Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 67–78

Károly Simon,[*] Institute of Mathematics, Technical University of Budapest, H-1529 B.O. Box 91, Hungary. email: simonk@math.bme.hu
Boris Solomyak,[†] Box 354350, Department of Mathematics, University of Washington, Seattle WA, 98195. email: solomyak@math.washington.edu

VISIBILITY FOR SELF-SIMILAR SETS OF DIMENSION ONE IN THE PLANE[‡]

Abstract

We prove that a purely unrectifiable self-similar set of finite 1-dimensional Hausdorff measure in the plane, satisfying the Open Set Condition, has radial projection of zero length from every point.

1 Introduction.

For $a \in \mathbb{R}^2$, let P_a be the radial projection from a,

$$P_a : \mathbb{R}^2 \setminus \{a\} \to S^1, \quad P_a(x) = \frac{(x-a)}{|x-a|}.$$

A special case of our theorem asserts that the "four corner Cantor set" of contraction ratio 1/4 has radial projection of zero length from all points $a \in \mathbb{R}^2$. See Figure **??**, where we show the second-level approximation of the four corner Cantor set and the radial projection of some of its points.

Denote by $\mathcal{H}^1$ the one-dimensional Hausdorff measure. A Borel set Λ is a 1-*set* if $0 < \mathcal{H}^1(\Lambda) < \infty$. It is said to be *invisible from a* if $P_a(\Lambda \setminus \{a\})$ has zero length.

Theorem 1.1. *Let Λ be a self-similar 1-set in $\mathbb{R}^2$ satisfying the Open Set Condition, which is not on a line. Then, Λ is invisible from every $a \in \mathbb{R}^2$.*

Key Words: Hausdorff measure, purely unrectifiable, self-similar set
Mathematical Reviews subject classification: Primary 28A80
Received by the editors October 23, 2005
Communicated by: Clifford E. Weil
[*]Supported in part by OTKA Foundation grant T42496.
[†]supported in part by NSF grant DMS-0355187.
[‡]This collaboration was supported by NSF-MTA-OTKA grant #77.

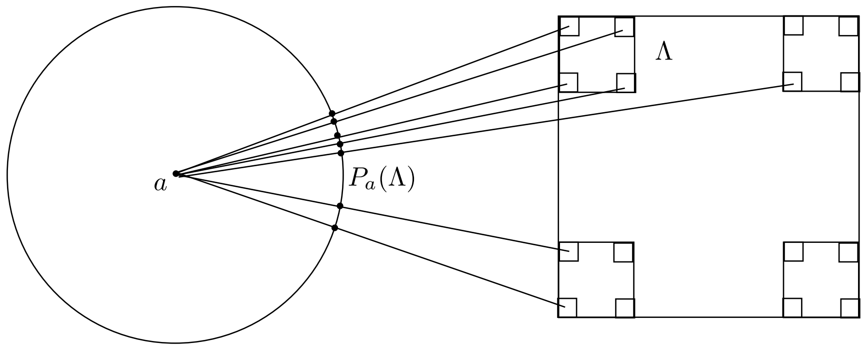

Figure 1: The radial projection of the four corner set.

Recall that a nonempty compact Λ is self-similar if $\Lambda = \bigcup_{i=1}^{m} S_i(\Lambda)$ for some contracting similitudes S_i. This means that $S_i(x) = \lambda_i \mathcal{O}_i x + b_i$, where $0 < \lambda_i < 1$, $\mathcal{O}_i$ is an orthogonal transformation of the plane, and $b_i \in \mathbb{R}^2$. The Open Set Condition holds if there exists an open set $V \neq \emptyset$ such that $S_i(V) \subset V$ for all i, and $S_i(V) \cap S_j(V) = \emptyset$ for all $i \neq j$. For a self-similar set satisfying the Open Set Condition, being a 1-set is equivalent to $\sum_{i=1}^{m} \lambda_i = 1$.

A Borel set Λ is *purely unrectifiable* (or *irregular*), if $\mathcal{H}^1(\Lambda \cap \Gamma) = 0$ for every rectifiable curve Γ. A set Λ satisfying the assumptions of Theorem 1.1 is purely unrectifiable by Hutchinson [5] (see also [8]). A classical theorem of Besicovitch [2] (see also [4, Theorem 6.13]) says that a purely unrectifiable 1-set has orthogonal projections of zero length on almost every line through the origin. We use it in our proof.

