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EXTENSIONS OF REAL AND VECTOR FUNCTIONS OF ONE VARIABLE WHICH PRESERVE DIFFERENTIABILITY

Abstract

Let $f : F \to X$ be a locally bounded function from a closed set $F \subset \mathbb{R}$ to a normed linear space. Then there exists its extension $f^* : \mathbb{R} \to X$ which is differentiable at all points at which f is differentiable. Moreover, f^* is Lipschitz if f is Lipschitz and, in the case $X = \mathbb{R}$, the extension "preserves Dini derivatives". The paper partly extends results proved by V. Jarník (1923), G. Petruska and M. Laczkovich (1974) and J. Mařík (1984).

1 Introduction.

In our paper [NZ] we needed the following assertion on extension of real functions with "preservation of differentiability and Lipschitzness".

Proposition 1.1. *Let $A \subset \mathbb{R}$ be an arbitrary set and $h : A \to \mathbb{R}$ be a Lipschitz function which has a derivative with respect to A at each non-isolated point of A. Then there exists a Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ which is an extension of h and $g'(x)$ exists for all $x \in A$.*

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There exist at least three papers ([J], [PL], [M]) which contain similar results but these results do not imply Proposition 1.1. On the other hand, it is not difficult to prove it using methods of [J] or [M].

In this note we slightly modify the method of [J] to obtain not only Proposition 1.1 (see Proposition 3.6) but also

a) a result on extensions of vector functions ($f : F \to X$ where $F \subset \mathbb{R}$ is closed and X is a normed linear space) with "preserving of differentiability and Lipschitzness" and

b) a result on extensions of real functions with "preservation of differentiability", "preservation of Dini derivatives" and "preservation of Lipschitzness".

Note that the result on vector functions seems to be new but extensions preserving Dini derivatives were considered in [J] (cf. also [Pr]). However, Jarník in [J] assumed that F is perfect and $f'_F(x)$ exists at each endpoint x of a component of $\mathbb{R} \setminus F$ and he did not consider the question of "preserving Lipschitzness". Moreover, the result in [J] is proved for the case of finite Dini derivatives only.

Note that in [PL] and [M] more precise results on "preserving Lipschitzness" are proved but the assumption that $f'_F(t)$ exists at each $t \in F$ is made.

In what follows we suppose that X is a fixed normed linear space and we use the following terminology.

Notation.

(a) If $M \subset X$ and $\omega > 0$, then we set

$$U_\omega(M) := \{x \in X; \operatorname{dist}(x, M) < \omega\}$$

and denote by $\operatorname{conv}(M)$ the convex hull of M. Observe that $U_\omega(M)$ is convex provided M is convex.

(b) If f is a Lipschitz mapping from a metric space to a metric space, then we denote by $\operatorname{Lip}(f)$ the minimal Lipschitz constant of f.

(c) By $f\restriction_P$ we denote the restriction of a mapping f to a set P.

(d) Let $M \subset \mathbb{R}$, t a right accumulation point of M and $f : M \to X$ be given. Then we put

$$f'_{+,M}(t) := \lim_{s \searrow t, s \in M} \frac{f(s) - f(t)}{s - t}.$$

Analogously we define the left relative derivative $f'_{-,M}(t)$ (the relative derivative $f'_M(t)$) if t is a left accumulation point of M (an accumulation point of M).

(e) Let $M \subset \mathbb{R}$, t a right accumulation point of M and $f : M \to \mathbb{R}$ be given. Then we define right (relative) Dini's derivatives of f at t by

$$D_{+,M} f(t) = \liminf_{s \searrow t, s \in M} \frac{f(s) - f(t)}{s - t}, \quad D^+_M f(t) = \limsup_{s \searrow t, s \in M} \frac{f(s) - f(t)}{s - t}.$$

Analogously we define the symbols $D_{-,M} f(t)$ and $D^-_M f(t)$.

Now we will define an auxiliary notion of a $\mathcal{D}$-set which enables us to deal with derivatives of vector functions and Dini derivatives of real functions at the same time.

Definition 1.2. *Let $M \subset \mathbb{R}$, $t \in M$ be a right accumulation point of M and $f : M \to X$ be a vector function. We say that a set $C \subset X$ is a right $\mathcal{D}$-set of f at the point t if*

(i) *C is non-empty, convex and closed and*

(ii) $\operatorname{dist}\Big(\dfrac{f(t_n) - f(t)}{t_n - t}, C\Big) \to 0$ *whenever $t_n \searrow t$, $t_n \in M$.*

The notion of a left $\mathcal{D}$-set is defined in an analogous fashion.

It is easy to see that condition (ii) is equivalent to the following condition. For each $\omega > 0$ there exists $T > t$ such that, for all $s \in (t, T) \cap M$,

$$\frac{f(s) - f(t)}{s - t} \in U_\omega(C). \tag{1.1}$$

We will apply the notion of a $\mathcal{D}$-set via the following obvious lemma which we state without proof.

