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THE EQUALITY BETWEEN BOREL AND BAIRE CLASSES

Abstract

In this paper, we study some properties of the Banach space $\beta_\alpha(X)$, which consists of all real Baire functions on a perfectly normal space X. We obtain the equality between Baire and Borel classes as a consequence of existence of an approximation property and a Tietze extension for these classes. Moreover, when Y is a zero dimensional topological space, we obtain a refinement of the known results for the equality between $\beta^\circ_\alpha(X,Y)$ and $B^\circ_\alpha(X,Y)$.

1 Introduction.

A topological space X is perfectly normal if it is Hausdorff, and every closed subset of X is a zero set of some real continuous function (cf. [1], [3], [6] and [10]).

We denote the Borel sets of multiplicative (additive) class α by $\mathcal{P}_\alpha$ ($\mathcal{S}_\alpha$) [10], beginning with $\mathcal{P}_0 = \mathcal{F}$ ($\mathcal{S}_0 = \mathcal{G}$), as the followings.

$$\begin{aligned} &\mathcal{P}_\alpha : \mathcal{F}, \mathcal{G}_\delta, \mathcal{F}_{\sigma\delta}, \dots \\ &\mathcal{S}_\alpha : \mathcal{G}, \mathcal{F}_\sigma, \mathcal{G}_{\delta\sigma}, \dots . \end{aligned}$$

We designate $\beta_0(X) = C(X)$, the set of all real valued continuous functions on X. For each finite ordinal α, we define Baire functions of class α as

$$\begin{aligned} \beta_\alpha(X) = \{f : X \to \mathbb{R} : &\text{there exists } (f_n)_{n=1}^\infty \subseteq \beta_{\alpha-1}(X) \text{ such that} \\ &\lim f_n(x) = f(x), \text{ for each } x \in X\} \end{aligned}$$

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We define Borel functions of class α as

$$B_\alpha(X) = \{f : X \to \mathbb{R}\colon \text{ for each closed set } F \text{ in } \mathbb{R}, f^{-1}(F) \in \mathcal{P}_\alpha\}$$

As X is perfectly normal, then by the same induction as in [10], $\beta_\alpha(X) \subseteq B_\alpha(X)$. It is obvious that $B_\alpha(X) \subseteq B_{\alpha+1}(X)$.

In general, β_α is not equal to B_α, see [5]. The Lebesgue-Hausdorff theorem [10, page 391] says that if X is a metric space and Y is either the n-dimensional cube $[0,1]^n$, or the Hilbert cube $[0,1]^{\mathbb{N}}$, then the first Baire and Borel classes of functions from X to Y are equal. Rolewicz [16] showed that if Y is a separable convex subset of a normal linear space, the first Baire and Borel classes of functions from X to Y are the same. In this paper we improve these results for perfectly normal spaces in the scaler case. Furthermore, we also extend our results to the case when the values of our functions are in a zero dimensional metric space. Our proofs are based on small modifications of the classical proofs of Lebesgue, Hausdorff and Banach for metric spaces [10 and 11].

Since X is perfectly normal, similar to the metric case [10], it is possible to see that $\mathcal{P}_\alpha$'s ($\mathcal{S}_\alpha$'s) form a chain and $\mathcal{F} \subseteq \mathcal{G}_\delta$, and for each $A \in \mathcal{P}_\alpha$ there exists a sequence $(G_n)_{n=1}^\infty \subseteq \mathcal{S}_{\alpha-1}$ such that $A = \cap_{n=1}^\infty G_n$. For additive sets, "$\mathcal{S}$", "$\mathcal{P}$", and "$\cap$" are replaced respectively by "$\mathcal{P}$", "$\mathcal{S}$", and "$\cup$". See [10, §-30] for details.

The ambiguous sets of class α is denoted by $\mathcal{H}_\alpha$ ([10]) and defined as

$$\mathcal{H}_\alpha = \mathcal{S}_\alpha \cap \mathcal{P}_\alpha.$$

In the following lemma, we mention some of the properties of perfectly normal topological spaces. (Throughout this paper except the remark after Theorem 1.9, we suppose that α is finite.)

