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REARRANGEMENTS OF TRIGONOMETRIC SERIES AND TRIGONOMETRIC POLYNOMIALS

Abstract

The paper is related to the following question of P. L. Ul'yanov. Is it true that for any 2π-periodic continuous function f there is a uniformly convergent rearrangement of its trigonometric Fourier series? In particular, we give an affirmative answer if the absolute values of Fourier coefficients of f decrease. Also, we study how to choose m terms of a trigonometric polynomial of degree n to make the uniform norm of their sum as small as possible.

1 Introduction

P. L. Ul'yanov [Ul] raised the following question. Is it true that for any 2π-periodic continuous function f there is a uniformly convergent rearrangement of its trigonometric Fourier series? The problem is still open.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $C(\mathbb{T})$ be the space of all continuous functions $f : \mathbb{T} \to \mathbb{C}$, $\|f\|$ be the uniform norm of $f \in C(\mathbb{T})$. We associate with every function $f \in C(\mathbb{T})$ its Fourier series in complex form

$$f \sim \sum_{k\in\mathbb{Z}} c_k e^{ikx}$$

and in real form

$$f \sim \sum_{k=0}^{\infty} A_k(x),\ A_k(x) = d_k \cos(kx + \phi_k).$$

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Observe that $A_k(x) = c_k e^{ikx} + c_{-k} e^{-ikx}$. It is easy to see that if Ul'yanov's conjecture is true for the series in a real form (that is, there is a permutation σ of $\mathbb{N}$ such that $\|f - d_0 - \sum_{k=1}^{n} A_{\sigma(k)}\| \to 0$ as $n \to \infty$), then it is also true for the series in complex form because for $n \to \infty$

$$\Big\|f - d_0 - \sum_{k=1}^{n} \left(c_{\sigma(k)} e^{i\sigma(k)x} + c_{-\sigma(k)} e^{-i\sigma(k)x}\right)\Big\| \to 0.$$

Sz.Gy. Révész[R, R2] proved that for any $f \in C(\mathbb{T})$ there is a rearrangement of its trigonometric Fourier series such that some subsequence of the sequence of partial sums of the rearranged series converges to f uniformly. Due to this result, Ul'yanov's conjecture is equivalent to the following. There is an absolute constant $C > 0$ such that for any trigonometric polynomial (with a zero constant term) $\sum_{k=1}^{n} A_k(x)$ there is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for $m = 1, \ldots, n$

$$\Big\|\sum_{k=1}^{m} A_{\sigma(k)}(x)\Big\| \le C\Big\|\sum_{k=1}^{n} A_k(x)\Big\|.$$

It is known that

$$\Big\|\sum_{k=1}^{m} A_k(x)\Big\| \le C\log(n+1)\Big\|\sum_{k=1}^{n} A_k(x)\Big\|$$

(see [Z][chapter 2, §12]). Let

$$\omega(f, \delta) = \sup_{\substack{x,y \in \mathbb{T} \\ |x-y| \le \delta}} |f(x) - f(y)|$$

be the modulus of continuity of f. By the Dini-Lipschitz theorem [Z] [chapter 2, §10], if $\omega(f, \delta) = o(1/\log 1/\delta)$ as $\delta \to 0$, then the Fourier series of f converges to f uniformly. Moreover, the condition on $\omega(f, \delta)$ is sharp and cannot be replaced by $\omega(f, \delta) = O(1/\log 1/\delta)$[Z] [chapter 8, §2].

The author[K, K2] proved the following results.

Theorem 1. *For any trigonometric polynomial $\sum_{k=1}^{n} A_k(x)$ there is a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for $m = 1, \ldots, n$*

$$\Big\|\sum_{k=1}^{m} A_{\sigma(k)}(x)\Big\| \le C\log\log(n+2)\Big\|\sum_{k=1}^{n} A_k(x)\Big\|.$$

Theorem 2. *Let $f \in C(\mathbb{T})$ and $\omega(f,\delta) = o(1/\log\log 1/\delta)$ as $\delta \to 0$. Then there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that*

$$\Big\| f - d_0 - \sum_{k=1}^{n} A_{\sigma(k)}(x) \Big\| \to 0 \ (n \to \infty).$$

Theorem 2 follows from Theorem 1 by using Theorem 5 from [R].

