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ON t-CONVEX FUNCTIONS*

Abstract

The main results of the paper, answering an open problem raised in [3], show that t-convexity can also be characterized in terms of a lower second-order generalized derivative. As a consequence, we obtain that t-convexity is also a localizable convexity property.

1 Introduction

A real-valued function $f : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is called *t-convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \text{ for } x, y \in I, \tag{1}$$

where t is a fixed element of the open unit interval $]0,1[$. If (1) holds with $t = 1/2$, then f is said to be *Jensen-convex* or *midpoint convex* (cf. [14]). Obviously, any convex function is t-convex, however there are nonconvex but t-convex functions. By a result of Kuhn [9], t-convexity always implies Jensen-convexity (cf. also [2] for a more elementary proof) but, for every irrational t, there exists a Jensen-convex but not t-convex function.

A related functional inequality is

$$f((1-t)x + ty) + f(tx + (1-t)y) \le f(x) + f(y) \text{ for } x, y \in I.$$

Functions satisfying the above inequality are called *t-Wright-convex* (see [16] for the origin of this notion). It is obvious that t-convex functions are also

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t-Wright-convex, however, depending on the algebraic character of t, t-Wright convexity can be equivalent and also non-equivalent to t-convexity. (See the paper [10] of Maksa, Nikodem, and Páles for further details.)

In a recent paper [3], Gilányi and Páles proved that t-Wright-convexity can be characterized in terms of a properly chosen generalized second-order derivative. Due to this characterization, it turns out that t-Wright-convexity is also localizable; i.e., a function is t-Wright-convex on I if and only if each point of I possesses a neighborhood such that the function restricted to this neighborhood is t-Wright-convex.

The main results of the paper, answering an open problems raised in [3], show that t-convexity can also be characterized in terms of a lower second-order generalized derivative. As a consequence, we obtain that t-convexity is also a localizable convexity property.

2 Second-Order Divided Differences

For an arbitrary function $f : I \to \mathbb{R}$ define the second-order divided difference of f at three pairwise distinct points x, y, z of I by

$$f[x,y,z] := \frac{f(x)}{(y-x)(z-x)} + \frac{f(y)}{(x-y)(z-y)} + \frac{f(z)}{(x-z)(y-z)}.$$

Obviously, the above expression is symmetric in x, y, z. The Mean Value Theorem of divided differences is recalled in the following Lemma (cf. [5], [6]).

Lemma 1. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I. Then, for all distinct elements x, y, z of I, there exists a point $\xi \in \operatorname{co}\{x,y,z\}$ such that $f[x,y,z] = \frac{f''(\xi)}{2}$.*

It is an immediate consequence of the above Lemma that if f is a second-degree polynomial of the form $f(x) = a + bx + cx^2$, then $f[x,y,z] = c$ for all pairwise distinct x, y, z in I.

The next result offers an identity called the *chain formula* for chains of divided differences of second-order. A generalization of this result for higher-order divided differences can be found in [8, Lemma XV.2.2, pp. 376–377.].

Lemma 2. (Chain Formula) *Let $x_0 < x_1 < \cdots < x_n$ $(n \geq 2)$ be arbitrary points in I. Then, for each fixed $0 < j < n$, there exist positive constants $\lambda_1, \ldots, \lambda_{n-1}$ with $\lambda_1 + \cdots + \lambda_{n-1} = 1$ such that*

$$\sum_{i=1}^{n-1} \lambda_i f[x_{i-1}, x_i, x_{i+1}] = f[x_0, x_j, x_n] \tag{2}$$

holds for all functions $f : I \to \mathbb{R}$. *Moreover,*

$$\lambda_i = \begin{cases} \dfrac{(x_{i+1} - x_{i-1})(x_i - x_0)}{(x_n - x_0)(x_j - x_0)} & \textit{if } 1 \le i < j, \\ \dfrac{x_{i+1} - x_{i-1}}{x_n - x_0} & \textit{if } i = j, \\ \dfrac{(x_{i+1} - x_{i-1})(x_n - x_i)}{(x_n - x_0)(x_n - x_j)} & \textit{if } j < i \le n - 1. \end{cases} \tag{3}$$

For the sake of completeness, we provide a simple proof for the above lemma which uses a completely different argument than that of [8, Lemma XV.2.2].