Lemma 1.1. *In a perfectly normal space X, we have:*

(a) Every set in $\mathcal{S}_\alpha$ ($\alpha \geq 1$) is a union of some countable disjoint sets in $\mathcal{H}_\alpha$.

(b) For each sequence $(G_n)_{n=1}^\infty \subseteq \mathcal{S}_\alpha$ ($\alpha \geq 1$), there exists a mutually disjoint sequence $(H_n)_{n=1}^\infty$ in $\mathcal{S}_\alpha$ such that $\cup_{i=1}^\infty H_i = \cup_{i=1}^\infty G_i$ and $H_i \subseteq G_i$ for each i. In addition, if $X = \cup_{i=1}^\infty H_i$, then each H_i belongs to $\mathcal{H}_\alpha$.

(c) For every sequence $(F_n)_{n=1}^\infty$ in $\mathcal{P}_\alpha$ ($\alpha \geq 1$) such that $\cap_{n=1}^\infty F_n = \emptyset$, there exists a sequence $(E_n)_{n=1}^\infty \subseteq \mathcal{H}_\alpha$ such that $\cap_{n=1}^\infty E_n = \emptyset$ and $F_n \subseteq E_n$ for each n. In particular, if A and B are two disjoint $\mathcal{P}_\alpha$ sets, then there exists a set E in $\mathcal{H}_\alpha$ such that $A \subseteq E$ and $B \cap E = \emptyset$. That is if $A \in \mathcal{P}_\alpha$, $C \in \mathcal{S}_\alpha$ and $A \subseteq C$, then there exists $E \in \mathcal{H}_\alpha$ such that $A \subseteq E \subseteq C$.

PROOF. The proof is similar to that of metric spaces. See [10, §-30, V, VII, VIII]. $\square$

In the following lemma, we give a representation of $\mathcal{S}_\alpha$ (resp. $\mathcal{P}_\alpha$) subsets of a subset A of X.

Lemma 1.2. *Let $B \subseteq A \subseteq X$. If B is $\mathcal{S}_\alpha$ (resp. $\mathcal{P}_\alpha$) in A, then there is an $\mathcal{S}_\alpha$ (resp. $\mathcal{P}_\alpha$) set G in X such that $A \cap G = B$. Consequently, if A in X and B in A are $\mathcal{P}_\alpha$ (resp. $\mathcal{S}_\alpha$ or $\mathcal{H}_\alpha$) sets, then B is $\mathcal{P}_\alpha$ (resp. $\mathcal{S}_\alpha$ or $\mathcal{H}_\alpha$) set in X.*

PROOF. We define the statement $\Delta(\alpha)$ for ordinal number α as the following composition:

If B is $\mathcal{S}_\alpha$ in A , then there exists an $\mathcal{S}_\alpha$ set G in X such that $A \cap G = B$ and

If D is $\mathcal{P}_\alpha$ in C , then there exists a $\mathcal{P}_\alpha$ set K in X such that $C \cap K = D$. Now, the proof will be completed by induction on α. $\square$

In Lemma 1.3, we obtain a representation for $\mathcal{H}_\alpha$ subsets of a subset A of X.

Lemma 1.3. *If A is $\mathcal{P}_\alpha$ in X and K is $\mathcal{H}_\alpha$ in A, then there exists $H \in \mathcal{H}_\alpha$ in X such that $K = A \cap H$.*

PROOF. It's obvious that K is $\mathcal{P}_\alpha$ in X. By applying the previous lemma, there exists $S \in \mathcal{S}_\alpha$, such that $K = S \cap A$. Thus $K \subseteq S$, and the above lemma completes the proof. $\square$

Here, the set of real functions, defined on X, is denoted by $\mathbb{R}^X$ and, $(\mathbb{R}^X)^\circ$ represents the set of bounded real functions on X. For $f \in (\mathbb{R}^X)^\circ$, we define