In [10, Problem 12] (see also [9, 10.12]), Mattila raised the following question. Let Λ be a Borel set in $\mathbb{R}^2$ with $\mathcal{H}^1(\Lambda) < \infty$. Is it true that for $\mathcal{H}^1$ almost all $a \in \Lambda$, the intersection $\Lambda \cap L$ is a finite set for almost all lines L through a? If Λ is purely unrectifiable, is it true that $\Lambda \cap L = \{a\}$ for almost all lines through a? Note that the latter property is equivalent to Λ being invisible from a. Thus, our theorem implies a positive answer for a purely unrectifiable self-similar 1-set Λ satisfying the Open Set Condition. The general case of a purely unrectifiable set remains open. On the other hand, M. Csörnyei and D. Preiss proved recently that the answer to the first part of the question is negative [personal communication].

Note that we prove a stronger property for our class of sets, namely, that the set is invisible from *every* point $a \in \mathbb{R}^2$. It is easy to construct examples of non-self-similar purely unrectifiable 1-sets for which this property fails. Marstrand [6, p. 281–284] has an example of a purely unrectifiable 1-set which is visible from a set of dimension one. It is obtained by an iterative construction which is far from being self-similar and is too complicated to describe here.

We do not discuss here other results and problems related to visibility; see [9, Section 6] for a recent survey. We only mention a result of Mattila [7, Th.5.1]. If a set Λ has projections of zero length on almost every line (which could have $\mathcal{H}^1(\Lambda) = \infty$), then the set of points Ξ from which Λ is visible is a purely unrectifiable set of zero 1-capacity. A different proof of this and a characterization of such sets Ξ is due to Csörnyei [3].

2 Preliminaries.

We have $S_i(x) := \lambda_i \mathcal{O}_i x + b_i$, where $0 < \lambda_i < 1$,

$$\mathcal{O}_i = \begin{bmatrix} \cos(\varphi_i) & -\varepsilon_i \sin(\varphi_i) \\ \sin(\varphi_i) & \varepsilon_i \cos(\varphi_i) \end{bmatrix},$$

$\varphi_i \in [0, 2\pi)$, and $\varepsilon_i \in \{-1, 1\}$ shows whether $\mathcal{O}_i$ is a rotation through the angle φ_i or a reflection about the line through the origin making the angle $\varphi_i/2$ with the x-axis.

Let $\Sigma := \{1, \ldots, m\}^{\mathbb{N}}$ be the symbolic space. The natural projection $\Pi : \Sigma \to \Lambda$ is defined by

$$\Pi(\mathbf{i}) = \lim_{n\to\infty} S_{i_1\ldots i_n}(x_0), \text{ where } \mathbf{i} = (i_1 i_2 i_3 \ldots) \in \Sigma, \tag{1}$$

and $S_{i_1\ldots i_n} = S_{i_1} \circ \cdots \circ S_{i_n}$. The limit in (1) exists and does not depend on x_0. Let $\lambda_{i_1\ldots i_n} = \lambda_{i_1} \cdots \lambda_{i_n}$ and $\varepsilon_{i_1\ldots i_k} = \varepsilon_{i_1} \cdots \varepsilon_{i_k}$. We can write

$$S_{i_1\ldots i_n}(x) = \lambda_{i_1\ldots i_n} \mathcal{O}_{i_1\ldots i_n} x + b_{i_1\ldots i_n},$$

where

$$\mathcal{O}_{i_1\ldots i_n} := \mathcal{O}_{i_1} \circ \cdots \circ \mathcal{O}_{i_n} = \begin{bmatrix} \cos(\varphi_{i_1\ldots i_n}) & -\varepsilon_{i_1\ldots i_n} \sin(\varphi_{i_1\ldots i_n}) \\ \sin(\varphi_{i_1\ldots i_n}) & \varepsilon_{i_1\ldots i_n} \cos(\varphi_{i_1\ldots i_n}) \end{bmatrix},$$

$$\varphi_{i_1\ldots i_n} := \varphi_{i_1} + \varepsilon_{i_1}\varphi_{i_2} + \varepsilon_{i_1 i_2}\varphi_{i_3} + \cdots + \varepsilon_{i_1\ldots i_{n-1}}\varphi_{i_n},$$

and

$$b_{i_1\ldots i_n} = b_{i_1} + \lambda_{i_1}\mathcal{O}_{i_1} b_{i_2} + \cdots + \lambda_{i_1\ldots i_{n-1}}\mathcal{O}_{i_1\ldots i_{n-1}} b_{i_n}.$$

Since $\sum_{i=1}^m \lambda_i = 1$, we can consider the probability product measure $\mu = (\lambda_1, \ldots, \lambda_m)^{\mathbb{N}}$ on the symbolic space Σ and define the *natural measure* on Λ, $\nu = \mu \circ \Pi^{-1}$. By a result of Hutchinson [5, Theorem 5.3.1(iii)], as a consequence of the Open Set Condition, we have

$$\nu = c\mathcal{H}^1|_\Lambda, \text{ where } c = (\mathcal{H}^1(\Lambda))^{-1}. \tag{2}$$