To approach Ul'yanov's conjecture, one can try to prove that there is an absolute constant $C > 0$ such that for any trigonometric polynomial (with a zero constant term) $\sum_{k=1}^{n} A_k(x)$ and for any $m \leq n$ there is an injection $\sigma : \{1,\ldots,m\} \to \{1,\ldots,n\}$ such that

$$\Big\| \sum_{k=1}^{m} A_{\sigma(k)}(x) \Big\| \leq C \Big\| \sum_{k=1}^{n} A_k(x) \Big\|.$$

I cannot prove this either.

Theorem 3. *For any trigonometric polynomial $\sum_{k=1}^{n} A_k(x)$ and for any $m \leq n$ there is a set $K \subset \{1,\ldots,n\}$ such that $|K| = m$ and*

$$\Big\| \sum_{k\in K} A_k(x) \Big\| \leq C \log\log\log(n+20) \Big\| \sum_{k=1}^{n} A_k(x) \Big\|.$$

Theorem 4. *Let $f \in C(\mathbb{T})$,*

$$f \sim \sum_{k=0}^{\infty} A_k(x), \ A_k(x) = d_k \cos(kx + \phi_k),$$

and $d_k = O(k^{-1/2})$. Then there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\Big\| f - d_0 - \sum_{k=1}^{n} A_{\sigma(k)}(x) \Big\| \to 0 \quad (n \to \infty).$$

In particular, Theorem 4 works if the sequence $\{|d_k|\}$ is nonincreasing. Note that, by a theorem of Salem [S], there exists an even continuous function such that its Fourier series diverges at $x = 0$ and the sequence $\{|d_k|\}$ is nonincreasing,

By $C, C', C_1, C_2, \ldots$ we denote positive constants. Let $[u]$ and $\{u\}$ be the integer and the fractional part of the real number u, respectively.

2 Proof of Theorem 3

Let $n \in \mathbb{N}$, T be a trigonometric polynomial,

$$T(x) = \sum_{k=1}^{n} A_k(x) = \sum_{k=1}^{n} d_k \cos(kx + \phi_k).$$

We use the following lemmas from [K2].

Lemma 1. *Let $\|T\| \le 1$, $l \in \mathbb{N}$, $j \in \mathbb{Z}$, $K_{l,j} = \{k : \ 1 \le k \le n, \ k \equiv \pm j (\mathrm{mod}\ l)\}$. Then*

$$\Big\| \sum_{k \in K_{l,j}} A_k \Big\| \le 2.$$

Lemma 2. *Let $\|T\| \le 1$. Then there exists an odd prime $p \le 2\log^3(n+3)$ such that*

$$\sum_{\substack{k_1 \ne k_2 \\ k_1 \equiv k_2 (\bmod\ p)}} |d_{k_1}|^2 |d_{k_2}|^2 \le \frac{C_1}{\log^2(n+1)}. \tag{1}$$

Lemma 3. *Let p be a prime satisfying (1), $j \in \mathbb{Z}$, $K_{p,j} = \{k : \ 1 \le k \le n, \ k \equiv \pm j (\mathrm{mod}\ p)\}$, $N_j = |K_{p,j}|$. Then there exists a bijection $\tau : \{1, \dots, N_j\} \to K_{p,j}$ such that for any $m = 1, \dots, N_j$ the inequality*

$$\Big\| \sum_{j=1}^{m} A_{\tau(j)} \Big\| \le C_2(1 + \|T\|)$$

holds.

In the proof of Theorem 3 we assume that n is sufficiently large and $\|T\| \le 1$. We can also assume that $m \le n/2$; otherwise we can take the complement to a set constructed for $n - m < n/2$ instead of m. Also, it is sufficient to construct a set $K' \subset \{1, \dots, n\}$ such that $|K'| = m'$ for some $m' \le m$, $m - m' \le 0.2n/\log^3 n$, and

$$\Big\| \sum_{k \in K'} A_k(x) \Big\| \le C' \log\log\log n.$$

Indeed, take an odd prime $p \le 2\log^3(n+3)$ satisfying Lemma 2. Define the sets $K_{p,j}$ as in Lemma 3. Since $|K'| \le n/2$, we can find j so that

$$|K_{p,j} \setminus K'| \ge (n - |K'|)/p \ge n/(4\log^3(n+3)) \ge 0.2n/\log^3 n$$

provided that $n \geq 20$. Applying Lemma 3 to the polynomial

$$\sum_{k\in\{1,\dots,n\}\setminus K'} A_k,$$

we can define the set K as $K' \cup \{\tau(1),\dots,\tau(m)\}$ where m is such that

$$\{\tau(1),\dots,\tau(m)\} \setminus K' = m - m'.$$

By the above arguments we can assume that $m > 0.2n/\log^3 n$; otherwise, we take $m' = 0$ and $K' = \emptyset$.

