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GÂTEAUX DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS VIA DIRECTIONAL DERIVATIVES

Abstract

Let X be a separable Banach space, Y a Banach space and $f : X \rightarrow Y$ a Lipschitz function. We show that the set of all Gâteaux non-differentiability points at which f has all one-sided or two-sided directional derivatives can be covered by (special subsets of) Lipschitz surfaces of codimension 1 or codimension 2, respectively. Further results indicate that these results are close to the best possible ones. Our results are new also for Lipschitz functions $\mathbb{R}^n \rightarrow \mathbb{R}$; for these functions Gâteaux differentiability is the classical (total) differentiability.

1 Introduction

In this paper we investigate sets of points x at which a Lipschitz function on a Banach space X is directionally differentiable but f is not Gâteaux differentiable at x . These and related sets were treated in a number of articles ([K], [LW], [Fa], [Zh1], [Zh2], [BC] and others; see below for more information). Our results are new also for Lipschitz functions $\mathbb{R}^n \rightarrow \mathbb{R}$. For these functions Gâteaux differentiability is the classical (total) differentiability. This simple result is well-known for (different) proofs see e.g. [EG] (p. 83, the proof of the Claim 3) or [NZ1].

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We will use the following terminology.

If X, Y are Banach spaces, $x, u \in X$ and f is a function from X to Y , we consider the one-sided directional derivative of f at x in direction u

$$f'_+(x, u) := \lim_{t \rightarrow 0_+} \frac{1}{t} (f(x + tu) - f(x))$$

and the (two-sided) directional derivative of f at x in direction u

$$f'(x, u) := \lim_{t \rightarrow 0} \frac{1}{t} (f(x + tu) - f(x)).$$

If X, Y are Banach spaces, $G \subset X$ is an open set and $f : G \rightarrow Y$ is a Lipschitz function, we will investigate the sets $S_+(f)$, $S(f)$ of “singular points” of f defined by

- (i) $S_+(f)$ is the set of all points $x \in G$ such that $f'_+(x, u)$ exists for each $u \in X$ but f is not Gâteaux differentiable at x .
- (ii) $S(f)$ is the set of all points $x \in G$ such that $f'(x, u)$ exists for each $u \in X$ but f is not Gâteaux differentiable at x .

In particular, we consider the following problems. (They are formulated for *real* functions, but our results hold for *vector* functions as well.)

Problem A.

- (i) Characterize the system $\mathcal{S}_+(X)$ of all sets which are of the form $S_+(f)$ for a Lipschitz real function f on a Banach space X .
- (ii) Characterize the system $\tilde{\mathcal{S}}_+(X)$ of all sets which are of the form $S_+(f)$, where f is a real Lipschitz function which has all one-sided directional derivatives $f'_+(x, u)$ at all points $x \in X$.

Problem B.

- (i) Characterize the system $\mathcal{S}(X)$ of all sets which are of the form $S(f)$ for a Lipschitz real function f on a Banach space X .
- (ii) Characterize the system $\tilde{\mathcal{S}}(X)$ of all sets which are of the form $S(f)$, where f is a real Lipschitz function which has all (two-sided) directional derivatives $f'(x, u)$ at all points $x \in X$.

Note that especially Problem A (ii) is quite natural and therefore we consider Problem A more thoroughly than Problem B. The point is that there exist important types of Lipschitz functions which have all *one-sided* derivatives at all points. For example, such are quasi-differentiable functions (cf. e.g. [DV]) which are used in optimization theory. Further, such functions arise as a supremum (or infimum) of special families of Lipschitz functions. For example ([Z2]), if X is a Banach space with a uniformly Gâteaux differentiable norm and $F \subset X$ is a closed set, then the distance function $x \rightarrow \text{dist}(x, F)$ has all one-sided derivatives at all points $x \in G := X \setminus F$. Results of [K], [Fa], [Zh1], [Zh2], [FZ], [FP], [G] imply that each set from $\mathcal{S}_+(X)$ is a first category set in all separable spaces and also in many non-separable spaces. Results of [BC], [PZ], [Z4], [Z5] even imply that each set from $\mathcal{S}_+(X)$ is σ -porous (in a strong sense) in all separable spaces and in some non-separable spaces. In the case of a separable Banach space we obtain here stronger results.

If X is a separable Banach space, we show that each set of type $S_+(f)$ can be covered by countably many Lipschitz surfaces of codimension 1 and each set of type $S(f)$ can be covered by countably many Lipschitz surfaces of codimension 2.

These results were already proved (for real-valued f) in [N]. Note that the result on $S_+(f)$ for distance functions in a separable Banach space with a uniformly Gâteaux differentiable norm is a special case of Theorem 4 from [Z2].

