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THE LEBESGUE DIFFERENTIATION THEOREM VIA THE RISING SUN LEMMA

Abstract

A complete version of Lebesgue's differentiation theorem, including the image of the exceptional set, is proved in an elementary way.

1 Introduction

Lebesgue's differentiation theorem [5] says that every monotone function is differentiable almost everywhere. Most proofs use Vitali's covering theorem (see for instance the classics [4] and [7]) or a slightly complicated version of the rising sun lemma for semicontinuous functions (see for instance the books [3] and [9]). Proofs of other types can be found in [1] or [6]. In this paper two simplifications are proposed. First, we use a brilliant idea of Rubel [10] to reduce the result to the case of continuous monotone functions. This case is then treated as in a preceding paper [2], by means of a new version of the rising sun lemma (for continuous functions). The proof will use the following properties of the Lebesgue outer measure m^*. Properties (P3) and (P4) can be taken as definition of the outer measure m^*.

(P1) $A \subseteq B \;\Rightarrow\; m^*(A) \leq m^*(B)$,

(P2) $m^*\big(\bigcup_{n=1}^{\infty} A_n\big) \leq \sum_{n=1}^{\infty} m^*(A_n)$,

(P3) if $U = \bigcup_n (c_n, d_n)$ is an open set, then $m^*(U) = \sum_n (d_n - c_n)$,

(P4) $m^*(A) = \inf\{m^*(U)/A \subseteq U \text{ and } U \text{ is open}\}$,

(P5) $m^*([c, d]) = d - c$ and $m^*(\{c\}) = 0$.

The following lemma is a slight modification of Riesz's rising sun lemma [8].

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Lemma 1. *Let $G : [a,b] \to \mathbb{R}$ be a continuous function and $U \subseteq (a,b)$ some open set. Then the set*

$$U_G := \{x \in U : \text{ there exists } y < x \text{ with } (y,x) \subseteq U \text{ and } G(y) < G(x)\}$$

is also open. Moreover, if (c,d) is a component of U_G, then $G(c) \le G(d)$.

PROOF. Trivially, U_G is open. Let (c,d) be any component of U_G. We show that $G(c) \le G(x)$ for all $x \in (c,d)$. Let $\gamma := \min\{y \in [c,x] : G(y) \le G(x)\}$, and suppose that $c < \gamma$. Hence $G(c) > G(x)$ and $\gamma \in U_G$. There exists $z < \gamma$ with $(z,\gamma) \subseteq U$ and $G(z) < G(\gamma)$. If $z < c$, then $G(z) < G(\gamma) \le G(x) < G(c)$ implies that $c \in U_G$, a contradiction. And if $z \ge c$, then $G(z) < G(\gamma) \le G(x)$ contradicts the minimality of γ. Therefore $c = \gamma$ and $G(c) \le G(x)$. □

Proposition 2. *Let $F : [a,b] \to \mathbb{R}$ be a continuous increasing function and let $R > 0$. If the set $E \subseteq (a,b)$ is such that*

$$\overline{D}_- F(x) := \limsup_{y \nearrow x} \frac{F(x) - F(y)}{x - y} > R$$

for every $x \in E$, then $m^*\big(F(E)\big) \ge R\, m^*(E)$.

PROOF. Let $\varepsilon > 0$. By (P4) there exists an open set V such that $F(E) \subseteq V$ and $m^*(V) < m^*\big(F(E)\big) + \varepsilon$. We put $U = F^{-1}(V) \cap (a,b)$ and consider the function $G(x) := F(x) - R\,x$. Then $\overline{D}_- G(x) > 0$ for every $x \in E$, and hence $E \subseteq U_G$. Let (c_k, d_k) denote the components of U_G. By Lemma 1 $G(c_k) \le G(d_k)$, which implies that $R\,(d_k - c_k) \le F(d_k) - F(c_k)$. Since F is continuous, it follows that $\bigcup_k \big(F(c_k), F(d_k)\big) \subseteq F\big(\bigcup_k (c_k, d_k)\big) = F(U_G)$. By using (P3) twice and (P1) once, one obtains

$$R\, m^*(U_G) = R \textstyle\sum_k (d_k - c_k) \le \sum_k \big(F(d_k) - F(c_k)\big) \le m^*\big(F(U_G)\big).$$

Then $R\, m^*(E) \le m^*\big(F(U)\big) \le m^*(V) < m^*\big(F(E)\big) + \varepsilon$ (by definition of U), and the assertion follows because $\varepsilon > 0$ is arbitrary. □

