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ON FUNCTIONS DIFFERENTIABLE ON COMPLEMENTS OF COUNTABLE SETS

Abstract

We will prove the following.

Theorem. *If f is a nondecreasing function with zero derivative almost everywhere on $\mathbb{R}$, and if there are at most countably many points where f has an infinite derivate, then f is the sum of nondecreasing functions that assume no more than 3 values. Furthermore, in every open interval on which f is not constant, there is an open subinterval in which f has exactly one point of discontinuity and f assumes no more than 3 values.*

We will also prove the existence of a function g with zero derivative at every irrational point but whose range has the power of the continuum.

We mean by a *jump function* centered at a point u in the real line $\mathbb{R}$, a real valued function h on $\mathbb{R}$ that is constant on intervals $(-\infty,\, u)$ and $(u,\, \infty)$, and

$$h(u-) < h(u+),\ h(u-) \le h(u) \le h(u+).$$

In [P], George Piranian proved that if E is a countable G_δ-set in $\mathbb{R}$, then there exists a bounded nondecreasing function f on $\mathbb{R}$ such that

$$f'(x) = \begin{cases} \infty & \text{for } x \in E \text{ and} \\ 0 & \text{for } x \notin E. \end{cases}$$

He made f the sum of (countably many) jump functions centered at the points of E.

So now let f be a bounded nondecreasing function on $\mathbb{R}$ with zero derivative almost everywhere. Let the set

$$S = \left\{x \in \mathbb{R} : D^+ f(x) + D^- f(x) = +\infty\right\}$$

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be at most a countable set. (Here D^+f, D^-f, D_+f, D_-f are the usual Dini derivates of f.) In Theorem 1 we will prove that f is the sum of jump functions centered at (some of) the points of S. Moreover, we will prove that the set of points u contained in some interval $(u-\epsilon,\, u+\epsilon)$ such that f is constant on $(u-\epsilon,\, u)$ and on $(u,\, u+\epsilon)$, and $f(u-) < f(u+)$, must be dense in S. (The idea is that f is locally like a jump function around the point u.)

We begin with a lemma that may be familiar to readers interested in point-set topology.

Lemma 1. *Let X be a G_δ-subset of $\mathbb{R}$ and let Y be a nonvoid subset of X dense in itself. Then X has the power of the continuum.*

Proof. Let $X = \cap_n U_n$ where U_n is an open subset of $\mathbb{R}$. For each finite sequence z of 0's and 1's, we will use induction on the number of components of z to define a point $P(z) \in Y$ and a compact neighborhood $V(z)$ of $P(z)$.

Choose distinct points $P(0)$ and $P(1)$ in Y and choose disjoint compact neighborhoods $V(0)$ and $V(1)$ of $P(0)$ and $P(1)$, respectively, so that

$$V(0) \cup V(1) \subset U_1.$$

Now assume that $P(z) \in Y$ and $V(z)$ have been chosen for every sequence z with $n-1$ components $(n > 1)$. For b with $n-1$ components, choose distinct points $P(b,0)$ and $P(b,1)$ in $\big(\textit{interior } V(b)\big) \cap Y$ whose distance from $P(b)$ is less than 2^{-n}, and choose disjoint compact neighborhoods $V(b,0)$ and $V(b,1)$ of $P(b,0)$ and $P(b,1)$, respectively, such that

$$V(b,0) \;\cup\; V(b,1) \subset \Big(\text{interior}\, V(b)\Big) \cap U_n.$$

The induction is complete.

For any infinite sequence A of 0's and 1's, let $x_n = P(b)$ where b is the initial segment of A with n components. Clearly (x_n) is a Cauchy sequence. Moreover, x_j $(j \geq n)$ lie in a compact subset $V(b)$ of $U_1 \cap \ldots \cap U_n$, and it follows that the limit of (x_n) lies in $X = \cap_n U_n$.

If A and A' are different sequences of 0's and 1's, let b and b' be the shortest initial segments in which A and A' differ. Say b and b' have n components. Then there are disjoint compact sets $V(b)$ and $V(b')$ containing x_j $(j \geq n)$ and x'_j $(j \geq n)$, respectively, and the sequences associated with A and A' have different limits in X.

Finally, we have associated with each infinite sequence of 0's and 1's a unique point in X; no two such sequences are associated with the same point. There must be as many points in X as there are such sequences. But there are clearly continuum many such sequences. $\square$

This also has analogues where $\mathbb{R}$ is replaced by a complete metric space or a compact Hausdorff space. However, we will not address these issues here.