PROOF. First of all observe that if (2) holds for some function f and $f^* : I \to \mathbb{R}$ is a function satisfying $f(x_i) = f^*(x_i)$ for $i = 0, 1, \dots, n$, then (2) is also satisfied by f^* instead of f. To utilize this observation, we show that, for every function $f : I \to \mathbb{R}$, there exist constants $a, c_0, c_1, \dots, c_{n-1}$ such that the function $f^* : I \to \mathbb{R}$ defined by

$$f^*(x) := a + \sum_{i=0}^{n-1} c_i (x - x_i)^+ \tag{4}$$

satisfies

$$f(x_i) = f^*(x_i) \qquad (i = 0, 1, \dots, n), \tag{5}$$

where the *positive part* t^+ of a real number t is defined by $t^+ := \max(0, t)$. Indeed, we can easily see that (5) is equivalent to the following system of linear equations

$$\begin{aligned} f(x_0) &= a, \\ f(x_1) &= a + c_0(x_1 - x_0), \\ f(x_2) &= a + c_0(x_2 - x_0) + c_1(x_2 - x_1), \\ &\vdots \\ f(x_n) &= a + c_0(x_2 - x_0) + c_1(x_2 - x_1) + \dots + c_{n-1}(x_n - x_{n-1}), \end{aligned}$$

which can be solved recursively for $a, c_0, c_1, \dots, c_{n-1}$. Thus, due to the above observation, (2) is satisfied for all functions f if and only if it is valid for all functions f^* of the form (4). Since the identity (2) is linear in f, it is sufficient to show that (2) is valid for the functions

$$f^*_{-1}(x) := 1,\ f^*_0(x) := (x - x_0)^+,\ \dots,\ f^*_{n-1}(x) := (x - x_{n-1})^+.$$

Clearly, (2) holds with $f = f_{-1}^*$ and $f = f_0^*$ identically (because these functions are polynomials of degree at most one on the interval $[x_0, x_n]$ and hence their second-order divided differences equal 0). Observe also that f_k^* is an at most first degree polynomial on the interval $[x_{i-1}, x_{i+1}]$ if k is different from i. Therefore, substituting $f = f_k^*$ into (2), we obtain $\lambda_k f_k^*[x_{k-1}, x_k, x_{k+1}] = f_k^*[x_0, x_j, x_n]$ for $(k = 1, \ldots, n-1)$. Hence, with the choice $\lambda_k := \frac{f_k^*[x_0,x_j,x_n]}{f_k^*[x_{k-1},x_k,x_{k+1}]}$ for $(k = 1, \ldots, n-1)$, (2) holds for $f = f_k^*$ $(k = 1, \ldots, n-1)$. Thus it holds also for all functions of the form (4). Now a simple computation yields that $\lambda_1, \ldots, \lambda_k$ are of the form (3) and then the inequalities $\lambda_k > 0$ can be checked directly. Finally, substituting $f(x) = x^2$ into (2), it follows that $\lambda_1 + \cdots + \lambda_{n-1} = 1$ also holds. □

An obvious consequence of the previous lemma is the following result which we call the *chain inequality* in the sequel.