To $\theta \in [0, \pi)$, we associate the unit vector $e_\theta = (\cos\theta, \sin\theta)$, the line $L_\theta = \{te_\theta : \ t \in \mathbb{R}\}$, and the orthogonal projection onto L_θ given by $x \mapsto (e_\theta \cdot x)e_\theta$. It is more convenient to work with the signed distance of the projection to the origin, which we denote by p_θ,

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}, \ p_\theta x = e_\theta \cdot x.$$

Let $\mathcal{A} := \{1, \ldots, m\}$ and let $\mathcal{A}^* = \bigcup_{i=1}^\infty \mathcal{A}^i$ be the set of all finite words over the alphabet $\mathcal{A}$. For $u = u_1 \ldots u_k \in \mathcal{A}^k$ we define the corresponding "symbolic" cylinder set by

$$[u] = [u_1 \ldots u_k] := \{\mathbf{i} \in \Sigma : \ i_\ell = u_\ell, \, 1 \le \ell \le k\}.$$

We also let

$$\Lambda_u = S_u(\Lambda) = \lambda_u \mathcal{O}_u \Lambda + b_u$$

and call Λ_u the cylinder set of Λ corresponding to the word u. Let d_Λ be the diameter of Λ. Then $\mathrm{diam}(\Lambda_u) = \lambda_u d_\Lambda$. For $\rho > 0$, consider the "cut-set"

$$\mathcal{W}(\rho) = \{u \in \mathcal{A}^* : \lambda_u \le \rho, \ \lambda_{u'} > \rho\}$$

where u' is obtained from u by deleting the last symbol. Observe that for every $0 < \rho < \lambda_{\min}$,

$$\Lambda = \bigcup_{u \in \mathcal{W}(\rho)} \Lambda_u,$$

where we denote $\lambda_{\min} := \min\{\lambda_i : \ 1 \le i \le m\}$. In view of (2), we have $\nu(\Lambda_u \cap \Lambda_v) = 0$ for distinct $u, v \in \mathcal{W}(\rho)$. Hence

$$\nu(\Lambda_u) = \lambda_u \text{ for all } u \in \mathcal{A}^*.$$

We identify the unit circle S^1 with $[0, 2\pi)$ and use additive notation $\theta_1 + \theta_2$ understood mod 2π for points on the circle. For a Radon measure η on the line or on S^1, the upper density of η with respect to $\mathcal{H}^1$ is defined by

$$\overline{D}(\eta, t) = \limsup_{r \to 0} \frac{\eta([t - r, t + r])}{2r}.$$

The open ball of radius r centered at x is denoted by $B(x, r)$.

3 Proof of the Main Theorem.

In the proof of Theorem 1.1, we may assume, without loss of generality, that $a \notin \Lambda$, and

$$P_a(\Lambda) \text{ is contained in an arc of length less than } \pi. \tag{3}$$

Indeed, $\Lambda \setminus \{a\}$ can be written as a countable union of self-similar sets Λ_u for $u \in \mathcal{A}^*$, of arbitrarily small diameter. If each of them is invisible from a, then Λ is invisible from a. We denote the usual left shift on Σ by σ. Let

$$\Omega := \{\mathbf{i} \in \Sigma : \forall\, u \in \mathcal{A}^*\ \exists\, n \text{ such that } \sigma^n \mathbf{i} \in [u]\};$$

that is, Ω is the set of sequences which contain each finite word over the alphabet $\mathcal{A} = \{1, \ldots, m\}$. It is clear that every $\mathbf{i} \in \Omega$ contains each finite word infinitely many times and $\mu(\Sigma \setminus \Omega) = 0$.

Lemma 3.1 (Recurrence Lemma)**.** *For every $\mathbf{i} \in \Omega$, $\delta > 0$, and $j_1, \ldots, j_k \in \{1, \ldots, m\}$, there are infinitely many $n \in \mathbb{N}$ such that*

$$\varphi_{i_1 \ldots i_n} \in [0, \delta],\ \ \varepsilon_{i_1 \ldots, i_n} = 1,\ \ \textit{and } \sigma^n \mathbf{i} \in [j_1 \ldots j_k]. \tag{4}$$

If the similitudes have no rotations or reflections; that is, $\varphi_i = 0$ and $\varepsilon_i = 1$ for all $i \le m$ (as in the case of the four corner Cantor set), then the conditions on φ and ε in (4) hold automatically and the lemma is true by the definition of Ω. The proof in the general case is not difficult, but requires a detailed case analysis, so we postpone it to the next section. Let

$$\Theta := \{\theta \in [0, \pi) :\ \mathcal{H}^1(p_\theta(\Lambda)) = 0\} \text{ and } \Theta' := (\Theta + \pi/2) \cup (\Theta + 3\pi/2).$$

(Recall that addition is considered mod 2π.) Since Λ is purely unrectifiable, $\mathcal{H}^1([0, \pi) \setminus \Theta') = 0$ by Besicovitch's Theorem [2]. The following proposition is the key step of the proof. We need the following measures,

$$\nu_a := \nu \circ P_a^{-1} \text{ and } \nu_\theta := \nu \circ p_\theta^{-1},\ \ \theta \in [0, \pi).$$

We also let $\Lambda' = \Pi(\Omega)$.