Lemma 1.3. *Let $M \subset \mathbb{R}$ and $t \in M$. Then the following assertions hold:*

(i) *Let $x \in X$. Then $f'_{+,M}(t) = x$ if and only if the singleton $\{x\}$ is a right $\mathcal{D}$-set of f at t.*

(ii) *Let $X = \mathbb{R}$, $D_{+,M} f(t) < D^+_M f(t)$ and $-\infty \le a < b \le \infty$. Then the set $C := \{x \in \mathbb{R}; a \le x \le b\}$ is a right $\mathcal{D}$-set of f at t if and only if $a \le D_{+,M} f(t)$, $D^+_M f(t) \le b$.*

(iii) *Let $X = \mathbb{R}$. Then $f'_{+,M} = \infty$ $(-\infty)$ if and only if $[K, \infty)$ $((-\infty, K])$ is a right $\mathcal{D}$-set of f at t for each $K \in \mathbb{R}$.*

Analogous assertions hold for the left derivative and for left Dini derivatives.

We will need also the following easy fact.

Lemma 1.4. *Let $F \subset \mathbb{R}$ be closed, let $f : \mathbb{R} \to X$ be a continuous function and $K \geq 0$. Assume that* $\operatorname{Lip}(f\restriction_F) \leq K$ *and* $\operatorname{Lip}\left(f\restriction_{(a,b)}\right) \leq K$ *for each component (a,b) of $\mathbb{R} \setminus F$. Then* $\operatorname{Lip}(f) \leq K$.

Proof. Let $t < s$ be given. Consider the case $t \in F$ and $s \in (a,b)$ where (a,b) is a component of $\mathbb{R} \setminus F$. Then clearly

$$\|f(s)-f(t)\| \leq \|f(s)-f(a)\|+\|f(a)-f(t)\| \leq K(s-a)+K(a-t) = K(s-t).$$

Analogously we can handle with other cases. □

2 An Auxiliary Function.

Recall that X is a fixed normed linear space.

The basic building block of our construction is the function p_z constructed in the following lemma. Put

$$\mathcal{Z} = \{z = (a,b,\delta,A,B,A',B') \in \mathbb{R}^3 \times X^4;\ a < b,\ 0 < \delta < (b-a)/2\}.$$

Lemma 2.1. *Let $z = (a,b,\delta,A,B,A',B') \in \mathcal{Z}$ and set $k = \frac{B-A}{b-a}$. Then there exists a function $p(t) = p_z(t) : [a,b] \to X$ which satisfies:*

$$p'(t) \text{ exists for all } t \in (a,b), \tag{2.1}$$

$$p(a) = A,\ p(b) = B,\ p'_+(a) = A',\ p'_-(b) = B', \tag{2.2}$$

$$\operatorname{Lip}(p) \leq 5\max\{\|A'\|, \|B'\|, \|k\|\}. \tag{2.3}$$

$$\frac{p(t)-A}{t-a} \in \operatorname{conv}\{A',k\} \text{ for } t \in (a,a+\delta],\quad \frac{p(t)-B}{t-b} \in \operatorname{conv}\{B',k\} \text{ for } t \in [b-\delta,b), \tag{2.4}$$

$$p(t) = A + k(t-a) \text{ if } t \in (a+\delta, b-\delta), \text{ or } t \in [a,a+\delta] \text{ and } A' = k, \text{ or } t \in [b-\delta,b] \text{ and } B' = k. \tag{2.5}$$

Proof. Define a function p on $[a,b]$ by

$$p(t) = \begin{cases} A + (t-a)\left((A'-k)\frac{(a+\delta-t)^2}{\delta^2} + k\right) & t \in [a,a+\delta], \\ A + k(t-a) & t \in (a+\delta, b-\delta), \\ B + (t-b)\left((B'-k)\frac{(b-\delta-t)^2}{\delta^2} + k\right) & t \in [b-\delta,b]. \end{cases} \tag{2.6}$$

Clearly (2.5) holds. It is easy to calculate

$$p'_{[a,b]}(t) = \begin{cases} (A'-k)\frac{(a+\delta-t)(3a+\delta-3t)}{\delta^2} + k & t \in [a, a+\delta], \\ k & t \in (a+\delta, b-\delta), \\ (B'-k)\frac{(b-\delta-t)(3b-\delta-3t)}{\delta^2} + k & t \in [b-\delta, b]. \end{cases} \tag{2.7}$$

Obviously (2.6) and (2.7) imply (2.1) and (2.2).

If $a \le t \le a+\delta$, then $|3a+\delta-3t| \le 2\delta$,

$$\left| \frac{(a+\delta-t)(3a+\delta-3t)}{\delta^2} \right| \le 2$$

and consequently

$$\mathrm{Lip}(p\restriction_{[a,a+\delta]}) \le 2\|A'-k\| + \|k\| \le 5\max\{\|A'\|, \|B'\|, \|k\|\}.$$

Analogously,

$$\mathrm{Lip}(p\restriction_{[b-\delta,b]}) \le 2\|B'-k\| + \|k\| \le 5\max\{\|A'\|, \|B'\|, \|k\|\}.$$

This gives with Lemma 1.4 and $\mathrm{Lip}(p\restriction_{[a+\delta,b-\delta]}) = \|k\|$ the assertion (2.3).

Clearly, setting $\lambda = \left(\frac{a+\delta-t}{\delta}\right)^2$, for $t \in (a, a+\delta]$ we have

$$\frac{p(t)-A}{t-a} = (A'-k)\lambda + k = \lambda A' + (1-\lambda)k,$$

which proves the first part of (2.4). The proof of the second part is quite analogous. □

3 Extensions.

Recall that X is a fixed normed linear space. It will be useful to use the following terminology.