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

For definitions of $\mathbb{R}$-module, $\mathbb{R}$-algebra, lattice and uniformly closed $\mathbb{R}$-modules, we refer to [11] and [12]. Let $\mathcal{U}$ be a subset of the power set of X and $\mathcal{A}$ be a subset of $\mathbb{R}^X$. We say that $\mathcal{A}$ separates the points of $\mathcal{U}$ whenever for each two disjoint sets, A and B in $\mathcal{U}$, there exists f in $\mathcal{A}$ with $0 \leq f \leq 1$ such that

$$f(A) = \{0\}, \quad f(B) = \{1\}.$$

If $\mathcal{A}$ is a lattice $\mathbb{R}$-module, then it suffices that there exists an f in $\mathcal{A}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Let $\hat{e} : X \to \mathbb{R}$ be the constant function that assigns e (in $\mathbb{R}$) to each member of X.

Suppose that $\mathcal{A} \subseteq \mathbb{R}^X$, we define

$$\mathcal{U}_1(\mathcal{A}) = \{f^{-1}((-\infty, a]), f^{-1}([a, \infty)) : f \in \mathcal{A}, a \in \mathbb{R}\},$$
$$\mathcal{U}_2(\mathcal{A}) = \mathcal{U}_1(\mathcal{A}) \cup \{f^{-1}(\{-1, 1\}) : f \in \mathcal{A}\}.$$

Assume that $\psi_0 : \mathbb{R} \to (-1, 1)$ is a function such that $\psi_0(r) = r(1 + |r|)^{-1}$ for each r in $\mathbb{R}$. We say that $\mathcal{A}$ is closed under composition from left with ψ_0 when $\psi_0 \circ f$ and $\psi_0^{-1} \circ f$ are in $\mathcal{A}$ for every f in $\mathcal{A}$(The latter case is well-defined when $range(f) \subseteq (-1, 1)$). As usual the restriction of $f : B \to \mathbb{R}$ to A is denoted by $f|_A$.

Here we give a Baire-α characterization of $\mathcal{H}_\alpha$ elements in X.

Lemma 1.4. *We have $H \in \mathcal{H}_\alpha$ if and only if $\chi_H \in \beta_\alpha(X)$.*

Proof. We prove by induction. Suppose the statement holds for $(\alpha - 1)$. Let H be in $\mathcal{H}_\alpha$. As X is perfectly normal, then it is normal and the statement is true for $\alpha = 1$. By Lemma 1.1.(b), there are a nondecreasing sequence $(F_n)_{n=1}^\infty$ of elements $\mathcal{P}_{\alpha-1}$ in X and a nonincreasing sequence $(G_n)_{n=1}^\infty$ of elements of $\mathcal{S}_{\alpha-1}$ such that

$$\cup_{n=1}^\infty F_n = H = \cap_{n=1}^\infty G_n.$$

For each positive integer n, $F_n \subseteq G_n$. By use of induction, there is an $H_n \in \mathcal{H}_{\alpha-1}$ such that $f_n = \chi_{H_n} \in \beta_{\alpha-1}(X)$, $f_n(F_n) = \{1\}$ and $f_n(G_n^c) = \{0\}$. Obviously, χ_H is the point-wise limit of f_n. The proof of the other side is obvious and is omitted. □

The Tietze extension theorem has been generalized for many cases of continuous functions (cf. [2], [14] and [15]). In order to give a Tietze extension theorem for real Baire-α functions, we need some more lemmas.

Lemma 1.5. *Let $\mathcal{A}$ be a lattice $\mathbb{R}$-module, uniformly closed and $\hat{1} \in \mathcal{A}$ such that $\mathcal{A}$ separates the points of $\mathcal{U}$ and $\mathcal{U}_1(\mathcal{A}) \subseteq \mathcal{U}$. Suppose that $A \in \mathcal{U}$ and $f \in (\mathbb{R}^A)^\circ$ such that for every set $D \subseteq \mathbb{R}$ of the form $(-\infty, a]$ or $[a, \infty)$ ($a \in \mathbb{R}$) and every $h \in \mathcal{A}$, we have $(f - h|_A)^{-1}(D) \in \mathcal{U}$. Then, there exists $g \in \mathcal{A}^\circ$ such that $g|_A = f$.*