Proposition 3.2. *If $\theta' \in P_a(\Lambda') \cap \Theta'$, then $\overline{D}(\nu_a, \theta') = \infty$.*

PROOF OF THEOREM 1.1 ASSUMING PROPOSITION 3.2. By Proposition 3.2 and [9, Lemma 2.13] (a corollary of the Vitali covering theorem), we obtain that $\mathcal{H}^1(P_a(\Lambda') \cap \Theta') = 0$. As noted above, Θ' has full $\mathcal{H}^1$ measure in S^1. On the other hand,

$$\mu(\Sigma \setminus \Omega) = 0 \ \Rightarrow\ \nu(\Lambda \setminus \Lambda') = 0 \ \Rightarrow\ \mathcal{H}^1(\Lambda \setminus \Lambda') = 0 \ \Rightarrow\ \mathcal{H}^1(P_a(\Lambda \setminus \Lambda')) = 0,$$

and we conclude that $\mathcal{H}^1(P_a(\Lambda)) = 0$, as desired. □

PROOF OF PROPOSITION 3.2. Let $x \in \Lambda'$ and $\theta' = P_a(x) \in \Theta'$. Let $\theta := \theta' - \pi/2 \mod [0,\pi)$. By the definition of Θ', we have $\mathcal{H}^1(p_\theta(\Lambda)) = 0$.

First, we sketch the idea of the proof. Since $\mathcal{H}^1(p_\theta(\Lambda)) = 0$, we have $\nu_\theta \perp \mathcal{H}^1$, and this implies that for every $N \in \mathbb{N}$ there exist N cylinders of Λ approximately the same diameter (say, $\sim r$), such that their projections to L_θ are r-close to each other. Then, there is a line parallel to the segment $[a,x]$, whose Cr-neighborhood contains all $\Lambda_{u_j}, j = 1,\dots,N$. By the definition of $\Lambda' = \Pi(\Omega)$, we can find similar copies of this picture near $x \in \Lambda'$ at arbitrarily small scales. The Recurrence Lemma 3.1 guarantees that these copies can be chosen with a small relative rotation. This will give N cylinders of Λ of diameter $\sim r_0 r$ contained in a $C' r_0 r$-neighborhood of the ray obtained by extending $[a,x]$. Since a is assumed to be separated from Λ, we will conclude that $\overline{D}(\nu_a,\theta') \geq C'' N$, and the proposition will follow. Now we make this precise. The proof is illustrated in Figure 2.

CLAIM. *For each $N \in \mathbb{N}$, there exists $r > 0$ and distinct $u^{(1)},\dots,u^{(N)} \in \mathcal{W}(r)$ such that*

$$|p_\theta(b_{u^{(j)}} - b_{u^{(i)}})| \leq r, \ \forall\, i,j \leq N. \tag{5}$$

Indeed, for every $u \in \mathcal{A}^*$,

$$\Lambda_u = \lambda_u \mathcal{O}_u \Lambda + b_u \ \Rightarrow \ \Lambda_u \subset B(b_u, d_\Lambda \lambda_u).$$

Hence for every interval $I \subset \mathbb{R}$ and $r > 0$,

$$\nu_\theta(I) \leq \sum_{u \in \mathcal{W}(r)} \{\lambda_u : \ \mathrm{dist}(p_\theta(b_u), I) \leq d_\Lambda r\}.$$

If the claim does not hold, then there exists $N \in \mathbb{N}$ such that for every $t \in \mathbb{R}$ and $r > 0$,

$$\nu_\theta([t-r,t+r]) \leq N(2(1+d_\Lambda)+1)r.$$

Then ν_θ is absolutely continuous with respect to $\mathcal{H}^1$, which is a contradiction. The claim is verified. □

We are given that $x \in \Lambda' = \Pi(\Omega)$, which means that $x = \pi(\mathbf{i})$ for an infinite sequence $\mathbf{i}$ containing all finite words. We fix $N \in \mathbb{N}$ and find $r > 0$, $u^{(1)},\dots,u^{(N)} \in \mathcal{W}(r)$ from the Claim. Then we apply Recurrence Lemma 3.1 with $j_1 \dots j_k := u^{(1)}$ and $\delta = r$ to obtain infinitely many $n \in \mathbb{N}$ satisfying (4). Fix such an n. Let

$$w := i_1 \dots i_n \text{ and } v^{(j)} = w u^{(j)}, \ j = 1,\dots,N.$$

Observe that $\mathbf{i}$ starts with $v^{(1)}$, so $x = \Pi(\mathbf{i}) \in \Lambda_{v^{(1)}}$. Hence

$$|p_\theta(x - b_{v^{(1)}})| \leq |x - b_{v^{(1)}}| \leq d_\Lambda \lambda_{v^{(1)}} \leq d_\Lambda \lambda_w r.$$