We shall use the following known fact.

Lemma 4. *For any real $\alpha \in (0,1]$ there exist positive integers $l_1, l_2, \dots,$ such that for any positive integer s*

$$0 < \alpha - \sum_{j=1}^{s} \frac{1}{l_j} \leq 2^{-2^{s-1}}. \tag{2}$$

Proof. We construct l_s inductively by

$$l_s = \min\{l : \alpha - \sum_{j=1}^{s-1} \frac{1}{l_j} - \frac{1}{l} > 0\}.$$

The inequalities (2) can be checked by induction on s. The proof of the first inequality is straightforward. The induction base for the second inequality holds: $\alpha - 1/l_1 \leq 1/2$.

By the induction supposition (2), we have $l_{s+1} - 1 \geq 2^{2^{s-1}}$. Also, by the definition of l_{s+1}, $\alpha - \sum_{j=1}^{s} \frac{1}{l_j} - \frac{1}{l_{s+1}-1} \leq 0$. Therefore,

$$\alpha - \sum_{j=1}^{s+1} \frac{1}{l_j} \leq \frac{1}{l_{s+1}-1} - \frac{1}{l_{s+1}} < \frac{1}{(l_{s+1}-1)^2} \leq 2^{-2^{s}},$$

and (2) is established for $s+1$. Lemma 4 is proved. □

Take $s = [2\log\log\log n]$. Note that for sufficiently large n we have

$$2^{-2^{s-1}} \leq 0.05/\log^3 n. \tag{3}$$

One can try to define the numbers $l_1, \dots, l_s$ by Lemma 3 with α close to m/n and to take, for example,

$$K' = \bigcup_{j=1}^{s} K_j, \quad K_j = \{k \equiv \pm 1 (\mathrm{mod}\ 2l_j)\},$$

By Lemma 1,

$$\Big\| \sum_{k\in K_j} A_k \Big\| \le 2$$

and $\sum_j |K_j|$ is close to m. However, the sets K_j might have common points, and in general we cannot give good estimates for $\|\sum_{k\in K'} A_k\|$ and for $|K'|$. We show how to correct the construction.

Let $l_0 = [5\log\log\log n]$, $\gamma = l_0 m/n - 0.1/\log^3 n$, $g = [\gamma]$, $\alpha = \{\gamma\}$. Note that $g \ge 0$. Because of our supposition $m > 0.2n/\log^3 n$. Take the numbers $l_1, \dots, l_s$ in accordance with Lemma 4 and define

$$K' = \bigcup_{j=1}^{g} K_j \cup \bigcup_{j=1}^{s} K'_j,$$

where $K_j = \{k \equiv \pm j (\mathrm{mod}\ 2l_0)\}$, $K'_j = \{k \pm (g+j) \equiv 0 (\mathrm{mod}\ 2l_0 l_j)\}$. Note that the residues classes $\pm j (\mathrm{mod}\ 2l_0)$ $(j = 1, \dots, g+s)$, are all distinct since $g + s \le l_0/2 + s < l_0 - 1$. Therefore, the sets K_j, K'_j are pairwise disjoint. Further, by Lemma 1,

$$\Big\| \sum_{k\in K_j} A_k \Big\| \le 2, \ \Big\| \sum_{k\in K'_j} A_k \Big\| \le 2.$$

Hence,

$$\Big\| \sum_{k\in K'} A_k \Big\| \le 2(g+s) \le 10\log\log\log n.$$

Also, it is not difficult to check that

$$||K_j| - n/l_0| \le 1, ||K'_j| - n/(l_0 l_j)| \le 1.$$

Therefore,

$$|K'| = ng/l_0 + \sum_{j=1}^{s} n/(l_0 l_j) + O(\log\log\log n).$$

Taking (2) and (3) into account, we get

$$ng/l_0 + \sum_{j=1}^{s} n/(l_0 l_j) \le m - 0.1n/\log^3 n$$

$$ng/l_0 + \sum_{j=1}^{s} n/(l_0 l_j) \ge m - 0.1n/\log^3 n - 0.05n/\log^3 n.$$

Combining three last inequalities, we obtain

$$m \geq |K'| \geq m - 0.2n/\log^3 n,$$

as required. This completes the proof of Theorem 3. □

3 Spencer's Theorem and Its Corollaries

Let u be a vector $u = (u^1, \dots, u^n) \in \mathbb{R}^n$ and let $|u|_\infty = \max_k |u^k|$. J. Spencer [Sp] actually proved the following theorem.