Definition 3.1. *Let $A \subset B \subset \mathbb{R}$ and $g : B \to X$ be an extension of $f : A \to X$. We say that g is an (DLCB)-extension (i.e., extension preserving differentiability, Lipschitzness, continuity and boundedness) of f if it has the following properties:*

(i) *If $t \in A$ and $f'_A(t)$ exists, then $g'_B(t) = f'_A(t)$. If $t \in B$ is an accumulation point of B but not of A, then $g'_B(t)$ exists.*

(ii) *If $t \in A$ and C is a right(left) $\mathcal{D}$-set of f at t, then C is a right(left) $\mathcal{D}$-set of g at t.*

(iii) *If f is Lipschitz, then g is Lipschitz.*

(iv) *If f is continuous at $t \in A$, then g is continuous at t.*

(v) *If f is (locally) bounded, then g is (locally) bounded.*

We will use the following useful obvious fact.

If g is an (DLCB)-extension of f and h is an (DLCB)-extension of g, then h is an (DLCB)-extension of f.

We will apply the property (ii) via the following lemma.

Lemma 3.2. *Let $A \subset B \subset \mathbb{R}$ and $g : B \to X$ be an (DLCB)-extension of $f : A \to X$. Then the following assertions hold:*

(i) *If $t \in A$ and $f'_{+,A}(t)(f'_{-,A}(t))$ exists, then $g'_{+,B}(t) = f'_{+,A}(t)(g'_{-,B}(t) = f'_{-,A}(t))$.*

(ii) *If $X = \mathbb{R}$ and $t \in A$ is a right (left) accumulation point of A, then $D_{+,A}f(t) = D_{+,B}g(t)$, $D^+_A f(t) = D^+_B g(t)$ ($D_{-,A}f(t) = D_{-,B}g(t)$, $D^-_A f(t) = D^-_B g(t)$).*

Proof. To prove (i), it suffices to apply property (ii) of Definition 3.1 and Lemma 1.3 (i) to $C := \{f'_{A,+}(t)\}$ ($C := \{f'_{A,-}(t)\}$).

To prove (ii), suppose that $t \in A$ is a right accumulation point of A. Clearly $D_{+,B}g(t) \le D_{+,A}f(t) \le D^+_A f(t) \le D^+_B g(t)$. To prove $D_{+,A}f(t) = D_{+,B}g(t)$ and $D^+_A f(t) = D^+_B g(t)$, it is sufficient to use (i) if $D_{+,A}f(t) = D^+_A f(t) \in \mathbb{R}$ and to apply the property (ii) of Definition 3.1 and Lemma 1.3 (ii), (iii) to

(α) $C = \{x \in \mathbb{R}; a \le x \le b\}$, where $a := D_{+,A}f(t)$, $b := D^+_A f(t)$, if $D_{+,A}f(t) < D^+_A f(t)$,

(β) $C = [K, \infty)$ for all $K \in \mathbb{R}$, if $D_{+,A}f(t) = D^+_A f(t) = \infty$,

(γ) $C = (-\infty, -K]$ for all $K \in \mathbb{R}$, if $D_{+,A}f(t) = D^+_A f(t) = -\infty$. □

Lemma 3.3. *Let X be a Banach space, $A \subset \mathbb{R}$ and $f : A \to X$ be a Lipschitz function. Then there exists a unique (DLCB)-extension $f^* : \overline{A} \to X$ of f.*

Proof. Since $f : A \to X$ is Lipschitz and consequently, uniformly continuous, there is a unique continuous extension $f^* : \overline{A} \to X$ of f. It is easy to see that $\mathrm{Lip}(f^*) = \mathrm{Lip}(f)$ and thus the properties (iii), (iv) and (v) of (DLCB)-extension are satisfied for $B := \overline{A}$ and $g := f^*$.

Let us prove (ii). Let $t \in A$ and C be a right $\mathcal{D}$-set of f at t. Consider an arbitrary $\omega > 0$. By (1.1) we can choose $T > t$ such that $\frac{f(s)-f(t)}{s-t} \in U_{\omega/2}(C)$

for each $s \in (t,T) \cap A$. If $s \in (t,T) \cap \overline{A}$ then there exists a sequence $s_n \in (t,T) \cap A$ such that $s_n \to s$. Clearly $\frac{f(s_n)-f(t)}{s_n-t} \to \frac{f^*(s)-f(t)}{s-t}$ and therefore $\frac{f^*(s)-f(t)}{s-t} \in \overline{U_{\omega/2}(C)} \subset U_\omega(C)$, which shows that C is a right $\mathcal{D}$-set of f^* at t. The "left case" is quite symmetrical.

Condition (i) easily follows from (ii) (cf. the proof of Lemma 3.2 (i)). □

The basic properties of our (modification of Jarník's) extension construction which can be applied both to vector and real case are formulated in the following proposition.

Proposition 3.4. *Let $\emptyset \neq F \subset \mathbb{R}$ be a closed set and $f : F \to X$ be a locally bounded vector function. Then there exists $g : \mathbb{R} \to X$ which is a (DLCB)-extension of f.*

PROOF. We will divide the proof into two steps.
Step 1. Assume that F is perfect. We may suppose that $\inf F = -\infty$, $\sup F = \infty$. (Indeed, it suffices to observe that the set $F^* = F \cup (-\infty, \inf F - 1] \cup [\sup F + 1, \infty)$ is also perfect and the function $f^* : F^* \to X$ defined by $f^*\restriction_F = f$ and $f^* = 0$ on $(\infty, \inf F - 1] \cup [\sup F + 1, \infty)$ is clearly a (DLCB)-extension of f.)