Proof. The proof is similar to that of [10, §-14, IV, Tietze theorem]. We will mention it for the sake of completeness. It's obvious that for every two disjoint set A and B in $\mathcal{U}$, and for each real number a and b, there exists a function h in $\mathcal{A}$ such that h is bounded $h(A) = \{a\}$ and $h(B) = \{b\}$. Now suppose that $\|f\|_\infty \leq c$ and define

$$A_1 = \{x \in A : f(x) \leq -c/3\}, A_2 = \{x \in A : f(x) \geq c/3\},$$

$$A_3 = \{y \in \mathbb{R} : |y| \leq c/3\}.$$

Since $\mathcal{A}$ is an $\mathbb{R}$-module, then $\hat{0} \in \mathcal{A}$. Consequently, for each closed set D in $\mathbb{R}$ of the form $(-\infty, a]$, $[a, \infty)$ $(a \in \mathbb{R})$, we have

$$(f - 0|_A)^{-1}(D) = f^{-1}(D) \in \mathcal{U}.$$

Now, since A_1 and A_2 are two disjoint members of $\mathcal{U}$, therefore, there exists g in $\mathcal{A}$ such that

$$g(A_1) = \{-c/3\}, g(A_2) = \{c/3\}, \ \|g\|_\infty \leq c/3,$$

$$|f(x) - g(x)| \leq 2c/3 \text{ for each } x \in X. \tag{1}$$

Let $g_0 = 0$. We construct by induction the sequence $(g_n)_{n=1}^\infty$ in $\mathcal{A}$ such that

$$|f(x) - \sum_{i=0}^{n} g_i(x)| \leq (2/3)^n \text{ for each } x \in A. \tag{2}$$

As $\mathcal{A}$ is an $\mathbb{R}$-module and $(f - h|_A)^{-1}(D) \in \mathcal{U}$, hence $h(x) = \sum_{i=0}^n g_i(x)$ is in $\mathcal{A}$. Now by setting $f(x) - \sum_{i=0}^n g_i(x)$ and $(2/3)^n c$ instead of $f(x)$ and c in (1) respectively, we choose g_{n+1} in $\mathcal{A}$ such that $\|g_{n+1}\|_\infty \leq 1/3(2/3)^n c$ and

$$|f(x) - \sum_{i=0}^{n+1} g_i(x)| \leq (2/3)^{n+1} c, \quad \text{for each } x \in A.$$

We define $g(x) = \sum_{i=0}^\infty g_i(x)$. By our construction, this series converges uniformly, therefore g is in $\mathcal{A}$. Also, for each x in A, we have $f(x) = g(x)$ by (2), and it is obvious that $\|g\|_\infty \leq c$. $\square$

As a consequence of the Lemma 1.5, we obtain the following corollary.

Corollary 1.6. *Let $\mathcal{A}$ be a lattice $\mathbb{R}$-algebra, contain $\hat{1}$, closed under composition by ψ_0 from left, which is uniformly closed, and separates the members of $\mathcal{U}$ and $\mathcal{U} \supseteq \mathcal{U}_2(\mathcal{A})$. Suppose $A \in \mathcal{U}$ and $f \in (\mathbb{R}^A)^\circ$ such that for each closed set $D \subseteq \mathbb{R}$ of the form $(-\infty, a]$, $[a, \infty)$, and $\{-1, 1\}$ and every h in $\mathcal{A}$, we have $(\psi_0 \circ f - h|_A)^{-1}(D) \in \mathcal{U}$. Then there is an element g in $\mathcal{A}$ such that $g|_A = f$.*

For the Baire extension, we obtain the following result which improves the extension theorem of [8] for Borel functions.