Theorem A. *Let $r \leq n$ be a positive integer, $u_j \in \mathbb{R}^n$, $|u_j|_\infty \leq 1$. Then for some choice of signs*

$$|\pm u_1 \pm \cdots \pm u_r|_\infty \leq C_3 (r \log(2n/r))^{1/2}.$$

Corollary 1. *Let $r \leq n$ be positive integers and $K \subset \{1, \dots, n\}$, $|K| = r$. Consider a trigonometric polynomial*

$$\sum_{k\in K} A_k(x),\ A_k(x) = d_k \cos(kx + \phi_k).$$

Then there are sets $K_+ \subset K$ and $K_- \subset K$ such that

$$K_+ \cup K_- = K,\ K_+ \cap K_- = \emptyset,\ |K_+| = [|K|/2] \tag{4}$$

and

$$\Big\| \sum_{k\in K_+} A_k - \sum_{k\in K_-} A_k \Big\| \leq C_4 (r \log(2n/r))^{1/2} \max_{k\in K} |d_k|. \tag{5}$$

Proof. Let $d = \max_{k\in K} |d_k|$. We apply Theorem A to the vectors $u_k \in \mathbb{R}^{20n+1}$, $k \in K$, defined as

$$u_k = \big(\Re(A_k(\pi l/(5n))/d)_{l=0,\dots,10n-1}, \Im(A_k(\pi l/(5n))/d)_{l=0,\dots,10n-1}, 1\big).$$

Then there exist numbers $\sigma_k = \pm 1$ $(k \in K)$ such that

$$\Big\| \sum_{k\in K} \sigma_k A_k \Big\| \leq 3\sqrt{2} C_3 (r \log((40n+2)/r))^{1/2} d \tag{6}$$

and

$$\Big| \sum_{k\in K} \sigma_k \Big| \leq C_3 (r \log((40n+2)/r))^{1/2}. \tag{7}$$

For the proof of (6) we use that for any trigonometric polynomial T of order n

$$\|T\| \leq 3 \max_{l=0,\dots,10n-1} |T(\pi l/(5n))|$$

(see, for example, [Kl]). Without loss of generality we can assume that $\sum_{k\in K} \sigma_k \leq 0$. Take $K'_+ = \{k \in K : \ \sigma_k = 1\}$, $K'_- = \{k \in K : \ \sigma_k = -1\}$. We have

$$2|K'_+| = |K| + \sum_{k\in K} \sigma_k \leq 2[|K|/2].$$

Take an arbitrary set $K_1 \subset K'_-$ such that $|K_1| = [|K|/2] - |K'_+|$. By (7), $|K_1| \leq C_3(r\log((40n+2)/r))^{1/2}/2$. Hence,

$$\Big\| \sum_{k\in K_1} A_k \Big\| \leq C_3(r\log((40n+2)/r))^{1/2} d/2. \tag{8}$$

Denote $K_+ = K'_+ \cup K_1$, $K_- = K'_- \setminus K_1$. The conditions (4) are satisfied. By (6) and (8) we get

$$\Big\| \sum_{k\in K_+} A_k - \sum_{k\in K_-} A_k \Big\| \leq 6C_3\big(r\log((40n+2)/r)\big)^{1/2} d.$$

Therefore, (5) also holds, and Corollary 1 is proved. □

Corollary 2. *Let $r \leq n$ be positive integers and $K \subset \{1,\dots,n\}$, $|K| = r$. Consider a trigonometric polynomial*

$$\sum_{k\in K} \alpha_k A_k(x), \ A_k(x) = d_k \cos(kx + \phi_k),$$

where α_k are real numbers. Then there are numbers $\beta_k \in \{[\alpha_k], [\alpha_k]+1\}$ such that

$$\Big\| \sum_{k\in K} \alpha_k A_k - \sum_{k\in K} \beta_k A_k \Big\| \leq C_4(r\log(2n/r))^{1/2} \max_{k\in K} |d_k|.$$

In fact, the deduction of Corollary 2 from Corollary 1 is exhibited in [Kl].