Our next lemma does not require monotone functions.

Lemma 2. *Let g be a real valued function (monotone or not) and let*

$$S = \Big\{x \in \mathbb{R} : \big|D^+g(x)\big| + \big|D^-g(x)\big| + \big|D_+g(x)\big| + \big|D_-g(x)\big| = \infty\Big\}.$$

Then S is a G_δ-set. (Here D^+, D^-, D_+, D_- denote the usual Dini derivates.)

PROOF. For each positive integer n, let

$$U_n = \Big\{u \in \mathbb{R} : \text{there is an } x \in \mathbb{R} \text{ such that } 0 < |x-u| < \frac{1}{n} \text{ and } \Big|\frac{g(u)-g(x)}{u-x}\Big| > n\Big\}.$$

Fix $u \in U_n$. We consider two cases.

CASE 1. g is continuous from the left at u.
It follows that there is a number $d > 0$ such that if $0 < u - y < d$, then

$$0 < |x-y| < \frac{1}{n} \text{ and } \Big|\frac{g(y)-g(x)}{y-x}\Big| > n.$$

It follows that $(u-d,\, u) \subset U_n$.

CASE 2. g is not continuous from the left at u.
There is a number $w < u$ such that

$$\Big|\frac{g(u)-g(w)}{u-w}\Big| > n \text{ and } |w-u| < \frac{1}{n}.$$

Now select $y \in (w,u)$. Say

$$\frac{g(u)-g(w)}{u-w} = r,\ \frac{g(u)-g(y)}{u-y} = s,\ \frac{g(y)-g(w)}{y-w} = t.$$

It follows that

$$g(u)-g(w) = r(u-w),\ g(u)-g(y) = s(u-y),\ g(y)-g(w) = t(y-w)$$

and hence

$$r(u-w) = s(u-y) + t(y-w) = r(u-y) + r(y-w).$$

So either

$$s(u-y) \geq r(u-y) \text{ and } t(y-w) \leq r(y-w)$$

or

$$s(u-y) \le r(u-y) \text{ and } t(y-w) \ge r(y-w).$$

But $u-w$, $u-y$, $y-w$ all have the same sign, so either

$$s \ge r \ge t \text{ or } s \le r \le t.$$

Hence, either

$$|t| \ge |r| > n \text{ or } |s| \ge |r| > n.$$

We conclude that $y \in U_n$. Thus $(w,u) \subset U_n$. In either Case 1 or 2, there is a $d > 0$ such that $(u-d,\, u) \subset U_n$.

By essentially the same argument (from the right) there is a number $d_1 > 0$ such that $(u,\, u+d_1) \subset U_n$. Hence $(u-d,\, u+d_1) \subset U_n$ and U_n is an open set for each index n. So $\cap_n U_n$ is a G_δ-set. Obviously any point in $\cap_n U_n$ lies in S, and any point in S lies in $\cap_n U_n$. Finally, $S = \cap_n U_n$. □

We say that a set $P \subset \mathbb{R}$ has *Jordan content* zero if for each $\epsilon > 0$ we can cover P with finitely many intervals the sum of whose lengths does not exceed ϵ. Clearly P has (Lebesgue) measure zero if P has Jordan content zero. There exist bounded sets of measure zero that do not have Jordan content zero, for example the set of rational numbers in $(0,1)$. But such a set can not be the range of a nondecreasing function, as we now see.

Lemma 3. *Let g be a bounded nondecreasing function on $\mathbb{R}$, and let T be a closed subset of $\mathbb{R}$. Then the difference of the sets $g(T)$ and closure $g(T)$ is at most a countable set. Also $g(T)$ has Jordan content zero if $g(T)$ has measure zero.*

Proof. Let S_1 denote the set of points not in $g(T)$ that are accumulation points of $g(T)$ from the left, and let S_2 denote the set of points not in $g(T)$ that are accumulation points of $g(T)$ from the right. For any $y \in S_1$ let $x(y)$ denote

$$\sup\Big\{x \in \mathbb{R} : g(x) \le y\Big\}.$$

Then g is discontinuous at $x(y)$, and $y \mapsto x(y)$ is a one-to-one mapping of S_1 to the set of discontinuities of g. It follows that S_1 is at most a countable set. Similarly, S_2 is at most a countable set. So the difference between the sets $g(T)$ and closure $g(T)$ is at most a countable set.