Theorem 1.7. *Let A be in $\mathcal{P}_\alpha$. Every f in $\beta_\alpha(A)$ has an extension to a member of $\beta_\alpha(X)$. Moreover, if f is bounded, then this extension can be bounded too.*

PROOF. It is easily deduced from Lemma 1.4, Corollary 1.6 and Lemma 1.1(c). □

Now, we obtain an approximation theorem for $\beta^\circ_\alpha(X)$. We define

$$\Sigma_{\alpha,\mathbb{R}} = \Big\{ \sum_{i=1}^{n} e_i \chi_{H_i} : n \in \mathbb{N}, e_i \in \mathbb{R} \text{ and } H_i \in \mathcal{H}_\alpha \text{ for each } i \le n \Big\}.$$

Theorem 1.8. *The uniform closure of* $\Sigma_{\alpha,\mathbb{R}}$ *is* $\beta^\circ_\alpha(X)$.

PROOF. One can prove by a similar proof as that of Theorem 7.29 of [9]. Suppose f_0 is a nonconstant function in $\beta^\circ_\alpha(X)$. We set

$$d = \sup\{f_0(x) : x \in X\} > \inf\{f_0(x) : x \in X\} = c.$$

Define

$$f = (2/(d-c))(f_0 - d) + 1.$$

It is enough to prove that f is in the uniform closure of $\Sigma_{\alpha,\mathbb{R}}$. But we have $f(X) \subseteq [-1, 1]$. Let

$$E = \{x \in X : f(x) \le -1/3\}, \ F = \{x \in X : f(x) \ge 1/3\}.$$

The sets E and F are two disjoint sets in $\mathcal{P}_\alpha$. Therefore, by lemma 1.1(c), there exists $K_1 \in \mathcal{H}_\alpha$ such that $E \subseteq K_1$ and $F \cap K_1 = \emptyset$. Define $g_1 = 1/3(1-2\chi_{K_1})$. Consequently, $g_1(E) = \{-1/3\}$, $g_1(F) = \{1/3\}$ and $g_1(X) \subseteq [-1/3, 1/3]$. Also, it is obvious that $g_1 \in \Sigma_{\alpha,\mathbb{R}}$ and $\|f - g_1\|_\infty = 2/3$.

In this way, there exists g_2 in $\Sigma_{\alpha,\mathbb{R}}$ such that $3/2(f - g_1)$ satisfies the equation

$$\|3/2(f - g_1) - g_2\|_\infty = 2/3.$$

Thus,

$$\|f - g_1 - 2/3g_2\|_\infty = (2/3)^2 \ , \ g_1 + 2/3g_2 \in \Sigma_{\alpha,\mathbb{R}}.$$

Now, there is $g_3 \in \Sigma_{\alpha,\mathbb{R}}$ such that

$$\|(3/2)^2(f - g_1 - 2/3g_2) - g_3\|_\infty = 2/3.$$

Here

$$g_1 + 2/3g_2 + (2/3)^2 g_3 \in \Sigma_{\alpha,\mathbb{R}},$$

and

$$\|f - g_1 - 2/3g_2 - (2/3)^2 g_3\|_\infty = (2/3)^3.$$

By induction, for each n, there exist $g_1, g_2, \ldots, g_n$ in $\Sigma_{\alpha,\mathbb{R}}$ such that

$$\|f - \sum_{i=1}^{n} (2/3)^{i-1} g_i\|_\infty = (2/3)^n,$$

and

$$h_n = \sum_{i=1}^{n} (2/3)^n g_i \in \Sigma_{\alpha,\mathbb{R}}.$$

Therefore, h_n's converge uniformly to f. Hence we obtain $\beta^{\circ}_{\alpha}(X)$ is a subset of the uniform closure of $\Sigma_{\alpha,\mathbb{R}}$. The other direction of proof is obvious. $\square$

For general real Baire and Borel classes, we obtain the following result.

Theorem 1.9. *For each finite ordinal number α, we have $\beta_\alpha(X) = B_\alpha(X)$.*

PROOF. By virtue of Theorem 1.8, bounded real valued Baire and Borel functions coincide. The rest of the proof is straightforward by use of the ψ_0 function. $\square$

Remark. By similar arguments, one can easily see that for an infinite ordinal number α, we have $\beta_\alpha(X) = B_{\alpha+1}(X)$.

Despite the example given in [5], we give another example to show that in general, the equality between $\beta_1(X)$ and $B_1(X)$ does not hold.