Corollary 3. *Let r, n be positive integers, $r \leq n/5$ and $K \subset \{1,\dots,n\}$, $|K| = r$. Consider a trigonometric polynomial*

$$\sum_{k\in K} A_k(x), \ A_k(x) = d_k \cos(kx + \phi_k).$$

Then there exists a bijection $\sigma : \{1, \dots, r\} \to K$ *such that for any* $m = 1, \dots, r$ *the inequality*

$$\Big\| \sum_{j=1}^{m} A_{\sigma(j)} - \frac{m}{r} \sum_{k\in K} A_k \Big\| \le (4C_4 + 4)(r \log(2n/r))^{1/2} \max_{k\in K} |d_k| \tag{9}$$

holds.

PROOF. Let $d = \max_{k\in K} |d_k|$. We fix n and use induction on r. If $r \le 8$ then we take an arbitrary bijection σ. For any $m \le r$ we have

$$\begin{aligned}\Big\| \sum_{j=1}^{m} A_{\sigma(j)} - \frac{m}{r} \sum_{k\in K} A_k \Big\| &\le md + \frac{m}{r}(rd) \le 2md \\ &\le 2rd = (2r)^{1/2}(2r)^{1/2} d \le 4(r\log(2n/r))^{1/2} d,\end{aligned}$$

and (9) holds. Let us assume that $9 \le r \le n/5$ and that the statement of the corollary is satisfied for all $r' < r$.

By Corollary 1, we split the sets K into the sets K_+ and K_-. The inequality (5) can be rewritten as

$$\Big\| \sum_{k\in K_+} A_k - \frac{1}{2} \sum_{k\in K} A_k \Big\| \le \frac{C_4}{2}(r\log(2n/r))^{1/2} d.$$

We have

$$\begin{aligned}\Big\| \sum_{k\in K_+} A_k - \frac{[r/2]}{r} \sum_{k\in K} A_k \Big\| &\le \frac{C_4}{2}(r\log(2n/r))^{1/2} d \\ &\quad + \left(\frac{1}{2} - \frac{[r/2]}{r}\right) \Big\| \sum_{k\in K} A_k \Big\| \\ &\le \frac{C_4}{2}(r\log(2n/r))^{1/2} d + \frac{1}{2r}(rd) \\ &= \frac{C_4}{2}(r\log(2n/r))^{1/2} d + d/2 \\ &\le \frac{C_4 + 1}{2}(r\log(2n/r))^{1/2} d.\end{aligned} \tag{10}$$

By the induction supposition, there exist bijections $\sigma_+ : \{1, \dots, [r/2]\} \to K_+$ and $\sigma_- : \{1, \dots, r - [r/2]\} \to K_-$ such that for any $m \le [r/2]$

$$\Big\| \sum_{j=1}^{m} A_{\sigma_+(j)} - \frac{m}{r_1} \sum_{k\in K_+} A_k \Big\| \le (4C_4 + 4)(r_1 \log(2n/r_1))^{1/2} d, \ r_1 = [r/2], \tag{11}$$

and for any $m \le r - [r/2]$

$$\Big\|\sum_{j=1}^{m} A_{\sigma_-(j)} - \frac{m}{r_1}\sum_{k\in K_-} A_k\Big\| \le (4C_4+4)(r_1\log(2n/r_1))^{1/2}d,\ r_1 = r-[r/2]. \tag{12}$$

We take $\sigma(j) = \sigma_+(j)$ for $j \le [r/2]$ and $\sigma(j) = \sigma_-(r+1-j)$ for $j > [r/2]$. If $m \le [r/2]$ then we have, by (10) and (11),

$$\begin{aligned}\Big\|\sum_{j=1}^{m} A_{\sigma(j)} - \frac{m}{r}\sum_{k\in K} A_k\Big\| &\le \Big\|\sum_{j=1}^{m} A_{\sigma_+(j)} - \frac{m}{r_1}\sum_{k\in K_+} A_k\Big\| \\ &\quad + \Big\|\frac{m}{r_1}\sum_{k\in K_+} A_k - \frac{m}{r}\sum_{k\in K} A_k\Big\| \\ &\le \Big\|\sum_{j=1}^{m} A_{\sigma_+(j)} - \frac{m}{r_1}\sum_{k\in K_+} A_k\Big\| \\ &\quad + \Big\|\sum_{k\in K_+} A_k - \frac{[r/2]}{r}\sum_{k\in K} A_k\Big\| \\ &\le (4C_4+4)(r_1\log(2n/r_1))^{1/2}d \\ &\quad + \frac{C_4+1}{2}(r\log(2n/r))^{1/2}d,\ r_1 = [r/2].\end{aligned} \tag{13}$$