Here we obtain also considerably finer results (Theorem 3.3 and Theorem 3.6) which show that the sets of type $S_+(f)$ or $S(f)$ can be covered by countably many “special subsets” of Lipschitz surfaces.

As a consequence we obtain e.g. (cf. Proposition 5.7) that (in each separable space with $\dim X \geq 2$) there exists a Lipschitz surface of codimension 1 which cannot be covered by countably many sets of type $S_+(f)$. On the other hand, if A is a countable union of “unilaterally smooth” Lipschitz surfaces of codimension 1 in a separable Banach space X , then $A \in \tilde{\mathcal{S}}_+(X)$ (cf. Theorem 4.7 for a more precise result). If $\dim X > 2$, we do not know whether each set from $\tilde{\mathcal{S}}_+(X)$ can be covered by countably many of “unilaterally smooth” Lipschitz surfaces of codimension 1.

We give a complete solution to Problem A (ii) in the case $X = \mathbb{R}^2$. Namely, we obtain (Proposition 5.4) that $\tilde{\mathcal{S}}_+(\mathbb{R}^2)$ is the system of all F_σ sets which can be covered by countably many “unilaterally smooth Lipschitz curves”. (Note that $\tilde{\mathcal{S}}_+(\mathbb{R}) = \mathcal{S}_+(\mathbb{R})$ is the system of all countable subsets of \mathbb{R} , which is an easy and well-known result.)

We also completely solve Problem B (ii) in the case $X = \mathbb{R}^3$. Namely (Proposition 5.4) $\tilde{\mathcal{S}}(\mathbb{R}^3)$ is the system of all F_σ sets which can be covered

by countably many “smooth Lipschitz curves”. (Our results also show that $\tilde{\mathcal{S}}(\mathbb{R}^2) = \mathcal{S}(\mathbb{R}^2)$ is the system of all countable sets, which is an easy result.) For a further result concerning Problem B see Theorem 4.10.

We are not able to characterize the systems $\mathcal{S}_+(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^3)$, but we give a characterization of σ -ideals generated by these systems (see Proposition 5.5).

Our results easily imply that in each separable Banach space X there exists a “Lipschitz curve” which cannot be covered by countably many sets from the system $\tilde{\mathcal{S}}_+(X)$ (cf. Proposition 5.8). A further result (see Proposition 5.7) says that, in each separable Banach space with $\dim X \geq 2$, the σ -ideals generated by the systems $\mathcal{S}_+(X)$ and $\tilde{\mathcal{S}}_+(X)$ do not coincide.

Now we describe briefly the structure of the paper. In Section 2 we work with Lipschitz surfaces of finite codimension and their special subsets. This relatively technical but indispensable section is based on a standard application of Bouligand’s notion of the contingent (cone) of a set at a point.

In Section 3 we prove that sets of “singular points” of vector valued Lipschitz functions on separable Banach spaces can be covered by special subsets of Lipschitz surfaces (of codimension 1 or 2). Some lemmas (especially Lemma 3.5) in this section need relatively delicate arguments.

In Section 4 we present some constructions of Lipschitz functions, which have prescribed “singular points”. In the proof of the basic Lemma 4.1 we use the method of B. Kirchheim’s proof of Proposition 14 from [PZ].

Consequences of the main results are presented in the last section, Section 5.

We finish the introduction with definitions, notation and well-known facts which we will need.

If $f : X \rightarrow Y$ is a mapping and $F \subset X$, then we denote by $f|_F$ the restriction of f to F . We use the notation $B(x, r)$ for the open ball with center x and radius r .

Definition 1.1. Let X be a Banach space, $A \subset X$ and $x \in X$. Then we denote by $\text{contg}(A, x)$ (the contingent of A at x) the set of all $0 \neq u \in X$ for which there exist sequences $x \neq x_i \in A$ and $\lambda_i > 0$ such that $\lim \lambda_i = 0$ and $\lim \frac{1}{\lambda_i} \|x + \lambda_i u - x_i\| = 0$.

For some properties and applications of the contingent see [Sa]. The set $\text{contg}(A, x) \cup \{0\}$ is frequently called the (Bouligand) tangent cone of A at x and denoted by $\text{Tan}(A, x)$ (see [Fe]). Note that clearly $tu \in \text{contg}(A, x)$ whenever $t > 0$ and $u \in \text{contg}(A, x)$. It is also easy to see that $\text{contg}(A, x) = \text{contg}(\bar{A}, x)$ and that a is an accumulation point of A whenever $\text{contg}(A, x) \neq \emptyset$.