Now any covering of closure $g(T)$ by open intervals has a finite subcovering by compactness. It follows easily that $g(T)$ has Jordan content zero if $g(T)$ has measure zero. □

Lemma 4. *Let f be any bounded nondecreasing real valued function nonconstant on $\mathbb{R}$. Then f is the sum of jump functions if and only if the range of f has Jordan content zero.*

PROOF. Without loss of generality we let $\inf f = 0$. (Use an additive constant if necessary.) Let u be a point of discontinuity of f. Put

$$f_u(x) = \begin{cases} 0 & \text{if} \quad x < u, \\ f(u) - f(u-) & \text{if} \quad x = u, \\ f(u+) - f(u-) & \text{if} \quad x > u. \end{cases}$$

Then f_u is a jump function centered at u.

Fix $x \in \mathbb{R}$ and let $u_1 < u_2 < u_3 < \cdots < u_n \leq x$ where the u_j are points of discontinuity of f. Then

$$\begin{aligned} 0 &\leq f(u_1-) < f(u_1+) \leq f(u_2-) < f(u_2+) \\ &\quad \leq \cdots \leq f(u_n-) \leq f(u_n) = f(x) \text{ if } u_n = x, \text{ and} \\ 0 &\leq f(u_1-) < f(u_1+) \leq f(u_2-) < f(u_2+) \\ &\quad \leq \cdots \leq f(u_n-) < f(u_n+) \leq f(x) \text{ if } u_n < x. \end{aligned} \tag{1}$$

Then

$$\begin{aligned} 0 &< \Big(f(u_1+) - f(u_1-)\Big) + \Big(f(u_2+) - f(u_2-)\Big) \\ &\quad + \cdots + \Big(f(u_n) - f(u_n-)\Big) \leq f(x) \text{ if } u_n = x, \text{ and} \\ 0 &< \Big(f(u_1+) - f(u_1-)\Big) + \Big(f(u_2+) - f(u_2-)\Big) \\ &\quad + \cdots + \Big(f(u_n+) - f(u_n-)\Big) \leq f(x) \text{ if } u_n < x. \end{aligned} \tag{2}$$

Now

$$f_{u_j}(x) = f(u_j+) - f(u_j-) \text{ for } u_j < x$$

and

$$f_{u_n}(x) = f(u_n) - f(u_n-) \text{ for } u_n = x.$$

From (2) we obtain in either case

$$f_{u_1}(x) + f_{u_2}(x) + \cdots + f_{u_n}(x) \leq f(x). \tag{3}$$

We deduce from (3) that $g(x) \leq f(x)$ where $g(x) = \sum_u f_u(x)$ and u runs over all the points of discontinuity of f. But x is arbitrary, so $g \leq f$.

Now let $f = g$. We must prove that $f(\mathbb{R})$ has Jordan content zero. Select $x \in \mathbb{R}$ and $\epsilon > 0$. Let u_j in (3) be chosen so that

$$f_{u_1}(x) + f_{u_2}(x) + \cdots + f_{u_n}(x) \ > \ f(x) - \epsilon. \tag{4}$$

The intervals

$$\Big[f(u_1-),\ f(u_1+)\Big], \Big[f(u_2-),\ f(u_2+)\Big], \ldots, \tag{5}$$

have respective lengths $f_{u_1}(x),\ f_{u_2}(x), \ldots$, and we deduce from (4) that the sum of their lengths exceeds $f(x)-\epsilon$. So the sum of the lengths of the intervals complementary (in $[0,\ f(x)]$) to their union can not exceed ϵ. But each interval in (5) contains at most 3 points in $f(\mathbb{R})$. Here ϵ is arbitrary, so it follows that $f(\mathbb{R}) \cap \big[0,\ f(x)\big]$ has Jordan content zero.

The difference of the sets $f(\mathbb{R})$ and $f(\mathbb{R}) \cap \big[0,\ f(x)\big]$ can be covered by an interval of length $\sup f(\mathbb{R}) - f(x)$. But

$$\lim_{x\to\infty} f(x) = \sup f(\mathbb{R}).$$

Clearly $f(\mathbb{R})$ has Jordan content zero.