Let $\{(a_n, b_n) : 1 \leq n < \alpha\}$ (where $\alpha \in \mathbb{N}$ or $\alpha = \infty$) be an ordering of all components of $\mathbb{R} \setminus F$. Let $F_1 = \{a_n;\ f'_F(a_n) \text{ exists}\}$, $F_2 = \{b_n;\ f'_F(b_n) \text{ exists}\}$, $k_n = \frac{f(b_n)-f(a_n)}{b_n-a_n}$ and set

$$A'_n = \begin{cases} f'_F(a_n) & \text{if } a_n \in F_1, \\ k_n & \text{if } a_n \notin F_1; \end{cases} \qquad B'_n = \begin{cases} f'_F(b_n) & \text{if } b_n \in F_2, \\ k_n & \text{if } b_n \notin F_2. \end{cases}$$

For each $1 \leq n < \alpha$, find $\varepsilon_n > 0$ such that

$$\varepsilon_n < \min\left\{1, \frac{b_n - a_n}{n+1}\right\}; \tag{3.1}$$

$$\left\| \frac{f(s) - f(a_n)}{s - a_n} - f'_F(a_n) \right\| < \frac{1}{n} \text{ for } s \in [a_n - \varepsilon_n, a_n) \cap F \text{ if } a_n \in F_1; \tag{3.2}$$

$$\left\| \frac{f(s) - f(b_n)}{s - b_n} - f'_F(b_n) \right\| < \frac{1}{n} \text{ for } s \in (b_n, b_n + \varepsilon_n] \cap F \text{ if } b_n \in F_2. \tag{3.3}$$

Further choose $\delta_n > 0$ such that

$$5 \max\{\|A'_n\|, \|B'_n\|, \|k_n\|, 1\}\, \delta_n < \min\{\varepsilon_n^3, \frac{1}{n}\}. \tag{3.4}$$

It is easy to see that this choice with (3.1) guarantees $\delta_n < (b_n - a_n)/2$. Set $p_n := p_{z_n}$ where $z_n := (a_n, b_n, \delta_n, f(a_n), f(b_n), A'_n, B'_n)$ and the function p_{z_n} is taken from Lemma 2.1. Set

$$g(t) = \begin{cases} f(t) & \text{if } t \in F, \\ p_n(t) & \text{if } t \in (a_n, b_n). \end{cases} \tag{3.5}$$

If f is Lipschitz, then clearly $\max\{\|A'_n\|, \|B'_n\|, \|k_n\|\} \le \mathrm{Lip}(f)$. Consequently, (2.3) and Lemma 1.4 imply that g is Lipschitz which proves (iii) in Definition 3.1.

By (2.3) and (3.4) we have

$$\|g(t) - f(a_n)\| \le \min\{\varepsilon_n^3, \frac{1}{n}\} \text{ for } t \in [a_n, a_n + \delta_n], \tag{3.6}$$

and

$$\|g(t) - f(b_n)\| \le \min\{\varepsilon_n^3, \frac{1}{n}\} \text{ for } t \in [b_n - \delta_n, b_n]. \tag{3.7}$$

Consequently (2.5) gives

$$\mathrm{dist}\big(g(s), \mathrm{conv}\{f(a_n), f(b_n)\}\big) \le \frac{1}{n} \quad \text{for each} \quad s \in [a_n, b_n], \tag{3.8}$$

which easily implies that (iv) and (v) in Definition 3.1 hold.

To prove (ii) in Definition 3.1, it suffices to verify only the "right case", since the "left case" is quite analogous. Assume that $t \in F$ is a right accumulation point of F and $C \subset X$ is a right $\mathcal{D}$-set of f at t. Let us prove that C is also a right $\mathcal{D}$-set of g at t. Assume that t is also a right accumulation point of $\mathbb{R} \setminus F$; the opposite case is trivial. Consider for each $s > t$, $s \notin F$ the integer $n(s)$, $1 \le n(s) < \alpha$, for which $s \in (a_{n(s)}, b_{n(s)})$. We already have verified condition (v) in Definition 3.1; thus there exist $L > 0$ and $T_1 > t$ such that

$$\|g(s)\| \le L \text{ for all } s \in [t, T_1]. \tag{3.9}$$

Consider an arbitrary $\omega > 0$. By (1.1) we can choose $T_2 \in (t, T_1)$ such that

$$\frac{f(s) - f(t)}{s - t} \in U_{\frac{\omega}{2}}(C) \text{ for each } s \in (t, T_2) \cap F. \tag{3.10}$$

Further, since clearly $n(s) \to \infty$ and $b_{n(s)} \to t$ provided $s \to t_+$, $s \notin F$, it is possible to find $T_3 \in (t, T_2)$ such that

$$\max\left\{\varepsilon_{n(s)}(\varepsilon_{n(s)} + 2L), \frac{1}{n(s)}\right\} < \frac{\omega}{2} \text{ for each } s \in (t, T_3) \setminus F \tag{3.11}$$

and also

$$b_{n(s)} < T_2 \text{ for each } s \in (t, T_3) \setminus F. \tag{3.12}$$

Now fix an arbitrary $s \in (t, T_3) \setminus F$ and let $n := n(s),\ a_n := a_{n(s)},\ b_n := b_{n(s)},\ \varepsilon_n := \varepsilon_{n(s)},\ \delta_n := \delta_{n(s)},\ k_n := \dfrac{f(b_n) - f(a_n)}{b_n - a_n}$. Distinguish four possible cases.