Definition 1.2. Let Banach spaces X, Y , an open set $G \subset X$, $x \in G$ and $f : G \rightarrow Y$ be given. Then we put

$$\text{Lip}(f) = \sup \left\{ \frac{\|f(u) - f(v)\|}{\|u - v\|} : u, v \in G, u \neq v \right\}$$

and

$$\text{Lip}(f, x) = \limsup_{u \rightarrow x} \frac{\|f(u) - f(x)\|}{\|u - x\|}.$$

Clearly $\text{Lip}(f, x) \leq \text{Lip}(f)$ and f is Lipschitz on G iff $\text{Lip}(f) < \infty$.

We will also need the following easy well-known facts.

Lemma 1.3. *Let X, Y, G, f be as in Definition 1.2. Then the following assertions hold.*

- (i) *If f is Lipschitz on G , $x \in G$ and $f'_+(x, u)$ exists for each $u \in X$, then the mapping $u \rightarrow f'_+(x, u)$ is Lipschitz.*
- (ii) *If $G = X$ and $\text{Lip}(f, x) \leq K$ for each $x \in X$, then $\text{Lip}(f) \leq K$.*

For (i) see e.g. [A], p. 164 or [Sp], p. 483. For (ii) see e.g. Lemma 14.4. of [BL].

2 Lipschitz Surfaces and Their Subsets

We start with natural definitions of “relative Hadamard” directional derivatives $h'_{A,+}(a, v)$ and $h'_A(a, v)$.

Definition 2.1. Let Y, W be Banach spaces, $A \subset Y$, $a \in A$, $v \in Y$, $d \in W$ and $h : A \rightarrow W$ be a mapping.

- (i) We write $h'_{A,+}(a, v) = d$ if $v \in \text{contg}(A, a)$ and $(1/\lambda_i)(h(x_i) - h(a)) \rightarrow d$ whenever $x_i \in A$, $\lambda_i > 0$, $\lambda_i \rightarrow 0$ and $(1/\lambda_i)\|a + \lambda_i v - x_i\| \rightarrow 0$.

We say that h is unilaterally smooth at a w.r.t. A if $h'_{A,+}(a, v)$ exists for each $v \in \text{contg}(A, a)$.

- (ii) We write $h'_A(a, v) = d$ if $v \in \text{contg}(A, a) \cup (-\text{contg}(A, a))$ and $(1/\lambda_i)(h(x_i) - h(a)) \rightarrow d$ whenever $x_i \in A$, $\lambda_i \neq 0$, $\lambda_i \rightarrow 0$ and $(1/\lambda_i)\|a + \lambda_i v - x_i\| \rightarrow 0$.

We say that h is bilaterally smooth at a w.r.t. A if $h'_A(a, v)$ exists for each $v \in \text{contg}(A, a) \cup (-\text{contg}(A, a))$.

- (iii) If $A = Y$, we say that h is unilaterally (or bilaterally) smooth at a if it is unilaterally (or bilaterally) smooth at a w.r.t. $A = Y$.

We say that h is unilaterally (or bilaterally) smooth if it is unilaterally (or bilaterally) smooth at each point of $A = Y$.

Remark 2.2. (i) It is easy to see that $h'_A(a, v) = d$ iff the following conditions hold:

- (a) $v \in \text{contg}(A, a) \cup (-\text{contg}(A, a))$.
 - (b) If $v \in \text{contg}(A, a)$, then $h'_{A,+}(a, v) = d$.
 - (c) If $v \in -\text{contg}(A, a)$, then $h'_{A,+}(a, -v) = -d$.
- (ii) Suppose $a \in \text{int}A$. Remark that $h'_{A,+}(a, v)$ means in this case the (one-sided) Hadamard derivative (cf. [Sp], p. 480). Then clearly

$$h'_{A,+}(a, v) = d \Rightarrow h'_+(a, v) = d, \quad h'_A(a, v) = d \Rightarrow h'(a, v) = d.$$

Easy examples show that the opposite implications do not hold in general. On the other hand, it is easy to see that the opposite implications hold if h is Lipschitz (cf. [Sp], Proposition 3.5).

- (iii) If $Y = \mathbb{R}$ and $v = 1$, then $v \in \text{contg}(A, a)$ iff a is a right accumulation point of A . In this case $h'_{A,+}(a, v)$ clearly equals to $h'_{A,+}(a)$ – the right derivative of h at a w.r.t. A .

In the sequel we will need the following natural notion of a Lipschitz surface of codimension n (cf. e.g. [Z7]).

Moreover, we define classes $\mathcal{A}_n^+, \mathcal{A}_n, \mathcal{B}_n^+, \mathcal{B}_n$ of “special subsets” of Lipschitz surfaces of codimension n , which we will need in the cases $n = 1, 2$ only.