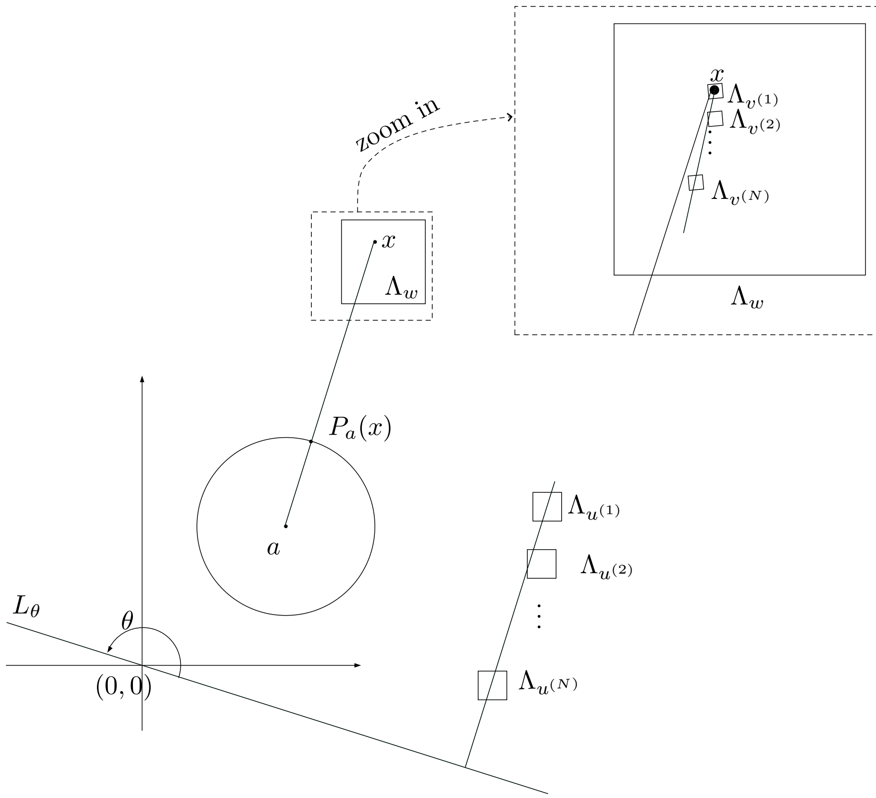

Figure 2: The cylinders of Λ causing high density.

Here we used that $u^{(1)} \in \mathcal{W}(r)$, so $\lambda_{v^{(1)}} = \lambda_w \lambda_{u^{(1)}} \leq \lambda_w r$. We have for $z \in \mathbb{R}^2$,

$\lambda_{v^{(j)}} \mathcal{O}_{v^{(j)}} z + b_{v^{(j)}} = S_{v^{(j)}}(z) = S_w \circ S_{u^{(j)}}(z) = \lambda_w \mathcal{O}_w (\lambda_{u^{(j)}} \mathcal{O}_{u^{(j)}} z + b_{u^{(j)}}) + b_w$.

Hence

$$b_{v^{(j)}} = \lambda_w \mathcal{O}_w b_{u^{(j)}} + b_w.$$

It follows that

$$p_\theta(b_{v^{(i)}} - b_{v^{(j)}}) = \lambda_w p_\theta \mathcal{O}_w (b_{u^{(i)}} - b_{u^{(j)}}).$$

By (4), we have $\varepsilon_w = 1$ and $\varphi := \varphi_w \in [0, r)$; therefore, $\mathcal{O}_w = R_\theta$ is the rotation through the angle φ. One can check that $p_\theta R_\varphi = p_{\theta-\varphi}$, which yields

$$|p_\theta(b_{v^{(i)}} - b_{v^{(j)}})| = \lambda_w |p_{\theta-\varphi}(b_{u^{(i)}} - b_{u^{(j)}})|. \tag{6}$$

Clearly, $\|p_\theta - p_{\theta-\varphi}\| \le |\varphi| \le r$, where $\|\cdot\|$ is the operator norm, so we obtain from (5) and (6) that

$$|p_\theta(b_{v^{(i)}} - b_{v^{(j)}})| \le \lambda_w(|b_{u^{(i)}} - b_{u^{(j)}}| r + r) \le \lambda_w(d_\Lambda + 1)r.$$