Lemma 3. *Let $F : [a,b] \to \mathbb{R}$ be an increasing function. Then the function $G : [F(a), F(b)] \to \mathbb{R}$ defined by $G(y) := \inf\{z \in [a,b] : F(z) \ge y\}$ is increasing, and $G\big(F(x)\big) \le x$ for every $x \in [a,b]$. Moreover, $G\big(F(x)\big) < x$ iff F is constant on some interval $[z,x]$. Finally, if the function F is strictly increasing, then G is a left inverse of F and is continuous.*

PROOF. All the assertions are trivial. Suppose that F is strictly increasing. To show that the function G is right continuous, let $x \in [a,b)$ and $\varepsilon > 0$ such that $x + \varepsilon \le b$. If $F(x) < y < F(x + \varepsilon)$, then $x \le G(y) \le x + \varepsilon$. □

Proposition 4. *Let $F : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $r > 0$. If the set $E \subseteq (a,b)$ is such that*

$$\underline{D}_- F(x) := \liminf_{y \nearrow x} \frac{F(x) - F(y)}{x - y} < r$$

for every $x \in E$, then $m^*\big(F(E)\big) \le r\, m^*(E)$.

PROOF. Following [10] we consider the function $G : [F(a), F(b)] \to \mathbb{R}$ of Lemma 3. Let D be the set of all left discontinuities of F. It is well known that D is at most denumerable. Now take a point $x \in E \setminus D$. Since $\underline{D}_- F(x) < r$, there exist $0 < \varepsilon < r$ and a sequence $x_n \nearrow x$ such that

$$\frac{F(x) - F(x_n)}{x - x_n} < r - \varepsilon, \text{ and hence } \frac{G\big(F(x)\big) - G\big(F(x_n)\big)}{F(x) - F(x_n)} > \frac{1}{r - \varepsilon}.$$

By continuity, the sequence $F(x_n)$ converges to $F(x)$. So one concludes that $\overline{D}_- G\big(F(x)\big) > \frac{1}{r}$. By Proposition 2 one obtains $m^*(E) \ge m^*(E \setminus D) \ge \frac{1}{r}\, m^*\big(F(E \setminus D)\big)$. Since $m^*\big(F(D)\big) = 0$, one gets $r\, m^*(E) \ge m^*\big(F(E \setminus D)\big) = m^*\big(F(E)\big)$. □

Lemma 5. *Let $F, G : [a,b] \to \mathbb{R}$ be two increasing functions and $E \subseteq (a,b)$. If $H(x) = F(x) + G(x)$, then $m^*\big(F(E)\big) + m^*\big(G(E)\big) \le m^*\big(H(E)\big)$.*

PROOF. Let $\varepsilon > 0$. By (P4) there exists an open set V such that $H(E) \subseteq V$ and $m^*(V) < m^*\big(H(E)\big) + \varepsilon$. Let (c_k, d_k) denote the components of V. For $x_1, x_2, y_1, y_2 \in H^{-1}(c_k, d_k)$ with $x_i < y_i$ note that

$$F(y_1) - F(x_1) + G(y_2) - G(x_2) \le H(\max y_i) - H(\min x_i) < d_k - c_k.$$

Hence $F\big(H^{-1}(c_k, d_k)\big)$ and $G\big(H^{-1}(c_k, d_k)\big)$ are contained in two intervals I_k and J_k such that $m^*(I_k) + m^*(J_k) \le d_k - c_k$. Since $E \subseteq \bigcup_k H^{-1}(c_k, d_k)$, it follows that $m^*\big(F(E)\big) + m^*\big(G(E)\big) \le \sum_k (d_k - c_k) < m^*\big(H(E)\big) + \varepsilon$. □

Proposition 6. *Let $F : [a,b] \to \mathbb{R}$ be an increasing function and let $r > 0$. If $E \subseteq (a,b)$ is such that*

$$\underline{D}_- F(x) := \liminf_{y \nearrow x} \frac{F(x) - F(y)}{x - y} < r$$

for every $x \in E$, then $m^*\big(F(E)\big) \le r\, m^*(E)$.

