i \neq \emptyset$

and $f(A_{k_i} \cap I_i) \subset J_i$, (for $i = 1, \ldots, n$). Let $I = I_n$. Then $I \cap C^\circ \neq \emptyset$ because $I \cap L_n \neq \emptyset$. Thus we have that L_{n+1} is dense in $C^\circ \cap I$. But

$$f(C^\circ \cap I) = f(\bigcup_{k=1}^{n} A_k \cap I) \subset \bigcup_{k=1}^{n} J_k$$

hence $L_{n+1} \cap C^\circ \cap I = \emptyset$. This is a contradiction. □

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