I. Let $g(s) = f(a_n) + k_n(s - a_n)$. Since $a_n < s < b_n$, there is $\lambda \in (0,1)$ such that

$$s = \lambda a_n + (1-\lambda) b_n. \tag{3.13}$$

Consequently, $s - a_n = (1-\lambda)(b_n - a_n)$ and

$$\begin{aligned} g(s) = f(a_n) + k_n(s - a_n) &= f(a_n) + \frac{f(b_n) - f(a_n)}{b_n - a_n}(1-\lambda)(b_n - a_n) \\ &= \lambda f(a_n) + (1-\lambda) f(b_n) \end{aligned}$$

which gives

$$\begin{aligned} \frac{g(s) - f(t)}{s - t} &= \frac{1}{s-t}\big(\lambda f(a_n) + (1-\lambda) f(b_n) - f(t)\big) \\ &= \lambda \frac{a_n - t}{s - t}\frac{f(a_n) - f(t)}{a_n - t} + (1-\lambda)\frac{b_n - t}{s - t}\frac{f(b_n) - f(t)}{b_n - t}. \end{aligned}$$

Since by (3.13)

$$\lambda \frac{a_n - t}{s - t} + (1-\lambda)\frac{b_n - t}{s - t} = \frac{1}{s-t}\big(\lambda a_n + (1-\lambda) b_n - t\big) = \frac{s-t}{s-t} = 1,$$

we obtain

$$\frac{g(s) - f(t)}{s - t} \in \operatorname{conv}\Big\{\frac{f(a_n) - f(t)}{a_n - t}, \frac{f(b_n) - f(t)}{b_n - t}\Big\}.$$

Now, by (3.12) and (3.10) we obtain $\dfrac{f(a_n) - f(t)}{a_n - t} \in U_{\frac{\omega}{2}}(C),\ \dfrac{f(b_n) - f(t)}{b_n - t} \in U_{\frac{\omega}{2}}(C)$ which gives with the convexity of $U_{\frac{\omega}{2}}(C)$

$$\frac{g(s) - f(t)}{s - t} \in U_{\frac{\omega}{2}}(C). \tag{3.14}$$

II. Let $g(s) \neq f(a_n) + k_n(s - a_n)$ and $s \in (b_n - \delta_n, b_n)$. Since $\delta_n < \dfrac{b_n - a_n}{2}$, we have by (3.1) $b_n - a_n - \delta_n > \dfrac{b_n - a_n}{2} > \varepsilon_n$, which gives with (3.7), (3.9), (3.4) and (3.11)

$$\left\|\frac{g(s)-f(t)}{s-t}-\frac{f(b_n)-f(t)}{b_n-t}\right\| = \left\|\frac{g(s)-f(b_n)}{s-t}+\frac{f(b_n)-f(t)}{b_n-t}\frac{b_n-s}{s-t}\right\|$$
$$\leq \left\|\frac{g(s)-f(b_n)}{s-t}\right\| + \|f(b_n)-f(t)\|\frac{b_n-s}{(b_n-t)(s-t)}$$
$$\leq \frac{\varepsilon_n^3}{b_n-a_n-\delta_n}+2L\frac{\delta_n}{(b_n-a_n-\delta_n)^2} < \frac{\varepsilon_n^3}{\varepsilon_n}+\frac{2L\varepsilon_n^3}{\varepsilon_n^2} = \varepsilon_n(\varepsilon_n+2L) < \frac{\omega}{2}.$$

Using this with $\frac{f(b_n)-f(t)}{b_n-t} \in U_{\frac{\omega}{2}}(C)$, we obtain

$$\frac{g(s)-g(t)}{s-t} \in U_\omega(C). \tag{3.15}$$

III. Let $g(s) \neq f(a_n)+k_n(s-a_n)$, $s \in (a_n, a_n+\delta_n)$ and $t \leq a_n - \varepsilon_n$. Since $s-t > a_n - t \geq \varepsilon_n$ we have by (3.6), (3.9), (3.4) and (3.11)

$$\left\|\frac{g(s)-f(t)}{s-t}-\frac{f(a_n)-f(t)}{a_n-t}\right\| = \left\|\frac{g(s)-f(a_n)}{s-t}+\frac{f(a_n)-f(t)}{a_n-t}\frac{a_n-s}{s-t}\right\|$$
$$\leq \left\|\frac{g(s)-f(a_n)}{s-t}\right\| + \|f(a_n)-f(t)\|\frac{s-a_n}{(a_n-t)(s-t)}$$
$$\leq \frac{\varepsilon_n^3}{\varepsilon_n}+2L\frac{\delta_n}{\varepsilon_n^2} \leq \varepsilon_n(\varepsilon_n+2L) < \frac{\omega}{2}.$$