PROOF. We consider the function $H(x) := F(x) + x$. Then $H(x)$ is strictly increasing and $\underline{D}_- H(x) < r + 1$ for every $x \in E$. According to the previous results one concludes that $m^*\big(F(E)\big) + m^*(E) \le m^*\big(H(E)\big) \le (r+1)\, m^*(E)$. □

Corollary 7. *Let $F:[a,b]\to\mathbb{R}$ be an increasing function and let $r>0$. If the set $E\subseteq(a,b)$ is such that*

$$\underline{D}_+F(x) := \liminf_{y\searrow x}\frac{F(y)-F(x)}{y-x} < r$$

for every $x\in E$, then $m^*\big(F(E)\big)\le r\,m^*(E)$.

PROOF. Consider the function $G(x) := -F(-x)$. $\square$

Proposition 8. *Let $F:[a,b]\to\mathbb{R}$ be an increasing function and let $R>0$. If the set $E\subseteq(a,b)$ is such that*

$$\overline{D}_+F(x) := \limsup_{y\searrow x}\frac{F(y)-F(x)}{y-x} > R$$

for every $x\in E$, then $m^*\big(F(E)\big)\ge R\,m^*(E)$.

PROOF. Consider the function $G:[F(a),F(b)]\to\mathbb{R}$ of Lemma 3. Let D_1 be the set of all right discontinuities of F, and D_2 the set of right end-points of intervals of constancy. It is well known that the set $D=D_1\cup D_2$ is at most denumerable. Now take a point $x\in E\setminus D$. Since $\overline{D}_+F(x)>R$, there exist $\varepsilon>0$ and a sequence $x_n\searrow x$ such that

$$\frac{F(x_n)-F(x)}{x_n-x} > R+\varepsilon, \text{ and hence } \frac{G\big(F(x_n)\big)-G\big(F(x)\big)}{F(x_n)-F(x)} < \frac{1}{R+\varepsilon}$$

by using the properties of G. By continuity, the sequence $F(x_n)$ converges to $F(x)$. So one concludes that $\underline{D}_+G\big(F(x)\big)<\frac{1}{R}$. By Corollary 7 one obtains

$$m^*(E\setminus D)\le \tfrac{1}{R}\,m^*\big(F(E\setminus D)\big)\le \tfrac{1}{R}\,m^*\big(F(E)\big).$$

Since $m^*(D)=0$, one gets $R\,m^*(E)=R\,m^*(E\setminus D)\le m^*\big(F(E)\big)$. $\square$

Theorem 9. *Let $F:[a,b]\to\mathbb{R}$ be an increasing function. Then*

$$\underline{D}_-F(x)=\overline{D}_+F(x)=\underline{D}_+F(x)=\overline{D}_-F(x)$$

for all $x\in(a,b)$ except on a set E such that $m^*(E)=m^*\big(F(E)\big)=0$. The set $Z:=\{x\in(a,b) : F'(x)=0\}$ satisfies the equality $m^*\big(F(Z)\big)=0$ and the set $I:=\{x\in(a,b) : F'(x)=\infty\}$ satisfies $m^*(I)=0$.

PROOF. This uses a classical argument. Given two rationals $R>r>0$ we consider

$$E_{rR} := \{x\in(a,b) : \underline{D}_-F(x)<r<R<\overline{D}_+F(x)\}.$$

By Propositions 6 and 8 we get $R\,m^*(E_{rR}) \le m^*\big(F(E_{rR})\big) \le r\,m^*(E_{rR})$, so $m^*(E_{rR}) = m^*\big(F(E_{rR})\big) = 0$. Then $E_1 := \{x \in (a,b) \,:\, \underline{D}_-F(x) < \overline{D}_+F(x)\}$ satisfies the equality $m^*(E_1) = m^*\big(F(E_1)\big) = 0$ by (P2), and the same holds for $E_2 := \{x \in (a,b) \,/\, \underline{D}_+F(x) < \overline{D}_-F(x)\}$ by considering $G(x) := -F(-x)$. Now let $E := E_1 \cup E_2$. For $x \in (a,b) \setminus E$ we remark that

$$\underline{D}_-F(x) \ge \overline{D}_+F(x) \ge \underline{D}_+F(x) \ge \overline{D}_-F(x) \ge \underline{D}_-F(x).$$

Finally, for every $n \in \mathbb{N}$ we obtain $m^*\big(F(Z)\big) \le \frac{1}{n}\,(b-a)$ by Proposition 6, and $m^*(I) \le \frac{1}{n}\,\big(F(b) - F(a)\big)$ by Proposition 8. $\square$

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