Recall that $\mathbf{i}$ starts with $v^{(1)}$, so $x = \Pi(\mathbf{i}) \in \Lambda_{v^{(1)}}$, hence for each $j \le N$, for every $y \in \Lambda_{v^{(j)}}$,

$$\begin{aligned} |p_\theta(x-y)| &\le |x - b_{v^{(1)}}| + |p_\theta(b_{v^{(1)}} - b_{v^{(j)}})| + |b_{v^{(j)}} - y| \\ &\le d_\Lambda(\lambda_{v^{(1)}} + \lambda_{v^{(j)}}) + \lambda_w(d_\Lambda + 1)r \le \lambda_w(3d_\Lambda + 1)r. \end{aligned} \tag{7}$$

Now we need a simple geometric fact: given that

$$P_a(x) = \theta', \;\; \theta = \theta' + \pi/2 \bmod [0,\pi), \;\; |p_\theta(x-y)| \le \rho, \;\; |y-a| \ge c_1, \;\text{ and (3) holds,}$$

we have

$$|P_a(y) - \theta'| = |P_a(y) - P_a(x)| = \arcsin \frac{|p_\theta(y-x)|}{|y-a|} \le \frac{\pi}{2c_1}\rho.$$

This implies, in view of (7), that for $c_2 = \pi(3d_\Lambda + 1)/(2c_1)$,

$$\nu_a([\theta' - c_2\lambda_w r, \theta' + c_2\lambda_w r]) \ge \sum_{j=1}^N \nu(\Lambda_{v^{(j)}}) = \sum_{j=1}^N \lambda_{v^{(j)}} = \lambda_w \sum_{j=1}^N \lambda_{u^{(j)}} \ge \lambda_w N \lambda_{\min} r,$$

by the definition of $\mathcal{W}(r)$. Recall that n can be chosen arbitrarily large, so λ_w can be arbitrarily small, and we obtain that $\overline{D}(\nu_a, \theta') \ge c_2^{-1}\lambda_{\min} N$. Since $N \in \mathbb{N}$ is arbitrary, the proposition follows. □

4 Proof of the Recurrence Lemma 3.1.

Let $K \in \{0, \ldots, m\}$ be the number of i for which $\varphi_i \notin \pi\mathbb{Q}$. Without loss of generality we may assume the following. If $K \ge 1$, then $\varphi_1, \ldots, \varphi_K \notin \pi\mathbb{Q}$.

We distinguish the following cases:

A $\varphi_i \in \pi\mathbb{Q}$ for all $i \le m$.

B there exists i such that $\varphi_i \notin \pi\mathbb{Q}$ and $\varepsilon_i = 1$.

C $K \geq 1$ and $\varepsilon_i = -1$ for all $i \leq K$.

- **C1** there exist $i, j \leq K$ such that $\varphi_i - \varphi_j \notin \pi\mathbb{Q}$.
- **C2** there exists $r_i \in \mathbb{Q}$ such that $\varphi_i = \varphi_1 + r_i\pi$ for $1 \leq i \leq K$.
  - **C2a** $K < m$ and there exists $j \geq K+1$ such that $\varepsilon_j = -1$.
  - **C2b** $K < m$ and for all $j \geq K+1$ we have $\varepsilon_j = 1$.
  - **C2c** $K = m$.

Denote by R_φ the rotation through the angle φ. We call it an irrational rotation if $\varphi \notin \pi\mathbb{Q}$. Consider the semigroup generated by $\mathcal{O}_i,\ i \leq m$, which we denote by $\mathcal{S}$. We begin with the following observation.

CLAIM. *Either $\mathcal{S}$ is finite, or $\mathcal{S}$ contains an irrational rotation.*

The semigroup $\mathcal{S}$ is clearly finite in Case A and contains an irrational rotation in Case B. In Case C1 we have $\mathcal{O}_i\mathcal{O}_j = R_{\varphi_i - \varphi_j}$, which is an irrational rotation. In Case C2a we also have that $\mathcal{O}_i\mathcal{O}_j = R_{\varphi_i - \varphi_j}$ is an irrational rotation, since $\varphi_i \notin \pi\mathbb{Q}$ and $\varphi_j \in \pi\mathbb{Q}$. We claim that in remaining Cases C2b and C2c the semigroup is finite. This follows easily; then $\mathcal{S}$ is generated by one irrational reflection and finitely many rational rotations.

