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PROPERTIES OF THE SPACE $\mathcal{DB}_1^{**}$ WITH THE METRIC OF UNIFORM CONVERGENCE.

Abstract

In this paper we shall show that the set $\mathcal{C}$ of all bounded continuous functions is superporous in the space $\mathcal{DB}_1^{**}$. Moreover, for an arbitrary function f defined on $\mathcal{C}$ there exists a quasi-continuous extension f_1 of this function on $\mathcal{DB}_1^{**}$, such that $\mathcal{C}$ is the set of all discontinuity points of f_1.

1 Introduction

This article contains some properties of the space of Darboux functions belonging to the class $\mathcal{B}_1^{**}$. The class $\mathcal{B}_1^{**}$ has been introduced by R. J. Pawlak in 2000 ([5]).

We will use mostly standard notations. In particular by the letter $\mathbb{R}$ we denote the set of all real numbers (as well as the space with the natural topology). By the letter $\mathcal{C}$ we shall denote the set of all bounded continuous functions. Let $f : X \to Y$, where X and Y are topological spaces. We say that f is Darboux functions if the image $f(C)$ is a connected set, for each connected set $C \subset X$.

The set of all discontinuity points of f we denote by D_f. If A is a subset of the domain of f, then the restriction of f to A we denote by $f \restriction A$. A function f belongs to the class B_1^{**} if either $D_f = \emptyset$ or $f \restriction D_f$ is the continuous function

By the symbol $\mathcal{DB}_1^{**}$ we shall denote the set of all bounded Darboux functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ belonging to the class $\mathcal{B}_1^{**}$, with the metric of the uniform convergence.

Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous mappings. We say that f and g are homotopic if there exists a continuous

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mapping $\xi : X \times [0,1] \to Y$ (the mapping ξ is called homotopy between f and g) such that $\xi(x,0) = f(x)$ and $\xi(x,1) = g(x)$ (for each $x \in X$). This relation we denote by $f \underset{\xi}{\simeq} g$.

The symbol $B(x,\varepsilon)$ denotes the ball with the centre at x and the radius $\varepsilon > 0$. The notions and symbols we use, connected with porosity, come from papers [9] and [10]. Let X be a metric space. Let $M \subset X$, $x \in X$ and $S > 0$. Then we denote by $\gamma(x,S,M)$ the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z,r) \subset B(x,S) \setminus M$. If $p(M,x) = 2 \cdot limsup_{S \to 0+} \frac{\gamma(x,S,M)}{S} > 0$, then we say that M is *porous* at x. If M is porous at each point $x \in X$ then we shall write $M \subset_p X$.

We say that the set C is superporous at x_0, if the set $C \cup A$ is porous at x_0, for each set A porous at x_0. We say that a set $C \subset X$ is a superporous set in X if C is superporous set at each point of X. This fact we denote by $C \subset_{sp} X$.

By a (topological) road in the topological space X we mean a set $f([0,1])$, where $f : [0,1] \longrightarrow X$ is a bounded continuous function. The point $f(0)$ is the initial point and $f(1)$ is the end-point of this road.

2 Main Results

The next theorem is a stronger version of the results from [4].

Theorem 1. $\mathcal{C} \subset_{sp} \mathcal{DB}_1^{**}$.

PROOF. Let $f \in \mathcal{DB}_1^{**}$ and let $A \subset \mathcal{DB}_1^{**}$ be an arbitrary set porous at f. Put $Z = \mathcal{C} \cup A$. We shall show, that Z is a porous set at f. Let now $R > 0$ be a fixed real number. Let us put $\sigma_0 = \frac{\gamma(f,R,A)}{2R} > 0$. Then there exists a real number $\sigma > \sigma_0$ and a function $g \in \mathcal{DB}_1^{**}$ such that

$$B(g, \sigma \cdot R) \subset B(f,R) \setminus A. \tag{1}$$

To prove our theorem it is sufficient to show that there exists a function $h \in \mathcal{DB}_1^{**}$ such that

$$B(h, \frac{\sigma \cdot R}{8}) \subset B(f,R) \setminus Z.$$

Let $x_0 \notin \overline{D_g}$ (observe, [5], that such a point exists). Let $\delta > 0$ be a number such that

$$[x_0 - \delta, x_0 + \delta] \cap \overline{D_g} = \emptyset \text{ and } g([x_0 - \delta, x_0 + \delta]) \subset (g(x_0) - \frac{\sigma \cdot R}{4}, g(x_0) + \frac{\sigma \cdot R}{4}).$$