Example 1.10. *(a) Let X be a countable set with co-finite topology. It is well known that X is not perfectly normal (and even regular). Furthermore, we have $\beta_1(X) \neq B_1(X)$.*
(b) Here we give an example to show that the equality between Baire and Borel functions can happen even when X is not regular. Let X be an uncountable set with co-countable topology, then we have $\beta_1(X) = B_1(X)$.

J. Fabrykowski [4] showed the existence of a sequence of continuous functions on $[0,1]$, whose pointwise limit is finite on the rational numbers and infinite on the irrational numbers. In [13], G. Myerson proved that one can replace rational numbers with an arbitrary $\mathcal{F}_\sigma$ set in $[0,1]$. In [7], R. W. Hansell improved these results by obtaining the following theorem.

Theorem A. *Let X be a perfectly normal space, and let Y be a complete separable metric space that is also an absolute retract. If p is any non-isolated point of Y, then a necessary and sufficient condition for the existence of a Baire class one function $f : X \to Y$ such that $S = \{x : f(x) \neq p\}$ for a given set $S \subseteq X$, is that S belongs to the class of $\mathcal{F}_\sigma$ subsets of X.*

In this direction, we give a similar result for Baire real functions of class α, when α is a finite ordinal number

Theorem 1.11. *Let X be a perfectly normal space. Suppose $p \in \mathbb{R}$, then a necessary and sufficient condition for the existence of a Baire class α function, $f : X \to \mathbb{R}$ such that $\{x \in X : f(x) \neq p\} = S$ for a given set $S \subseteq X$, is that S belongs to $\mathcal{S}_\alpha$.*

Proof. By linearity of $\mathbb{R}$, it is enough to prove theorem for $p = 0$. Suppose that S is an $\mathcal{S}_\alpha$ set in X. Thus there is a non-decreasing sequence, $\{P_i\}_{i=1}^\infty$ of $\mathcal{P}_{\alpha-1}(\subseteq \mathcal{P}_\alpha)$ set in X such that

$$S = \cup_{i=1}^\infty P_i.$$

The set $X - S = P$ belongs to $\mathcal{P}_\alpha$ and $P \cap P_i = \emptyset$ for each $i \in \mathbb{N}$. By Lemma 1.1. (c), for each i, there exists an $H_i \in \mathcal{H}_\alpha$ such that $P \cap H_i = \emptyset$ and $P_i \subseteq H_i$. Suppose $e \in \mathbb{R}$ is not equal to zero. Therefore, $f_i = \frac{e}{3^i}\chi_{H_i}$ belongs to $\beta^\circ_\alpha(X)$. We define f as $f = \Sigma_{n=1}^\infty f_i$. It is obvious that $|f(x)| > 0$ for each $x \in S$, and

$$f(X - S) = f(P) = \{0\}.$$

By the uniform convergence of $\Sigma_{n=1}^\infty f_i$, f belongs to $\beta^\circ_\alpha(X)$ and the proof is complete. The proof of the other side of the theorem is obvious. $\square$

Now, we extend the results of [4] and [13] for perfectly normal spaces.

Theorem 1.12. *Let X be a perfectly normal topological space and S be a subset of X. There is a sequence of functions in $\beta^\circ_{\alpha-1}(X)$ whose pointwise limit is finite on S and infinite on complement of S if and only if S belongs to $\mathcal{S}_\alpha$.*

2 Baire Functions with Ranges in a Zero-Dimensional Space.

In arbitrary topological spaces, the relation $\mathcal{P}_\beta \subseteq \mathcal{P}_\alpha$ may fail for $\beta < \alpha$, but we always have

$$\mathcal{P}_\alpha = X - \mathcal{S}_\alpha.$$

Suppose that Y is a topological space. For a finite ordinal number α, we define $B_\alpha(X, Y)$, the class of Borel-α functions from X to Y by

$$B_\alpha(X, Y) = \{f : X \to Y : f^{-1}(\mathcal{G}) \subseteq \mathcal{S}_\alpha\ \}.$$