Corollary 1. (Chain Inequality) *Let $f : I \to \mathbb{R}$ and $x_0 < x_1 < \cdots < x_n$ $(n \geq 2)$ be arbitrary points in I. Then, for all fixed $0 < j < n$,*

$$\min_{1\leq i\leq n-1} f[x_{i-1}, x_i, x_{i+1}] \leq f[x_0, x_j, x_n] \leq \max_{1\leq i\leq n-1} f[x_{i-1}, x_i, x_{i+1}].$$

3 Convexity Triplets

It is easy to check that a function $f : I \to \mathbb{R}$ is convex if and only if $f[x, y, z] \geq 0$ for $(x, y, z) \in I^3$ with $x < y < z$. Motivated by this characterization of convexity, a triplet (x, y, z) in I^3 with $x < y < z$ is called a *convexity triplet for the function* $f : I \to \mathbb{R}$ if $f[x, y, z] \geq 0$ and the set of all convexity triplets of f is denoted by $\mathcal{C}(f)$. Using this terminology, f is t-convex if and only if

$$\big(x, tx + (1-t)y, y\big),\ \big(x, (1-t)x + ty, y\big) \in \mathcal{C}(f) \text{ for } x, y \in I \text{ with } x < y.$$

Applying the chain inequality established in Corollary 1, we can deduce the following *chain rule* for convexity triplets.

Corollary 2. (Chain Rule) *Let $f : I \to \mathbb{R}$ and $x_0 < x_1 < \cdots < x_n$ $(n \geq 2)$ be arbitrary points in I such that (x_{i-1}, x_i, x_{i+1}) is in $\mathcal{C}(f)$ for all $i = 1, \ldots, n-1$. Then*

$$(x_0, x_j, x_n) \in \mathcal{C}(f) \tag{6}$$

for all $0 < j < n$.

PROOF. We have that $f[x_{i-1}, x_i, x_{i+1}] \geq 0$ for all $i = 1, \ldots, n-1$. Therefore, by the chain inequality, $f[x_0, x_j, x_n] \geq 0$; i.e., (6) holds for all $j = 1, \ldots, n-1$. □

As applications of the above Corollary, we derive two well known results on the connection of t- and Jensen-convexity. The second statement of the next Corollary was proved by Kuhn [9] by using Rodé's theorem. An elementary proof for this fact was first found by Daróczy and Páles [2].

Corollary 3. *Let $f : I \to \mathbb{R}$. If f is Jensen-convex, then it is t-convex for all rational t in $]0, 1[$. Conversely, if f is t-convex for some $t \in]0, 1[$, then it is also Jensen-convex.*

PROOF. For the first statement, assume f is Jensen-convex and let $t = j/n$ where $0 < j < n$ are integers. Let x, y be fixed and assume that $y < x$. (The case $x < y$ can be treated similarly.) Define x_i by $x_i := \frac{i}{n}x + \frac{n-i}{n}y$ for $(i = 0, \ldots, n)$. Then $x_0 = y < x_1 < \cdots < x_n = x$, furthermore, $\frac{x_{i-1}+x_{i+1}}{2} = x_i$ for all $i = 1, \ldots, n-1$. Therefore, by the Jensen-convexity of f, we have that (x_{i-1}, x_i, x_{i+1}) belongs to $\mathcal{C}(f)$. Hence, by the chain rule for convexity triplets, we get that (6) holds. Thus $f[x_0, x_j, x_n] \geq 0$; i.e., $f[y, tx + (1-t)y, x] \geq 0$, which shows that f is t-convex, indeed.

To prove the converse, assume that f is t-convex for some $t \in]0, 1[$. To prove the Jensen-convexity of f, let $x, y \in I$ with $x < y$ be arbitrary. Define the points x_0, x_1, x_2, x_3, x_4 by

$$x_0 := x,\ x_1 := tx+(1-t)\frac{x+y}{2},\ x_2 := \frac{x+y}{2},\ x_3 := t\frac{x+y}{2}+(1-t)y,\ x_4 := y.$$