Using this with $\dfrac{f(a_n)-f(t)}{a_n-t} \in U_{\frac{\omega}{2}}(C)$, we obtain

$$\frac{g(s)-g(t)}{s-t} \in U_\omega(C). \tag{3.16}$$

IV. Let $g(s) \neq f(a_n)+k_n(s-a_n)$, $s \in (a_n, a_n+\delta_n)$ and $a_n - \varepsilon_n < t < a_n$. Note that by (2.5) we have $a_n \in F_1$. Observe that

$$\frac{g(b_n)-g(t)}{b_n-t} = \frac{g(b_n)-g(a_n)}{b_n-a_n}\;\frac{b_n-a_n}{b_n-t}+\frac{g(a_n)-g(t)}{a_n-t}\;\frac{a_n-t}{b_n-t} \tag{3.17}$$

and by (2.4) there exists $0 \leq \lambda \leq 1$ such that

$$\frac{g(s)-g(t)}{s-t} = \frac{g(s)-g(a_n)}{s-a_n}\;\frac{s-a_n}{s-t}+\frac{g(a_n)-g(t)}{a_n-t}\;\frac{a_n-t}{s-t} \tag{3.18}$$
$$= \Big(\lambda f'_F(a_n)+(1-\lambda)k_n\Big)\;\frac{s-a_n}{s-t}+\frac{g(a_n)-g(t)}{a_n-t}\;\frac{a_n-t}{s-t}.$$

Setting

$$D := \Big(\lambda \frac{g(a_n)-g(t)}{a_n-t} + (1-\lambda)k_n\Big)\frac{s-a_n}{s-t} + \frac{g(a_n)-g(t)}{a_n-t}\ \frac{a_n-t}{s-t} \tag{3.19}$$

we obtain by (3.18), (3.19), (3.2) and (3.11) that

$$\Big\|\frac{g(s)-g(t)}{s-t} - D\Big\| = \lambda \Big\| f'_F(a_n) - \frac{g(a_n)-g(t)}{a_n-t}\Big\|\Big(\frac{s-a_n}{s-t}\Big) < \frac{1}{n} < \frac{\omega}{2}. \tag{3.20}$$

Since $k_n = \dfrac{g(b_n)-g(a_n)}{b_n-a_n}$, the equalities (3.17) and (3.19) can be rewritten in the form

$$\frac{g(b_n)-g(t)}{b_n-t} = k_n.\mu + \frac{g(a_n)-g(t)}{a_n-t}(1-\mu), \tag{3.21}$$

and

$$D = k_n.\mu^* + \frac{g(a_n)-g(t)}{a_n-t}(1-\mu^*) \tag{3.22}$$

where $\mu := \dfrac{b_n-a_n}{b_n-t}$ and $\mu^* = (1-\lambda)\dfrac{s-a_n}{s-t}$. Clearly, $0 \le \mu^* < \mu \le 1$, and so, (3.21) and (3.22) easily imply (it is geometrically obvious) that

$$D \in \operatorname{conv}\Big\{\frac{g(a_n)-g(t)}{a_n-t}, \frac{g(b_n)-g(t)}{b_n-t}\Big\}.$$

Using (3.10) and (3.12) we obtain $D \in U_{\frac{\omega}{2}}(C)$ and therefore (3.20) implies

$$\frac{g(s)-g(t)}{s-t} \in U_\omega(C).$$

Using also (3.14), (3.15) and (3.16), we obtain that C is a right $\mathcal{D}$-set of g at t, which proves (ii).

To prove the first part of (i) in Definition 3.1, suppose that $t \in F$ and $f'_F(t)$ exists. If $t \in F \setminus (F_1 \cup F_2)$, it is sufficient to apply (ii) to $C := \{f'_F(t)\}$; in the opposite case we use moreover (2.2) and the definition of A'_n and B'_n. The second part of (i) follows immediately from (2.1).

Step 2. Now suppose that $F \subset \mathbb{R}$ is a closed non-perfect set. According to Step 1 it suffices to find a perfect set $P \supset F$ and a function $h : P \to X$ which is a (DLCB)-extension of f.

Denote by F' the (closed) set of all accumulation points of F and suppose that $\{r_n; 1 \le n < \alpha\}$, $\alpha \in \mathbb{N} \cup \{\infty\}$ is an ordering of all isolated points of F. For each $1 \le n < \alpha$ set

$$d_n := \operatorname{dist}(r_n, F \setminus \{r_n\}), \quad \delta_n := \min\left\{\frac{1}{4}d_n, d_n^3\right\} \tag{3.23}$$

and $J_n := [r_n - \delta_n, r_n + \delta_n]$. Clearly, $J_n \cap F' = \emptyset$ for each $1 \le n < \alpha$. First we will show that

$$|s - t| \ge \frac{1}{2}|r_n - r_m| \text{ for } 1 \le n,\ m < \alpha,\ n \ne m \text{ and } t \in J_n,\ s \in J_m. \quad (3.24)$$

Indeed, the above inequality immediately follows from inequalities

$$\delta_n \le \frac{1}{4}|r_n - r_m| \ \ , \delta_m \le \frac{1}{4}|r_n - r_m| \quad (3.25)$$

and $|r_n - r_m| \le |t - s| + \delta_n + \delta_m$. In particular, the intervals $\{J_n; 1 \le n < \alpha\}$ are pairwise disjoint.

Moreover, we will show that for each $t \in \mathbb{R} \setminus F'$ there exists $\varepsilon_t > 0$ such that $(t - \varepsilon_t, t + \varepsilon_t)$ intersects at most one interval J_n and as a consequence we obtain that $P := F \cup \bigcup\{J_n; 1 \le n < \alpha\}$ is closed and therefore perfect.