Let us define required function $h : \mathbb{R} \to \mathbb{R}$ in the following way:

$$h(x) = \begin{cases} g(x) & if \quad x \in (-\infty, x_0 - \delta] \cup \{x_0\} \cup [x_0 + \delta, +\infty), \\ t(x) & if \quad x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta), \end{cases}$$

where t is a continuous function mapping $[x_0 - \delta, x_0) \cup (x_0, x_0 + \delta]$ into $\mathbb{R}$ such that $t(x_0 - \delta) = g(x_0 - \delta)$, $t(x_0 + \delta) = g(x_0 + \delta)$ and $t((x, x_0)) = t((x_0, y)) = [g(x_0) - \frac{\sigma \cdot R}{4}, g(x_0) + \frac{\sigma \cdot R}{4}]$, for each $x \in (x_0 - \delta, x_0)$ and $y \in (x_0, x_0 + \delta)$.

We shall show that $h \in \mathcal{DB}_1^{**}$. Remark that $h \restriction (-\infty, x_0 - \delta]$, $h \restriction [x_0 + \delta, +\infty)$, $h \restriction [x_0 - \delta, x_0 + \delta]$ are Darboux functions. Then (according to the proof of Lemma 1 from [8], see also Lemma 1.4 from [7]) h is a Darboux function. On the other hand

$$D_h = (D_g \cap (-\infty, x_0 - \delta) \cup (x_0 + \delta, +\infty)) \cup \{x_0\},$$

x_0 is an isolated point in the set D_h and h, g are agree on the set $D_h \setminus \{x_0\}$. So $h \in \mathcal{B}_1^{**}$, because $g \in \mathcal{B}_1^{**}$.

Obviously

$$\varrho(h, g) < \frac{\sigma \cdot R}{2}.$$

It is easy to see that

$$B(h, \frac{\sigma \cdot R}{8}) \subset B(g, \sigma \cdot R). \tag{2}$$

Now, we shall show that

$$B(h, \frac{\sigma \cdot R}{8}) \cap \mathcal{C} = \emptyset. \tag{3}$$

Indeed. Let $l \in B(h, \frac{\sigma \cdot R}{8})$ and let $\{x_n\} \subset (x_0 - \delta, x_0)$ be an increasing sequence converging to x_0 such that $h(x_n) = g(x_0) + \frac{\sigma \cdot R}{4}$. Clearly $l(x_n) > g(x_0) + \frac{\sigma \cdot R}{8}$ and $l(x_0) < g(x_0) + \frac{\sigma \cdot R}{8}$ and so x_0 is a discontinuity point of l. From (1),(2) and (3) it follows that

$$B(h, \frac{\sigma \cdot R}{8}) \cap (A \cup \mathcal{C}) = \emptyset$$

and so

$$B(h, \frac{\sigma \cdot R}{8}) \subset B(f, R) \setminus Z.$$

Consequently, $p(Z, f) > 0$, which finishes this proof. □

Definition 1. [3]. We say that a function $f : X \to Y$ is 2-continuous (briefly $f \in \mathcal{C}_2$) if there exist two sets A and B such that $X = A \cup B$ and the restrictions $f \restriction A$ and $f \restriction B$ are continuous functions.

Lemma 1. *[3].* $\mathcal{DB}_1^{**} = \mathcal{DC}_2$.

Lemma 2. *If* $f \in \mathcal{DB}_1^{**}$ *and* $g \in \mathcal{C}$*, then* $f + g \in \mathcal{DB}_1^{**}$.

Proof. Let $f \in \mathcal{DB}_1^{**}$ and $g \in \mathcal{C}$. From Lemma 1 we obtain that $f \in \mathcal{C}_2$. Let A, B be two sets such that $A \cup B = \mathbb{R}$ and the restrictions $f \restriction A$ and $f \restriction B$ are continuous functions. Obviously $g \restriction A$ and $g \restriction B$ are continuous functions, too. So, $f + g \restriction A$ and $f + g \restriction B$ are continuous functions. On the other hand $f + g \in \mathcal{DB}_1$ ([1], Theorem II.3.2, see also [6],[2]). Consequently, $f + g \in \mathcal{DC}_2$ and, according to Lemma 1 we have $f + g \in \mathcal{DB}_1^{**}$. □

Lemma 3. *For each* $\alpha \in \mathbb{R}$ *and for an arbitrary* $f \in \mathcal{DB}_1^{**}$ *we have* $\alpha \cdot f \in \mathcal{DB}_1^{**}$.