Finally, let $f(\mathbb{R})$ have Jordan content zero. We must prove that $f = g$. Choose $x \in \mathbb{R}$ and $\epsilon > 0$. Let $I_1, I_2, \ldots, I_n, I_{n+1}$ be compact mutually disjoint intervals covering $f(\mathbb{R}) \cap \big[0,\ f(x)\big]$, the sum of their lengths not exceeding ϵ. Say $I_1 < I_2 < \ldots < I_n < I_{n+1}$. For each index j, assume I_j meets $f(\mathbb{R}) \cap \big[0,\ f(x)\big]$. Put

$$y_k = \sup\Big\{t : f(t) \in I_1 \cup \ldots \cup I_k\Big\} \text{ for } k = 1, \ldots, n.$$

Then $y_1 < y_2 < \ldots < y_n$. Each subinterval of $\big[0,\ f(x)\big]$ complementary to the union $\cup_j I_j$ is contained in an interval of the form $\big[f(y_j-),\ f(y_j)\big]$ or $\big[f(y_j),\ f(y_j+)\big]$. It follows from the definition of f_u that

$$f_{y_1}(x) + f_{y_2}(x) + \cdots + f_{y_n}(x) \geq f(x) - \epsilon$$

and

$$g(x) = \sum_u f_u(x) \geq f(x) - \epsilon.$$

But ϵ was arbitrary, so $g(x) \geq f(x)$ and hence $g(x) = f(x)$. Here x was also arbitrary, so $g = f$. □

We are now ready for our main result.

Theorem 1. *Let f be a bounded nondecreasing real valued function nonconstant on $\mathbb{R}$. Let $f' = 0$ almost everywhere on $\mathbb{R}$. Let the set*

$$S = \Big\{x \in \mathbb{R} : D^+ f(x) + D^- f(x) = \infty\Big\}$$

be countable. Then f is the sum of jump functions. Furthermore, if I is any open interval on which f is not constant, then I contains a subinterval $(t-\epsilon,\, t+\epsilon)$ such that f is constant on the intervals $(t-\epsilon,\, t)$ and $(t,\, t+\epsilon)$ and $f(t-) < f(t+)$.

Proof. Let (a, b) be an open interval. Let

$$T_0 = \Big\{x \in (a, b) : f'(x) = 0\Big\},$$

and for each integer $n > 0$ let

$$T_n = \Big\{x \in (a, b) : 0 < Df(x) < n \text{ for some Dini derivate } Df \text{ of } f\Big\}.$$

By hypothesis, $m(T_n) = 0$ for all n, where m denoted the Lebesgue outer measure. From [HS, (17.25), p. 269], we obtain

$$m\big(f(T_n)\big) \leq n \cdot m(T_n) \text{ and } m\big(f(T_n)\big) = 0 \text{ for all } n > 0.$$

We deduce from the same reference $m\big(f(T_0)\big) = 0$. Of course $m\big(f(S)\big) = 0$. Consequently, f maps (a, b) to a set of measure zero. But (a, b) was arbitrary, so $m\big(f(\mathbb{R}) = 0$. By Lemma 3, $f(\mathbb{R})$ has Jordan content zero, and by Lemma 4, f is the sum of jump functions.

Now let f be nonconstant on the open interval I. If a jump function is not centered at a point in I, then it is constant on I. It follows that $S \cap I$ is nonvoid. By Lemma 2, S is a G_δ-set and therefore $S \cap I$ is a G_δ-set. But $S \cap I$ is also countable, and we deduce from Lemma 1 that $S \cap I$ contains an isolated point t. Say $(t-\epsilon,\, t+\epsilon)$ is a subinterval of I that contains no point of S other than t. Then f is constant on the intervals $(t-\epsilon,\, t)$ and $(t,\, t+\epsilon)$. Finally, $f(t-) < f(t+)$ because $t \in S$. □

The following proposition may be difficult to prove without our Lemma 4.

Proposition 1. *Let f and g be bounded nondecreasing functions nonconstant on $\mathbb{R}$ such that $f - g$ is also nondecreasing. Let f be the sum of jump functions. Then g is the sum of jump functions.*

Proof. By Lemma 4, $f(\mathbb{R})$ has Jordan content zero. Select $\epsilon > 0$. Let $I_1, \dots, I_n$ be intervals covering $f(\mathbb{R})$, with $\sum_j m(I_j) < \epsilon$.

Now for points u and x, $u < x$, we have

$$f(x) - g(x) \geq f(u) - g(u), \text{ and hence } f(x) - f(u) \geq g(x) - g(u).$$

It follows that $g\big(f^{-1}(I_j)\big)$ can be covered by an interval no longer than I_j. Consequently,

$$g(\mathbb{R}) = g\Big(\cup_j f^{-1}(I_j)\Big) = \cup_j g\big(f^{-1}(I_j)\big)$$

can be covered by n intervals, the sum of whose lengths is no longer than $\sum_j m(I_j) < \epsilon$. It follows that $g(\mathbb{R})$ has Jordan content zero. By Lemma 4 again, g is the sum of jump functions. □

Note that the function K need not be additive in the hypothesis of our next proposition. Possibly $K(a+b) \neq K(a) + K(b)$ for some numbers a and b.