Indeed, if $t = r_n$ for some $1 \le n < \alpha$, then we can choose $\varepsilon_t := \delta_n$. Otherwise, $t \notin F$ and it suffices to set $\varepsilon_t := \frac{1}{4}\text{dist}(t, F)$. To see it, suppose that, for some $n \ne m$,

$$J_n \cap (t - \varepsilon_t, t + \varepsilon_t) \ne \emptyset, \quad J_m \cap (t - \varepsilon_t, t + \varepsilon_t) \ne \emptyset.$$

Then $\varepsilon_t \le \frac{1}{4}|t - r_n|$, $|t - r_n| \le \varepsilon_t + \delta_n$ and consequently, $\delta_n \ge \frac{3}{4}|t - r_n|$. Analogously we have $\delta_m \ge \frac{3}{4}|t - r_m|$ and using (3.25) we obtain

$$|r_n - r_m| \le |t - r_n| + |t - r_m| \le \frac{4}{3}(\delta_n + \delta_m) \le \frac{2}{3}|r_n - r_m|,$$

which is a contradiction.

Define a function $h : P \to X$ by $h(t) = f(t)$ if $t \in F'$ and $h(t) = f(r_n)$ if $t \in J_n$ and let us prove that h is a (DLCB)-extension of f.

The properties (iv) and (v) in Definition 3.1 are easy to see.

To prove (iii) in Definition 3.1, let $K := \text{Lip}(f)$ and $t, s \in P$ be given. If $t \in J_n$ and $s \in F'$, then (3.23) implies

$$|t - s| \ge |r_n - s| - \delta_n \ge \frac{3}{4}|r_n - s|$$

and therefore

$$|h(t) - h(s)| = |f(r_n) - f(s)| \le K|r_n - s| \le \frac{4}{3}K|t - s|.$$

If $t \in J_n$, $s \in J_m$, $n \ne m$, then by (3.24) we obtain

$$|h(t) - h(s)| = |f(r_n) - f(r_m)| \le K|r_n - r_m| \le 2K|t - s|.$$

Consequently $\mathrm{Lip}(h) \le 2K$.

To prove (ii) in Definition 3.1, it suffices to verify only the "right case", since the "left case" is quite analogous. Assume that $t \in F$ is a right accumulation point of F and $C \subset X$ is a right $\mathcal{D}$-set of f at t. Let us prove that C is also a right $\mathcal{D}$-set of h at t. Assume that t is also a right accumulation point of $\mathbb{R} \setminus F$; the opposite case is trivial.

Let $\omega > 0$. By (1.1) there exists $T > t$ such that

$$\mathrm{dist}\Big(\frac{f(r)-f(t)}{r-t}, C\Big) < \frac{\omega}{2} \text{ for } r \in F \cap (t,T]. \tag{3.26}$$

Consider for each $s \in P \setminus F'$ the integer $n(s)$ for which $s \in J_{n(s)}$. Since h is locally bounded, there exists $t < T_1 < T$ and $L > 0$ such that

$$\|h(s)\| \le L \text{ for } s \in P \cap [t, T_1].$$

By (3.23) we have

$$|s-t| \ge |r_{n(s)} - t| - |r_{n(s)} - s| \ge \mathrm{d}_{n(s)} - \frac{1}{4}\mathrm{d}_{n(s)}$$

and

$$\Big|\frac{1}{s-t} - \frac{1}{r_{n(s)}-t}\Big| = \frac{|s - r_{n(s)}|}{(s-t)(r_{n(s)}-t)} \le \frac{\mathrm{d}^3_{n(s)}}{\frac{3}{4}\mathrm{d}^2_{n(s)}} = \frac{4}{3}\mathrm{d}_{n(s)}.$$

Since $\mathrm{d}_{n(s)} \to 0$ if $s \to t_+$, $s \in P \setminus F$, it is possible to find T_2, $t < T_2 < T_1$, such that for $s \in (P \setminus F) \cap (t, T_2]$ we have

$$2L\Big|\frac{1}{s-t} - \frac{1}{r_{n(s)}-t}\Big| < \frac{\omega}{2} \quad \text{and} \quad r_{n(s)} < T.$$

It gives with (3.26) that, for each $s \in (P \setminus F) \cap (t, T_2]$,

$$\begin{aligned}
&\mathrm{dist}\Big(\frac{h(s)-h(t)}{s-t}, C\Big) = \mathrm{dist}\Big(\frac{f(r_{n(s)})-f(t)}{s-t}, C\Big) \\
&\le \Big\|\frac{f(r_{n(s)})-f(t)}{s-t} - \frac{f(r_{n(s)})-f(t)}{r_{n(s)}-t}\Big\| + \mathrm{dist}\Big(\frac{f(r_{n(s)})-f(t)}{r_{n(s)}-t}, C\Big) \\
&\le 2L\Big|\frac{1}{s-t} - \frac{1}{r_{n(s)}-t}\Big| + \frac{\omega}{2} < \omega.
\end{aligned}$$

Therefore C is a right $\mathcal{D}$-set of h at t, which proves (ii). To prove the first part of (i) in Definition 3.1, it is sufficient to apply (i) to $C := \{f'_F(t)\}$. The second part of (i) is obvious from the construction of h. $\square$

As easy consequences of Proposition 3.4 we obtain the following main results of the paper.