PROOF OF LEMMA 3.1 WHEN $\mathcal{S}$ IS FINITE. A finite semigroup of invertible transformations is necessarily a group. Let $\mathcal{S} = \{s_1, \dots, s_t\}$. By the definition of the semigroup $\mathcal{S}$ we have $s_i = \mathcal{O}_{w^{(i)}}$ for some $w^{(i)} \in \mathcal{A}^*$, $i = 1, \dots, t$. For every $v \in \mathcal{A}^*$, we can find $\widehat{v} \in \mathcal{A}^*$ such that $\mathcal{O}_{\widehat{v}} = \mathcal{O}_v^{-1}$. Fix $u = j_1 \dots j_k$ from the statement of the lemma. Consider the following finite word over the alphabet $\mathcal{A}$.

$$\omega := \tau_1 \dots \tau_t, \quad \text{where } \ \tau_j = (w^{(j)}u)\,\widehat{(w^{(j)}u)},\ j = 1, \dots, t$$

Note that $\mathcal{O}_{\tau_j} = I$ (the identity). By the definition of Ω, the sequence $\mathbf{i} \in \Omega$ contains ω infinitely many times. Suppose that $\sigma^\ell \mathbf{i} \in [\omega]$. Put $\mathbf{i}|\ell := i_1 \dots i_\ell$. Since $\mathcal{O}_{\mathbf{i}|\ell} \in \mathcal{S}$, there exists $w^{(j)}$ such that $\mathcal{O}_{w^{(j)}} = \mathcal{O}_{\mathbf{i}|\ell}^{-1}$. Then, the occurrence of u in τ_j, the jth factor of ω, will be at the position n such that $\mathcal{O}_{\mathbf{i}|n} = I$, so we will have $\varphi_{\mathbf{i}|n} = 0 \in [0, \delta]$ and $\varepsilon_{\mathbf{i}|n} = 1$, as desired. □

PROOF OF LEMMA 3.1 WHEN $\mathcal{S}$ IS INFINITE. By the claim above, there exists $w \in \mathcal{A}^*$ such that $\varphi_w \notin \pi\mathbb{Q}$ and $\varepsilon_w = 1$. Fix $u = j_1 \dots j_k$ from the statement of the lemma. Let

$$v := \begin{cases} uu, & \text{if } \varphi_u \notin \pi\mathbb{Q}; \\ uuw, & \text{if } \varphi_u \in \pi\mathbb{Q}. \end{cases}$$

Observe that $\varphi_v \notin \pi\mathbb{Q}$ and $\varepsilon_v = 1$. Let $v^k = v \dots v$ (the word v repeated k times). Since φ_v/π is irrational, there exists an N such that every orbit of R_{φ_v} of length N contains a point in every subinterval of $[0, 2\pi)$ of length δ. Put

$$\omega := \begin{cases} v^N, & \text{if } \varepsilon_i = 1,\ \forall\, i \le m; \\ v^N j^* v^N, & \text{if } \exists\, j^* \text{ such that } \varepsilon_{j^*} = -1. \end{cases}$$

By the definition of Ω, the sequence $\mathbf{i} \in \Omega$ contains ω infinitely many times. Let $\ell \in \mathbb{N}$ be such that $\sigma^\ell \mathbf{i} \in [\omega]$. Suppose first that $\varepsilon_{\mathbf{i}|\ell} = 1$. Then we have, denoting the length of v by $|v|$,

$$\sigma^{\ell+k|v|}\mathbf{i} \in [u], \quad \varphi_{\mathbf{i}|(\ell+k|v|)} = \varphi_{\mathbf{i}|\ell} + k\varphi_v \pmod{2\pi}, \quad \varepsilon_{\mathbf{i}|(\ell+k|v|)} = 1, \tag{8}$$

for $k = 0, \dots, N-1$. By the choice of N, we can find $k \in \{0, \dots, N-1\}$ such that $\varphi_{\mathbf{i}|(\ell+k|v|)} \in [0, \delta]$, then $n = \ell + k|v|$ will be as desired. If $\varepsilon_{\mathbf{i}|\ell} = -1$, then we replace ℓ by $\ell^* := \ell + N|v| + 1$ in (8), that is, we consider the occurrences of u in the second factor v^N. The orientation will be switched by $\mathcal{O}_{j^*}$ and we can find the desired n analogously. □

5 Concluding Remarks.

Consider the special case when the self-similar set Λ is of the form

$$\Lambda = \bigcup_{i=1}^m (\lambda_i \Lambda + b_i), \quad b_i \in \mathbb{R}^2. \tag{9}$$

In other words, the contracting similitudes have no rotations or reflections, as for the four corner Cantor set. Then the projection $\Lambda^\theta := p_\theta(\Lambda)$ is itself a self-similar set on the line

$$\Lambda^\theta = \bigcup_{i=1}^m (\lambda_i \Lambda^\theta + p_\theta(b_i)), \quad \text{for } \theta \in [0, \pi).$$

Let $\Lambda_i^\theta = \lambda_i \Lambda^\theta + p_\theta(b_i)$. As above, ν is the natural measure on Λ. Let ν_θ be the natural measure on Λ^θ, so that $\nu_\theta = \nu \circ p_\theta^{-1}$.