Then, for $i = 1$ and for $i = 3$, we obviously have $x_i = tx_{i-1} + (1-t)x_{i+1}$. Furthermore $x_2 = (1-t)x_1 + tx_3$. Hence, due to the t-convexity of f, $(x_{i-1}, x_i, x_{i+1}) \in \mathcal{C}(f)$ for $i = 1, 2, 3$. Thus, by the chain rule, we get that $(x_0, x_2, x_4) \in \mathcal{C}(f)$; i.e., f is Jensen-convex. $\square$

4 Main Results

Our main results offer mean value theorems in terms of the lower 2nd-order generalized derivatives defined by

$$\underline{\delta}^2 f(\xi) := \liminf_{\substack{(x,y)\to(\xi,\xi) \\ \xi,u\in\mathrm{co}\{x,y\}}} 2f[x, u, y] \text{ for } \xi \in I, \tag{7}$$

$$\underline{\delta}_t^2 f(\xi) := \liminf_{\substack{(x,y)\to(\xi,\xi) \\ \xi\in\mathrm{co}\{x,y\}}} 2f[x, tx + (1-t)y, y] \text{ for } \xi \in I, \tag{8}$$

where, in the second definition, $t \in]0, 1[$ is a fixed parameter. Clearly,

$$\underline{\delta}_t^2 f(\xi) \geq \underline{\delta}^2 f(\xi) \tag{9}$$

for all $\xi \in I$ and $t \in]0,1[$. One can also easily show that if f is twice continuously differentiable at ξ, then $\underline{\delta}_t^2 f(\xi) = \underline{\delta}^2 f(\xi) = f''(\xi)$.

Theorem 1. (Mean Value Inequality for t-convexity) *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, $t \in]0,1[$, and let $x, y \in I$ with $x \neq y$. Then there exists a point $\xi \in \mathrm{co}\{x, y\}$ such that*

$$\frac{tf(x) + (1-t)f(y) - f(tx + (1-t)y)}{t(1-t)(x-y)^2} = f[x, tx + (1-t)y, y] \geq \frac{\underline{\delta}_t^2 f(\xi)}{2}. \tag{10}$$

Proof. In the sequel, a triplet $(x, u, z) \in I^3$ will be called a *t-triplet* if either $u = tx + (1-t)y$ or $u = (1-t)x + ty$. Let x and y be distinct elements of I. Without loss of generality, we may assume that $x < y$. In what follows, we intend to construct a sequence of t-triplets (x_n, u_n, y_n) such that

$$x = x_0 \leq x_1 \leq x_2 \leq \dots, \; y = y_0 \geq y_1 \geq y_2 \geq \dots, \; x_n < u_n < y_n \; (n \in \mathbb{N}), \tag{11}$$

$$|y_n - x_n| \leq \big(\max(t, 1-t)\big)^n |y - x| \; (n \in \mathbb{N}), \tag{12}$$

and

$$f[x, u, y] = f[x_0, u_0, y_0] \geq f[x_1, u_1, y_1] \geq f[x_2, u_2, y_2] \geq \dots . \tag{13}$$

Define $(x_0, u_0, y_0) = (x, tx + (1-t)y, y)$ and assume that we have constructed (x_n, u_n, y_n). Now set

$$z_{n,0} := x_n, \; z_{n,1} := (1-t)x_n + tu_n, \; z_{n,2} := u_n, \; z_{n,3} := tu_n + (1-t)y_n, \; z_{n,4} := y_n.$$

Then, clearly, $(z_{n,0}, z_{n,1}, z_{n,2})$ and $(z_{n,2}, z_{n,3}, z_{n,4})$ are t-triplets. On the other hand, we have that $u_n = s_n x_n + (1-s_n)y_n$, where either $s_n = t$ or $s_n = 1-t$. Thus,

$$\begin{aligned} s_n z_{n,1} + (1-s_n) z_{n,3} &= s_n\big((1-t)x_n + t(s_n x_n + (1-s_n)y_n)\big) \\ &\quad + (1-s_n)\big(t(s_n x_n + (1-s_n)y_n) + (1-t)y_n\big) = u_n; \end{aligned}$$

that is, $(z_{n,1}, z_{n,2}, z_{n,3})$ is also a t-triplet.