Theorem 3.5. *Let $\emptyset \neq F \subset \mathbb{R}$ be a closed set and X be a normed linear space. Then for each locally bounded $f : F \to X$ there exists $g : \mathbb{R} \to X$ which extends f and has the following properties:*

(i) *If $t \in F$ and $f'_F(t)$ exists, then $g'(t) = f'_F(t)$.*

(ii) *$g'(t)$ exists if $t \notin F$ or t is an isolated point of F.*

(iii) *If $t \in F$ and $f'_{+,F}(f'_{-,F})$ exists, then $g'_+(t) = f'_{+,F}(g'_-(t) = f'_{-,F})$.*

(iv) *If f is continuous (Lipschitz) on F, then g is continuous (Lipschitz) on $\mathbb{R}$.*

Proof. By Proposition 3.4 there exists a (DLCB)-extension $g : \mathbb{R} \to X$ of $f : F \to X$. By Definition 3.1 and Lemma 3.2 (i) g has all properties (i)-(iv). (Note that g is continuous at $t \notin F$ by (ii).) □

We obtain easily also the following generalization of Proposition 1.1.

Proposition 3.6. *Let $A \subset \mathbb{R}$ be an arbitrary set and X be a Banach space. Then for each Lipschitz $f : A \to X$ there exists a Lipschitz $g : \mathbb{R} \to X$ which extends f and has the following properties:*

(i) *If $t \in A$ and $f'_A(t)$ exists, then $g'(t) = f'_A(t)$.*

(ii) *$g'(t)$ exists if $t \notin \overline{A}$ or t is an isolated point of A.*

Proof. By Lemma 3.3 there is a (DLCB)-extension $f^* : \overline{A} \to X$ of $f : A \to X$. Since f^* is Lipschitz and so, locally bounded, by Proposition 3.4 there exists a (DLCB)-extension $g : \mathbb{R} \to X$ of f^*. Because g is a (DLCB)-extension of f, it has properties (i), (ii). □

For real functions we obtain the following results.

Theorem 3.7. *Let $F \subset \mathbb{R}$ be a closed set and $f : F \to \mathbb{R}$ be a locally bounded function. Then there exists $g : \mathbb{R} \to \mathbb{R}$ which extends f and has the following properties:*

(i) *If $t \in F$ and $f'_F(t) \in \mathbb{R}$ exists, then $g'(t) = f'_F(t)$.*

(ii) *$g'(t) \in R$ exists if $t \notin F$ or t is an isolated point of F.*

(iii) *If* $t \in F$ *and* $D^+_F f(t)$ $(D_{+,F} f(t), D^-_F f(t), D_{-,F} f(t))$ *is defined, then* $D^+ g(t) = D^+_F f(t)$ $(D_+ g(t) = D_{+,F} f(t), D^- g(t) = D^-_F f(t), D_- g(t) = D_{-,F} f(t))$.

(iv) *If* f *is continuous (Lipschitz) on* F, *then* g *is continuous (Lipschitz) on* $\mathbb{R}$.

PROOF. By Proposition 3.4 there exists a (DLCB)-extension $g : \mathbb{R} \to \mathbb{R}$ of $f : F \to \mathbb{R}$. By Definition 3.1 and Lemma 3.2 it has all properties (i)-(iv). □

Similarly we obtain the following result on extensions from arbitrary subsets of $\mathbb{R}$.

Proposition 3.8. *Let* $A \subset \mathbb{R}$ *be an arbitrary set and* $f : A \to \mathbb{R}$ *be a Lipschitz function. Then there exists a Lipschitz function* $g : \mathbb{R} \to \mathbb{R}$ *which extends* f *and has the following properties:*

(i) *If* $t \in A$ *and* $f'_A(t) \in \mathbb{R}$ *exists, then* $g'(t) = f'_A(t)$.

(ii) $g'(t)$ *exists if* $t \notin \overline{A}$ *or* t *is an isolated point of* A.

(iii) *If* $t \in A$ *and* $D^+_A f(t)$ $(D_{+,A} f(t), D^-_A f(t), D_{-,A} f(t))$ *is defined, then* $D^+ g(t) = D^+_A f(t)$ $(D_+ g(t) = D_{+,A} f(t), D^- g(t) = D^-_A f(t), D_- g(t) = D_{-,A} f(t))$.

PROOF. By Lemma 3.3 there exists a (DLCB)-extension $f^* : \overline{A} \to \mathbb{R}$ of $f : A \to \mathbb{R}$ and by Proposition 3.4 there exists a (DLCB)-extension $g : \mathbb{R} \to \mathbb{R}$ of f^*. Thus g is an (DLCB)-extension of f and by Definition 3.1 and Lemma 3.2 it has all properties (i)-(iii). □

References

[J] V. Jarník, *Sur l'extension du domaine de définition des fonctions d'une variable, qui laisse intacte la dérivabilité de la fonction*, Bulletin international de l'Académie des Sciences de Boheme, 1923, 1–5.

[M] J. Mařík, *Derivatives and closed sets*, Acta Math. Hungar., **43** (1984), 25–29.

[NZ] A. Nekvinda and L. Zajíček, *Gâteaux differentiability of Lipschitz functions via directional derivatives*, Real Anal. Exchange, **28** (2002-03), 287–320.

[PL] G. Petruska and M. Laczkovich, *Baire 1 functions, approximately continuous functions and derivatives*, Acta Math. Acad. Sci. Hungar., **25** (1974), 189–212.

[Pr] D. Preiss, *Bemerkung zu einem Problem von V. Jarník*, Čas. Pěst. Mat., **95** (1970), 146–149.