Proof.[1] Let $\alpha \in \mathbb{R}$ and $f \in \mathcal{DB}_1^{**}$. From Lemma 1 we obtain that $f \in \mathcal{C}_2$. Thus $\mathbb{R} = A \cup B$, where A, B are the subsets of $\mathbb{R}$ such that $f \restriction A$ and $f \restriction B$ are continuous functions. Obviously, $(\alpha \cdot f) \restriction A$ and $(\alpha \cdot f) \restriction B$ are continuous functions, too. So, $\alpha \cdot f \in \mathcal{C}_2$. Moreover, if $f \in \mathcal{D}$ then $\alpha \cdot f \in \mathcal{D}$, so $\alpha \cdot f \in \mathcal{DC}_2 = \mathcal{DB}_1^{**}$. □

Theorem 2. *Let* $j : \mathcal{C} \to \mathcal{DB}_1^{**}$ *be the identity mapping* ($j(f) = f$*, for each* $f \in \mathcal{C}$)*. Then there exists a continuous mapping* $t : \mathcal{C} \to \mathcal{DB}_1^{**}$ *and a homotopy* $h : \mathcal{C} \times [0,1] \to \mathcal{DB}_1^{**}$ *such that* $j \underset{h}{\simeq} t$ *and* $h(\mathcal{C} \times (0,1]) \cap \mathcal{C} = \emptyset$.

Proof. Let us define a function $\xi : \mathbb{R} \to [0,1]$ by letting

$$\xi(x) = \begin{cases} 0 & if \quad x \in (-\infty, 0] \cup \{\frac{1}{2n-1} :\ n = 1,2,...\} \cup (1, +\infty), \\ 1 & if \quad x \in \{\frac{1}{2n} :\ n = 1,2,...\}, \\ l_1(x) & if \quad x \in [\frac{1}{2n}, \frac{1}{2n-1}]\ (n = 1,2,...), \\ l_2(x) & if \quad x \in [\frac{1}{2n+1}, \frac{1}{2n}]\ (n = 1,2,...), \end{cases}$$

where $l_1 : [\frac{1}{2n}, \frac{1}{2n-1}] \to [0,1]$ is a linear function such that $l_1(\frac{1}{2n}) = 1$ and $l_1(\frac{1}{2n-1}) = 0$, for $n = 1,2,...$ and $l_2 : [\frac{1}{2n+1}, \frac{1}{2n}] \to [0,1]$ is a linear function such that $l_2(\frac{1}{2n+1}) = 0$ and $l_2(\frac{1}{2n}) = 1$, for $n = 1,2,....$ Of course $\mid \xi(x) \mid \leq 1$, for $x \in \mathbb{R}$.

It is easy to see that $\xi \in \mathcal{DB}_1^{**}$. Let us define $t : \mathcal{C} \to \mathcal{DB}_1^{**}$ by the formula

$$t(\mu) = \mu + \xi, \text{ for each } \mu \in \mathcal{C}.$$

Clearly, $\mu + \xi \in \mathcal{DB}_1^{**}$ (see Lemma 2). Moreover t is a continuous mapping. Let us define the homotopy $h : \mathcal{C} \times [0,1] \to \mathcal{DB}_1^{**}$ by letting

$$h(f, r) = f + r \cdot \xi.$$

[1]The Reviewer has remarked that this lemma can be proved in a straightforward manner independent of the result given in Lemma 1.

It is easy to see that h is a continuous function. By virtue of Lemma 3, $r \cdot \xi \in \mathcal{DB}_1^{**}$ and so according to Lemma 2, $f + r \cdot \xi \in \mathcal{DB}_1^{**}$.