Using the Chain Inequality, we get that there exists an index $i \in \{1,2,3\}$ such that $f[x_n, u_n, y_n] \geq f[z_{n,i-1}, z_{n,i}, z_{n,i+1}]$. Finally, let

$$(x_{n+1}, u_{n+1}, y_{n+1}) := (z_{n,i-1}, z_{n,i}, z_{n,i+1}).$$

The sequence so constructed clearly satisfies (11) and (13). We prove (12) by induction. It is obvious for $n = 0$. Assume that it holds for n. Then

$$\begin{aligned} |y_{n+1} - x_{n+1}| &\leq \max_{1 \leq i \leq 3} |z_{n,i+1} - z_{n,i-1}| = \max(1-s_n, 1-t, s_n)|y_n - x_n| \\ &= \max(t, 1-t)|y_n - x_n| \leq \big(\max(t, 1-t)\big)^{n+1} |y - x|. \end{aligned}$$

Thus (12) is also verified.

Due to the monotonicity properties of the sequences (x_n), (y_n) and also (12), there exists a unique element $\xi \in [x,y]$ such that $\bigcap_{i=0}^{\infty}[x_n, y_n] = \{\xi\}$. Then, by (13), we get that

$$f[x,u,y] \geq \liminf_{n\to\infty} f[x_n,u_n,y_n] \geq \liminf_{\substack{(v,w)\to(\xi,\xi)\\ \xi\in\mathrm{co}\{v,w\}}} f[v, tv+(1-t)w, w] = \frac{\underline{\delta}_t^2 f(\xi)}{2},$$

which completes the proof of the theorem. □

Corollary 4. (Mean Value Inequality for convexity) *Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and let $x,u,y \in I$ with $x < u < y$. Then there exists a point $\xi \in [x,y]$ such that $f[x,u,y] \geq \frac{\underline{\delta}^2 f(\xi)}{2}$.*

PROOF. Choose $t \in]0,1[$ so that $u = tx + (1-t)y$. Then, by Theorem 1, there exists $\xi \in [x,y]$ such that (10) holds. Therefore, by (9),

$$f[x,u,y] = f[x, tx+(1-t)y, y] \geq \frac{\underline{\delta}_t^2 f(\xi)}{2} \geq \frac{\underline{\delta}^2 f(\xi)}{2}. \qquad \square$$

If one replaces f by $-f$ in the above results, then mean value inequality for the upper 2nd-order generalized derivatives can be deduced that are defined via (7) and (8) with "limsup" instead of "liminf".

As an immediate consequence of the above theorem, we get the following characterization of convexity and t-convexity.

Corollary 5. *Let $t \in]0,1[$. A function $f : I \to \mathbb{R}$ is t-convex (resp. convex) on I if and only if $\underline{\delta}_t^2 f(\xi) \geq 0$ (resp. $\underline{\delta}^2 f(\xi) \geq 0$) for $\xi \in I$.*

PROOF. If f is t-convex, then, clearly $\underline{\delta}_t^2 f \geq 0$. Conversely, if $\underline{\delta}_t^2 f$ is nonnegative on I, then, by the previous Theorem $f[x, tx+(1-t)y, y] \geq 0$ for all $x,y \in I$; i.e., f is t-convex. A similar argument shows that the convexity of f is characterized by the nonnegativity of $\underline{\delta}^2 f$. □

Another obvious but interesting consequence of Corollary 5 is that the t-convexity property (and also convexity) is *localizable* in the following sense.