Let $\beta_0(X, Y) = C(X, Y)$ be the class of all of continuous functions from X to Y. We define Baire class α (α is a finite ordinal number) functions from X to Y by

$$\beta_\alpha(X, Y) = \{f : X \to Y\ : \exists (f_n)_{n=1}^\infty \subseteq \beta_{\alpha-1}(X, Y) \text{ such that } \lim_{n\to\infty} f_n(x) = f(x) \text{ for } \forall x \in X\ \}.$$

A topological space is said to be zero-dimensional, if it has a base consisting of only its clopen subsets. We suppose that Y is a zero dimensional metric space and X is an arbitrary topological space. The space Y can be written as the union of disjoint clopen sets with arbitrary small diameters [3, 10]. We denote the set of members of $B_\alpha(X,Y)$ with relatively compact ranges by $B^\circ_\alpha(X,Y)$. We define $\beta^\circ_\alpha(X,Y)$ in a similar way. The main result of this section shows the uniform density of simple functions with finite ranges in $\beta^\circ_\alpha(X,Y)$.

In [5], Fosgerau has proved the following theorem.

Theorem B. *Let Y be a complete metric space. Then the family of first Baire class functions coincides with the first Borel class functions from $[0,1]$ to Y if and only if Y is connected and locally connected.*

In order to obtain some other variant results similar to Theorem B, we need to introduce some notations. Let f be a function with finite values y_1, y_2,..., y_m on mutually disjoint sets A_1, A_2,..., A_m, respectively. For simplicity, we denote f in the form $f = \sum_{i=1}^m y_i \chi_{A_i}$. Therefore here, "summation" is only a formal notation. Now, we give the main theorem of this section. For a topological space, not necessary perfectly normal, we can define $\mathcal{P}_\alpha$, $\mathcal{S}_\alpha$ and $\mathcal{H}_\alpha$ similarly.

Theorem 2.1. *Suppose that Y is a zero-dimensional metric space and X is an arbitrary topological space. Let*

$$SIM_\alpha(X,Y) = \Big\{ \sum_{i=1}^{n} y_i \chi_{H_i} : n \in \mathbb{N},\ y_i \in Y,\quad \{H_i\}_{i=1}^n \subseteq \mathcal{H}_\alpha \text{ is a partition of } X \Big\}.$$

Then each element of $B^\circ_\alpha(X,Y)$ is the uniform limit of a sequence from the set $SIM_\alpha(X,Y)$.

Proof. Suppose that $f \in B^\circ_\alpha(X,Y)$. Since the range of f is relatively compact so there exists a sequence, $(y_n)_{n=1}^\infty$ in Y, which is dense in the range of f. Since Y is zero-dimensional, for each $\epsilon > 0$ there is a base for Y with diameter at most ϵ and so there exist mutually disjoint clopen sets $B_\epsilon(y_{i_\epsilon})$ with diameter at most ϵ containing y_{i_ϵ} such that

$$Range(f) \subseteq \cup_{i=1}^{n_\epsilon} B_\epsilon(y_{i_\epsilon}).$$

Let $T_{i_\epsilon} = f^{-1}(B_\epsilon(y_{i_\epsilon}))$. Each T_{i_ϵ} is an $\mathcal{S}_\alpha$ set in X and $X = \cup_{i=1}^{n_\epsilon} T_{i_\epsilon}$. Therefore, each T_{i_ϵ} is an $\mathcal{H}_\alpha$ set in X. Now define $f_\epsilon = \sum_{i=1}^{n_\epsilon} y_{i_\epsilon} \chi_{H_{i_\epsilon}}$. It is easy to see that

$$d(f, f_\epsilon) \le \epsilon.$$

Now, for each $\epsilon = \frac{1}{m}$, we obtain a function f_m such that the sequence $(f_m)_{m=1}^\infty$ converges uniformly to f. □

Let us recall that an ultra-normal topological space X is a Hausdorff space in which disjoint closed subsets may be separated by clopen sets ([14]).