Now, we shall show that $j \underset{h}{\simeq} t$ and $h(\mathcal{C} \times (0,1]) \cap \mathcal{C} = \emptyset$. First observe that

$$h(f,0) = f = j(f) \text{ and } h(f,1) = f + \xi = t(f).$$

Let $f \in \mathcal{C}$ be an arbitrary function and let $r \in (0,1]$. It is not difficult to see that $h(f,r) \notin \mathcal{C}$. □

Corollary 1. *For every continuous function k, there exists a (topological) road $\mathcal{R}$ with the initial point at k such that $\emptyset \neq \mathcal{R} \setminus \{k\} \subset \mathcal{DB}_1^{**} \setminus \mathcal{C}$.*

PROOF. Let k be an arbitrary continuous function. Using the terminology of the proof of Theorem 2 one can say that there exists a homotopy $h : \mathcal{C} \times [0,1] \to \mathcal{DB}_1^{**}$ such that $h(k,0) = k$, $h(k,1) = t(k)$, where $t(k)$ is some function from $\mathcal{DB}_1^{**}$. Moreover,

$$h_k = h \restriction \{k\} \times [0,1] \text{ is a continuous function.}$$

To the simplify notation we can assume that h_k is a function of a one variable $h_k : [0,1] \to \mathcal{DB}_1^{**}$ and so for each $r \in (0,1]$, $h_k(r) \in \mathcal{DB}_1^{**} \setminus \mathcal{C}$. To finish, observe that $h_k(0) = k$ and $h_k(1) = t(k)$. □

Theorem 3. *For each function $F : \mathcal{C} \to \mathbb{R}$ there exists an extension $F_1 : \mathcal{DB}_1^{**} \to \mathbb{R}$ of a function F, such that F_1 is a quasi-continuous function and $D_{F_1} = \mathcal{C}$. Moreover, if F is a Darboux function then F_1 is a Darboux function, too.*

PROOF. Let $F : \mathcal{C} \to \mathbb{R}$ be an arbitrary function. Define $F_1 : \mathcal{DB}_1^{**} \to \mathbb{R}$ by the formula

$$F_1(k) = \begin{cases} F(k) & if \quad k \in \mathcal{C}, \\ \frac{\sin \frac{1}{\varrho(k,\mathcal{C})}}{\varrho(k,\mathcal{C})} & if \quad k \in \mathcal{DB}_1^{**} \setminus \mathcal{C}. \end{cases}$$

We shall show that F_1 is a quasi-continuous function. First we can observe that F_1 is a continuous function on the set $\mathcal{DB}_1^{**} \setminus \mathcal{C}$. So, it suffices to prove that for every $k \in \mathcal{C}$, F_1 is quasi-continuous at k.

Fix $k \in \mathcal{C}$. According to Corollary 1 there exists a road $\mathcal{R}_k$ with the initial point at k such that $\emptyset \neq \mathcal{R}_k \setminus \{k\} \subset \mathcal{DB}_1^{**} \setminus \mathcal{C}$. Let $\delta > 0$, $\varepsilon > 0$. First, we shall show that there exists a road $\mathcal{R}'_k \subset \mathcal{R}_k$ with the initial point at k such that $\mathcal{R}'_k \subset B(k,\delta)$ and $\emptyset \neq \mathcal{R}'_k \setminus \{k\}$. Let $h_k : [0,1] \to \mathcal{DB}_1^{**}$ be a continuous function such that $h_k([0,1]) = \mathcal{R}_k$ $(h_k(0) = k)$. By the continuity of h_k there exists a positive number α such that $h_k([0,\alpha]) \subset B(k,\delta)$.

Consider the following cases:

1) $h_k([0,\alpha]) \neq \{k\}$. In this case we put $\alpha_0 = \alpha$.

2) $h_k([0,\alpha]) = \{k\}$.

Let $\alpha_1 = \sup\{\alpha' : h_k([0,\alpha']) = \{k\}\}$. By the continuity of h_k, $h_k(\alpha_1) = k$ and there exists $\alpha_0 \in [\alpha_1, 1]$ such that $h_k([0,\alpha_0]) = h_k([\alpha_1,\alpha_0]) \subset B(k,\delta)$. Observe that

$$h_k([0,\alpha_0]) \neq \{k\}. \tag{4}$$

In the both cases there exists a road

$$\mathcal{R}'_k = h_k([0,\alpha_0]) \subset B(k,\delta) \cap \mathcal{R}_k, \text{ such that } \mathcal{R}'_k \setminus \{k\} \neq \emptyset.$$