Corollary 6. *Let $t \in]0,1[$. A function $f : I \to \mathbb{R}$ is t-convex (resp. convex) on I if and only if, for each point $\xi \in I$, there exists a neighborhood U of ξ such that f is t-convex (resp. convex) on $I \cap U$.*

The localizability of convexity for upper semicontinuous functions was also proved in [11]. In the literature, there are some other definitions for local convexity and local Jensen-convexity. For instance, following Cardinali and Papalini [1], we call a function $f : I \to \mathbb{R}$ J^*-convex at a point $p \in I$ if there is a neighborhood U of p such that $f\left(\frac{x+p}{2}\right) \le \frac{f(x)+f(p)}{2}$ for $x \in U$. Another definition is motivated by Kostyrko [7]. We say that a function $f : I \to \mathbb{R}$ is locally Jensen-convex at a point $p \in I$ if there exists a positive number δ such that

$$f(p) \le \frac{f(p-h)+f(p+h)}{2} \text{ for } 0 < h < \delta.$$

Note, however, that neither J^*-convex functions, nor locally Jensen-convex functions in the sense of Kostyrko need not be Jensen-convex. For instance, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := |x|$ for $-1 \le x < 1$ and then extended periodically to $\mathbb{R}$ is J^*-convex but not Jensen-convex. The function $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) := x - [x]$ for noninteger x and $h(x) := 1/2$ for integer x is locally Jensen-convex in the sense of Kostyrko but it is not Jensen-convex (cf. [7]).

The following corollary derives t-convexity from a formally weaker property, namely from the local γ-th order approximate t-convexity. A function $f : I \to \mathbb{R}$ is called *approximately t-convex of order* γ on I (where $\gamma \ge 0$) if there exists a nonnegative constant c such that

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+ct(1-t)|x-y|^\gamma \tag{14}$$

for all $x, y \in I$. If the above inequality holds for all $t \in [0,1]$ with a constant c independent of t, then we say that f is an *approximately convex function of order* γ. If each point of I has a neighborhood such that f restricted to this neighborhood is approximately t-convex (resp. convex) of order γ, then we say that f is locally approximately t-convex (resp. convex) of order γ.

Approximately convex functions of first-order were introduced by Páles in [12]. First-order approximately Jensen-convex functions were investigated by Házy and Páles [4].

The next result shows that local approximate t-convexity (resp. local approximate convexity) of order higher than 2 is equivalent to t-convexity (resp. convexity). It is also related to a result of Rolewicz [15], stating that if a function f is γ-paraconvex; that is,

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+c|x-y|^\gamma$$

for $x, y \in I$, $t \in [0,1]$ and $\gamma > 2$, then it is convex.

Corollary 7. *Let $t \in]0,1[$. Assume that, for some $\gamma > 2$, $f : I \to \mathbb{R}$ is a locally approximately t-convex (resp. convex) function of order γ. Then f is t-convex (resp. convex).*

Proof. We prove the statement concerning t-convexity. Let $\xi \in I$ be arbitrary. By the assumption, there exists a neighborhood U of ξ and $c \geq 0$ such that (14) holds for all $x, y \in U \cap I$. Then

$$f[x, tx + (1-t)y, y] = \frac{tf(x) + (1-t)f(y) - f(tx + (1-t)y)}{t(1-t)(x-y)^2} \geq -c|x-y|^{\gamma-2}$$

for $x, y \in I$ with $x \neq y$. Thus, upon taking the liminf as $(x, y) \to (\xi, \xi)$, we get that

$$\underline{\delta}_t^2 f(\xi) = \liminf_{\substack{(x,y)\to(\xi,\xi) \\ \xi \in \mathrm{co}\{x,y\}}} 2f[x, tx + (1-t)y, y] \geq \liminf_{\substack{(x,y)\to(\xi,\xi) \\ \xi \in \mathrm{co}\{x,y\}}} -2c|x-y|^{\gamma-2} = 0.$$

Therefore, by Corollary 5, f is t-convex. □

We note that, in Corollary 7, the lower bound 2 for γ cannot be improved, because the function $f(x) = -x^2$ is obviously approximately convex of order 2 and not t-convex for any $t \in]0, 1[$.

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