Corollary 2.2. *Suppose that Y is a zero-dimensional metric space and X is an ultra-normal topological space. Then*

$$B_1^\circ(X,Y) = \beta_1^\circ(X,Y).$$

PROOF. It is well known that

$$\beta_1^\circ(X,Y) \subseteq B_1^\circ(X,Y).$$

By the previous theorem, as the range of Baire and Borel functions are separable, it is enough to show that for mutually disjoint sets $H_1, H_2, \dots, H_m \in \mathcal{H}_1$ and $y_1, y_2, \dots, y_m \in Y$ such that $\cup_{i=1}^m H_i = X$, we have $f = \sum_{i=1}^m y_i \chi_{H_i} \in \beta_1^\circ(X,Y)$. Notice that for each $k = 1, 2, \dots, m$, there exists a suitable sequence of non-decreasing closed sets $F_{k,n}$'s such that $\cup_{n=1}^\infty F_{k,n} = H_k$. Since the space X is an ultra-normal space, and $F_{1,i}$, $F_{2,i}$,..., $F_{m,i}$ are disjoint closed sets for each $i \in \mathbb{N}$, there exist disjoint clopen sets, $O_{k,i}$'s such that for each k, we have $F_{k,i} \subseteq O_{k,i}$. We define f_i by

$$f_i = \sum_{k=1}^m y_k \chi_{O_{k,i}} \in C(X,Y).$$

We must prove that

$$\lim_{i\to\infty} f_i(x) = f(x) \quad, x \in X.$$

If $x \in X$, then there exists an integer k such that $x \in H_k$. Therefore, there exists an integer i such that $x \in F_{k,i}$. But $F_{k,i}$'s are non-decreasing, thus for each $j \geq i$, $x \in F_{k,j}$ and so $f_j(x) = y_k$. It follows that every member of $SIM_1(X,Y)$ belongs to $\beta_1^\circ(X,Y)$. □

Lemma 2.3. *Let α be a finite ordinal number. For an ultra-normal , perfectly normal space X and a zero-dimensional metric space Y with distinct elements y_1 and y_2, we have $y_1\chi_H + y_2(1 - \chi_H)$ belongs to $\beta_\alpha^\circ(X,Y)$ if and only if $H \in \mathcal{H}_\alpha$.*

PROOF. Notice first that for each finite ordinal number α and any two disjoint sets P_1 and P_2 in $\mathcal{P}_\alpha$, there exists H in $\mathcal{H}_\alpha$ which separates these two sets. Now, the proof is by induction. As $H \in \mathcal{H}_\alpha$, therefore, there exist two monotone sequences of sets such that

$$P_1 \subseteq P_2 \subseteq \dots \subseteq P_n \subseteq \dots \text{ in } \mathcal{P}_{\alpha-1} , \quad \dots \subseteq S_n \subseteq \dots \subseteq S_2 \subseteq S_1 \text{ in } \mathcal{S}_{\alpha-1}$$

and

$$\cup_{n=1}^{\infty} P_n = H = \cap_{n=1}^{\infty} S_n.$$

Since for each n, we have $P_n \subseteq S_n$, there exists $H_n \in \mathcal{H}_{\alpha-1}$ which separates P_n and $X - S_n$. Now, we define $f_n = y_1\chi_{H_n} + y_2(1 - \chi_{H_n})$. It is clear that

$$\lim_{n\to\infty} f_n(x) = y_1\chi_H(x) + y_2(1 - \chi_H)(x) \quad \forall x \in X.$$

Hence $y_1\chi_H + y_2(1 - \chi_H) \in \beta_\alpha^\circ(X, Y)$. The proof of the other side is obvious. □

Remark. As in Corollary 2.2, we can prove the statement for any finite set.

Theorem 2.4. *Let α be a finite ordinal number. For an ultra-normal , perfectly normal space X and a zero-dimensional metric space Y, we have*

$$\beta_\alpha^\circ(X, Y) = B_\alpha^\circ(X, Y).$$

PROOF. It is a direct consequence of Theorem 2.2 and Lemma 2.4. □

Remark. By theorem B, for an arbitrary separable zero-dimensional metric space Y, the equality between 1-Baire and 1-Borel classes does not hold.

$$\beta_1([0, 1], Y) \neq B_1([0, 1], Y).$$

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