Further, for $r_1 = [r/2]$ we have

$$\begin{aligned}(r_1\log(2n/r_1))^{1/2} &\le \left(\frac{r}{2}\log(2n/r\times 9/4)\right)^{1/2} < \left(\frac{r}{2}\times\frac{3}{2}\log(2n/r)\right)^{1/2} \\ &< \left(\frac{3}{4}r\log(2n/r)\right)^{1/2} < \frac{7}{8}(r\log(2n/r))^{1/2}.\end{aligned}$$

Substituting the last inequality into (13) we get the required

$$\Big\|\sum_{j=1}^{m} A_{\sigma(j)} - \frac{m}{r}\sum_{k\in K} A_k\Big\| \le (4C_4+4)(r\log(2n/r))^{1/2}d.$$

If $m > [r/2]$, then, similarly to (13), we have

$$\begin{aligned}\Big\|\sum_{j=1}^{m} A_{\sigma(j)} - \frac{m}{r}\sum_{k\in K} A_k\Big\| &= \Big\|\sum_{j=1}^{r-m} A_{\sigma_-(j)} - \frac{r-m}{r}\sum_{k\in K} A_k\Big\| \\ &\le (4C_4+4)(r_1\log(2n/r_1))^{1/2}d \\ &\quad + \frac{C_4+1}{2}(r\log(2n/r))^{1/2}d,\ r_1 = r-[r/2].\end{aligned} \tag{14}$$

For $r_1 = [r/2]$ we have

$$(r_1 \log(2n/r_1))^{1/2} \le \left(\frac{5r}{9}\log(2n/r \times 2)\right)^{1/2} < \left(\frac{5r}{9} \times \frac{4}{3}\log(2n/r)\right)^{1/2}$$
$$< \left(\frac{3}{4}r\log(2n/r)\right)^{1/2} < \frac{7}{8}(r\log(2n/r))^{1/2}.$$

and after substitution of the last inequality into (14) we complete the proof of Corollary 3. □

4 Proof of Theorem 4

We use Vallée Poussin sums defined for positive integers $n > m$ as

$$V_{m,n}(x) = \sum_{k=0}^{m} A_k(x) + \sum_{k=m+1}^{n} \frac{n-k}{n-m} A_k(x).$$

It is known that for any $f \in C(\mathbb{T})$ there is a function $n : \mathbb{N} \to \mathbb{N}$ such that $n(m) > m$ for all m, $\lim_{m\to\infty} n(m)/m = 1$ and $\lim_{m\to\infty} \|V_{m,n} - f\| = 0$. (This follows, for example, from [D] or from [St].) We define the increasing sequence of positive integers $\{N_\lambda\}_{\lambda\in\mathbb{N}}$ by $N_1 = 1$, $N_{\lambda+1} = n(N_\lambda)$ for $\lambda \ge 1$.

We fix $\lambda \ge 1$, take $m = N_\lambda$, $n = N_{\lambda+1}$ and use Corollary 2 for $K_\lambda = \{m+1, \dots, n\}$, $\alpha_k = \frac{n-k}{n-m}$. We find that there are numbers $\beta_k \in \{0, 1\}$, $k \in K$, such that

$$\left\| V_{m,n} - \sum_{k=0}^{m} A_k - \sum_{k\in K} \beta_k A_k \right\|$$
$$\ll (((n-m)/n)\log((2n)/(n-m)))^{1/2} \to 0 \quad (\lambda \to \infty).$$

Also, by the choice of the sequence $\{N_\lambda\}\}$, we have $\lim_{\lambda\to\infty} \|V_{m,n} - f\| = 0$. Therefore, letting $L_\lambda = \{1, \dots, m\} \cup \{k \in K_\lambda : \ \beta_k = 1\}$ we get

$$\left\| f - d_0 - \sum_{k\in L_\lambda} A_k \right\| \to 0 \ (\lambda \to \infty). \tag{15}$$

To complete the proof, it is enough, by (15), to find a good permutation of the terms of the polynomials $\sum_{k\in L_{\lambda+1}\setminus L_\lambda} A_k$. We construct a permutation in such a way that the numbers from $L_\lambda \setminus L_{\lambda-1}$ precede the numbers from $L_{\lambda+1} \setminus L_\lambda$ for all λ for all $\lambda \in \mathbb{N}$; we consider that $L_0 = \emptyset$. The permutation can be constructed by Corollary 3, the partial sums can be estimated similarly to (15), and we are done.

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