Corollary 5.1. *Let Λ be a self-similar set of the form (9) that is not on a line, such that $\sum_{i=1}^m \lambda_i \le 1$. If Λ satisfies the Open Set Condition condition, then*

$$\nu_\theta(\Lambda_i^\theta \cap \Lambda_j^\theta) = 0, \ i \ne j, \quad \textit{for a.e. } \theta \in [0, \pi).$$

Proof. Let $s > 0$ be such that $\sum_{i=1}^m \lambda_i^s = 1$. By assumption, we have $s \le 1$. This number is known as the similarity dimension of Λ (and also of Λ^θ for all θ). Suppose first that $s = 1$. Then we are in the situation covered by Theorem 1.1, and ν is just the normalized restriction of $\mathcal{H}^1$ to Λ. Consider the product measure $\nu \times \mathcal{L}$, where $\mathcal{L}$ is the Lebesgue measure on $[0, \pi)$. Theorem 1.1 implies that

$$(\nu \times \mathcal{L})\{(x, \theta) \in \Lambda \times [0, \pi) : \ \exists\, y \in \Lambda,\ y \ne x,\ p_\theta(x) = p_\theta(y)\} = 0.$$

By Fubini's Theorem, it follows that for $\mathcal{L}$ a.e. θ, for ν_θ a.e. $z \in L^\theta$, we have that $p_\theta^{-1}(z)$ is a single point. This proves the desired statement, in view of the fact that $\nu(\Lambda_i \cap \Lambda_j) = 0$ for Λ satisfying the Open Set Condition.

In the case when $s < 1$, we can use [11, Proposition 1.3], which implies that the packing measure $\mathcal{P}^s(\Lambda^\theta)$ is positive and finite for $\mathcal{L}$ a.e. θ. By self-similarity and the properties of $\mathcal{P}^s$ (translation invariance and scaling), we have $\mathcal{P}^s(\Lambda_i^\theta \cap \Lambda_j^\theta) = 0$ for $i \ne j$. Then we use [11, Corollary 2.2], which implies that ν_θ is the normalized restriction of $\mathcal{P}^s$ to Λ^θ, to complete the proof. □

Remark. In [1, Proposition 2], it is claimed that if a self-similar set $\mathcal{K} = \bigcup_{i=1}^m \mathcal{K}_i$ in $\mathbb{R}^d$ has the Hausdorff dimension equal to the similarity dimension, then the natural measure of the "overlap set" $\bigcup_{i \ne j}(\mathcal{K}_i \cap \mathcal{K}_j)$ is zero. This would imply Corollary 5.1, since the Hausdorff dimension of Λ^θ equals s for $\mathcal{L}$ a.e. θ by Marstrand's Projection Theorem. Unfortunately, the proof in [1] contains an error, and it is still unknown whether the result holds [C. Bandt, personal communication]. (It should be noted that [1, Proposition 2] was not used anywhere in [1].)

Acknowledgment. We are grateful to M. Csörnyei, E. Järvenpää, and M. Järvenpää for helpful discussions. This work was done while K. S. was visiting the University of Washington.

References

[1] C. Bandt and S. Graf, *Self-similar Sets 7. A Characterization of Self-similar Fractals with Positive Hausdorff Measure*, Proc. Amer. Math. Soc., **114** (1992), 995–1001.

[2] A. S. Besicovitch, *On the Fundamental Geometric Properties of Linearly Measurable Plane Sets of Points II*, Math. Annalen, **115** (1938), 296–329.

[3] M. Csörnyei, *How to Make Davies' Theorem Visible*, Bull. London Math. Soc., **33** (2001), 59–66.

[4] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Pres, 1985.

[5] J. E. Hutchinson, *Fractals and Self-similarity*, Indiana Univ. Math. J., **30** (1981), 713–747.

[6] J. Marstrand, *Some Fundamental Geometrical Properties of Plane Sets of Fractional Dimension*, Proc. London Math. Soc., **4** (1954), 257–302.

[7] P. Mattila, *Integralgeometric Properties of Capacities*, Trans. Amer. Math. Soc., **266** (1981), 539–544.

[8] P. Mattila, *On the Structure of Self-similar Fractals*, Ann. Acad. Sci. Fenn. Ser. A I Math., **7** (1982), 189–195.

[9] P. Mattila, *The Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[10] P. Mattila, *Hausdorff Dimension, Projections, and the Fourier Transform*, Publ. Math., **48** (2004), 3–48.

[11] Y. Peres, K. Simon and B. Solomyak, *Self-similar Sets of Zero Hausdorff Measure and Positive Packing Measure*, Israel J. Math., **117** (2000), 353–379.