Proposition 2. *Let the bounded nondecreasing function g be the sum of jump functions. Let K be a strictly increasing function, absolutely continuous on an interval containing the range of g. For each $x \in \mathbb{R}$, let $f(x) = K\big(g(x)\big)$. Then f is the sum of jump functions.*

Proof. By Lemma 4, $g(\mathbb{R})$ has Jordan content zero and hence measure zero. Then $K\big(g(\mathbb{R})\big)$ has measure zero because K is absolutely continuous. For each $x \in \mathbb{R}$ we have $f(x) \in K\big(g(\mathbb{R})\big)$. It follows that $f(\mathbb{R})$ has measure zero, as well. Also, f is nonconstant on $\mathbb{R}$ because K is strictly increasing. By Lemma 3, $f(\mathbb{R})$ has Jordan content zero, and by Lemma 4, f is the sum of jump functions. □

So far, we have used reference [P] only to establish the existence of the kind of function described in the hypothesis of Theorem 1. Now we offer a more direct application of [P].

Proposition 3. *There exists a function f on $\mathbb{R}$ such that $f'(x) = 0$ for every irrational point x, but $f(\mathbb{R})$ has the power of the continuum.*

Proof. Let T denote the set of midpoints of all the intervals complementary (in $[0,1]$) to Cantor's ternary set. Then T consists exclusively of rational points. The points of T have mutually disjoint neighborhoods, and it easily follows that T is a G_δ-set. It follows from [P] that there exists a nondecreasing function f on $\mathbb{R}$ discontinuous at each $x \in T$, such that $f'(x) = 0$ for all $x \notin T$. Thus $f'(x) = 0$ for all irrational x.

Let V denote the set of all points of the Cantor ternary set E that are accumulation points from the left to E. Clearly V has the power of the continuum.

Now f is strictly increasing on T. Between any two points of V there are points of T. It follows that f is strictly increasing on V. Finally, $f(V)$ and likewise $f(K)$ have the power of the continuum. □

By an isolated point of discontinuity t of a function h we mean a point of discontinuity t of h such that there is a neighborhood of t containing no other point of discontinuity of h.

Next we have a variation on the theme that a function with zero derivative everywhere must be constant.

Proposition 4. *Let f be a function that has zero derivative at every point of $\mathbb{R}$ except possibly countably many points. Let f have no isolated point of discontinuity. Then f is constant on $\mathbb{R}$.*

PROOF. We consider two cases. Either f is everywhere continuous, or it is not.

CASE 1. f is not everywhere continuous.
Let Y denote the set of points where f is discontinuous, and let

$$X = \Big\{x \in \mathbb{R} : \big|D^+ f(x)\big| + \big|D^- f(x)\big| + \big|D_+ f(x)\big| + \big|D_- f(x)\big| = \infty\Big\}.$$

Then Y is a nonvoid subset of X by assumption. But X is a G_δ-set by Lemma 2, and it follows from the hypothesis that the set Y is dense in itself. By Lemma 1, then, X has the power of the continuum, contrary to our hypothesis.

CASE 2. f is everywhere continuous.
We employ a familiar argument. Suppose that $a < b$ and $f(a) \neq f(b)$. Say $f(a) < f(b)$ for definiteness. (For $f(a) > f(b)$, use the function $-f$.) Select a positive number r so small that $g(a) < g(b)$ where $g(x) = f(x) - rx$. Then $g'(x) = -r$ for all numbers x except possibly countably many. Select a number y, $g(a) < y < g(b)$, such that $g'(x) = -r$ for all x satisfying $y = g(x)$. Let u be the greatest point in the set

$$\Big\{x : a \leq x \leq b \text{ and } y = g(x)\Big\}.$$

It follows that $u < b$ and $g'(u) = -r < 0$, and this violates the intermediate value property of the continuous function g. □

A sum of jump functions need not have a zero derivative at all but countably many points. For example, if the summands are centered at a dense set of points, it easily follows from Lemmas 1 and 2 that the sum fails to be differentiable at continuum many points. However, we must have a zero derivative almost everywhere, as we now see.

Proposition 5. *Let f be the sum of jump functions. Then $f' = 0$ almost everywhere.*

Proof. Any jump function has zero derivative at every point but one. Now for nondecreasing functions, the derivative of the sum equals the sum of the derivatives almost everywhere (see for example [HS, (17.18), p. 267]). The conclusion follows from this. □

References

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