Now, we shall show that

$$F_1(\mathcal{R}'_k) = \mathbb{R}. \tag{5}$$

Indeed. Let $z \in \mathbb{R}$ and let $\xi_1 \in \mathcal{R}'_k \setminus \{k\} \subset \mathcal{DB}_1^{**} \setminus \mathcal{C}$. Let us assume that $s_1 = \varrho(\xi_1, \mathcal{C})$. Then there exists a real number $s_0 \in (0, s_1)$ such that $\sin\left(\frac{1}{s_0}\right) = z \cdot s_0$. Let us denote by $\varrho^*(\varphi) = \varrho(\varphi, \mathcal{C})$, for any $\varphi \in \mathcal{DB}_1^{**}$. So the set $\varrho^*(\mathcal{R}'_k)$ is connected (as a continuous image of the connected set $\mathcal{R}'_k$), $0 \in \varrho^*(\mathcal{R}'_k)$ (because $k \in \mathcal{R}'_k$) and $s_1 \in \varrho^*(\mathcal{R}'_k)$. Consequently, $s_0 \in \varrho^*(\mathcal{R}'_k)$. Thus there exists $\xi_0 \in \mathcal{R}'_k$ such that $s_0 = \varrho^*(\xi_0) = \varrho(\xi_0, \mathcal{C})$ (of course $\xi_0 \notin \mathcal{C}$). Therefore $F_1(\xi_0) = \frac{1}{\varrho(\xi_0,\mathcal{C})} \cdot \sin \frac{1}{\varrho(\xi_0,\mathcal{C})} = \frac{1}{s_0} \sin \frac{1}{s_0} = z$, and the condition (5) is proved.

To finish the proof of the quasi-continuity of F_1 at k let us consider a number $c \in (F_1(k) - \varepsilon, F_1(k) + \varepsilon)$. From the condition (5) one can deduce that there exists $c' \in \mathcal{R}'_k$ such that $F_1(c') = c$ and $c' \in \mathcal{DB}_1^{**} \setminus \mathcal{C}$, so c' is a continuity point of F_1.

Now, we assume that $F : \mathcal{C} \to \mathbb{R}$ is a Darboux function. We shall prove that F_1 is a Darboux function, too. Let A be a connected set in the space $\mathcal{DB}_1^{**}$. If $A \subset \mathcal{C}$ then $F_1(A) = F(A)$ is a connected set. If $A \subset \mathcal{DB}_1^{**} \setminus \mathcal{C}$ then, from the continuity of F_1 on the set $\mathcal{DB}_1^{**} \setminus \mathcal{C}$, it follows that A is a connected set.

Finally, suppose that $A \cap \mathcal{C} \neq \emptyset \neq A \setminus \mathcal{C}$. Then there exists $g_0 \in A \setminus \mathcal{C}$. Let $\beta_0 = \varrho(g_0, \mathcal{C}) > 0$. We shall show that

$$\forall_{\beta \in (0,\beta_0]} \ \exists_{g \in A} \ \ \varrho(g, \mathcal{C}) = \beta. \tag{6}$$

Let $\beta \in (0, \beta_0)$ (for $\beta = \beta_0$ we have $g = g_0$).

It suffices to show that $A \cap C_\beta \neq \emptyset$, where $C_\beta = \varrho^{*-1}(\beta)$ (ϱ^* is defined as in the proof of the condition (5)).

Conversely, suppose that $A \cap C_\beta = \emptyset$. Assume

$$A_1 = \varrho^{*-1}([0,\beta)) \cap A \text{ and } A_2 = \varrho^{*-1}((\beta,+\infty)) \cap A.$$

One can easily verify that $A = A_1 \cup A_2$. Additionally, $A_1 \neq \emptyset$ because $A \cap \mathcal{C} \neq \emptyset$ and $A_2 \neq \emptyset$ because $g_0 \in A$. Moreover, $\overline{A_1} \subset \varrho^{*-1}([0,\beta])$ and $\overline{A_2} \subset \varrho^{*-1}([\beta,+\infty))$. Since A_1 and A_2 are separated sets, then A is a disconnected set. The obtained contradiction proves that $A \cap C_\beta \neq \emptyset$ and so the condition (6) is true. Hence, $F_1(A)$ is a connected set. To conclude the proof it suffices to observe that $D_{F_1} = \mathcal{C}$. □

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