Definition 2.3. Let X be a Banach space and $0 < n \leq \dim X$ be an integer.

- (i) We say that $S \subset X$ is a Lipschitz surface of codimension n if there exists a closed linear subspace Y of X of codimension n , an n -dimensional topological complement W of Y and a Lipschitz mapping $h : Y \rightarrow W$ such that $S = \{y + h(y) : y \in Y\}$. Then we say that S is a Lipschitz surface associated with W .

If Y, W, h can be chosen in such a way that $h'_+(y, v)$ (or $h'(y, v)$) exists for each $y, v \in Y$, we say that S is a unilaterally smooth (or a bilaterally smooth) Lipschitz surface associated with W .

If $\dim(Y) = 1$, we will speak about Lipschitz curves, unilaterally smooth Lipschitz curves and smooth Lipschitz curves.

- (ii) Let $Z \subset X$ and there exist Y, W, h, S as above such that $Z \subset S$. Let π_Y be the projection of X on Y parallel to W . We say that
 - (a) $Z \in \mathcal{A}_n^+$ if h is unilaterally smooth w.r.t. $\pi_Y(Z)$ at each point $a \in \pi_Y(Z)$.
 - (b) $Z \in \mathcal{A}_n$ if h is bilaterally smooth w.r.t. $\pi_Y(Z)$ at each point $a \in \pi_Y(Z)$.
 - (c) $Z \in \mathcal{B}_n^+$ if h has all one-sided directional derivatives $h'_+(a, y)$, $y \in Y$, at each point $a \in \pi_Y(Z)$.
 - (d) $Z \in \mathcal{B}_n$ if h has all (two-sided) directional derivatives $h'(a, y)$, $y \in Y$, at each point $a \in \pi_Y(Z)$.

Remark 2.4. (i) We admit also the trivial case $n = \dim X$. In this case Lipschitz surfaces are singletons and $\mathcal{A}_n^+ = \mathcal{A}_n = \mathcal{B}_n^+ = \mathcal{B}_n$ is the system containing all singletons and \emptyset .

- (ii) Using Remark 2.2 (ii), we immediately obtain

$$\mathcal{B}_n^+ \subset \mathcal{A}_n^+, \mathcal{B}_n \subset \mathcal{A}_n, \mathcal{A}_n \subset \mathcal{A}_n^+, \mathcal{B}_n \subset \mathcal{B}_n^+.$$

- (iii) A Lipschitz surface of codimension n in X is clearly closed.

Let S, Y, W be as in Definition 2.3 (i) and π_Y be the projection of X on Y parallel to W . It is easy to see that $\pi_Y|_S: S \rightarrow Y$ is a bilipschitz bijection, in particular $\pi_Y(A)$ is closed iff $A \subset S$ is closed.

- (iv) We will also write $\mathcal{A}_n^+(X), \mathcal{A}_n(X), \dots$ instead of $\mathcal{A}_n^+, \mathcal{A}_n, \dots$ when it is not clear in which space X we are working.

Now we will show how extension theorems give some non-trivial relations among defined classes in the cases $\dim X = 2$ and $\dim X = 3$. We will use the terminology of Definition 2.1; cf. also Remark 2.2, (iii).

The first extension result is essentially contained in [J].

Lemma 2.5. *Let $A \subset \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ be a Lipschitz function which is unilaterally smooth w.r.t A at each point of A . Then there exists a Lipschitz function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F|_A = f$, $\text{Lip}(F) = \text{Lip}(f)$ and $F'_+(x), F'_-(x)$ exist for all $x \in A \cup (\mathbb{R} \setminus \overline{A})$.*

PROOF. Clearly there exists a unique function f^* defined on \overline{A} such that $\text{Lip}(f^*) = \text{Lip}(f)$ and $f^*|_A = f$. It is easy to see that f^* is unilaterally smooth w.r.t. \overline{A} at each point of A . Let $\{I_s = (a_s, b_s) : s \in S\}$ be the system of all components of $\mathbb{R} \setminus \overline{A}$. Let F be the (clearly unique) extension of f^* with

the following properties: If I_s is bounded, then F is linear on $\overline{I_s}$. If I_s is unbounded, then F is constant on $\overline{I_s}$. It is easy to verify $\text{Lip}(F) = \text{Lip}(f)$. To prove that $F'_+(x)$ and $F'_-(x)$ exist for all $x \in A$ we repeat the argument of Jarník [J]: It is sufficient to observe that if y belongs to a bounded (a_s, b_s) with $x \neq a_s, x \neq b_s$, then $\frac{F(y)-F(x)}{y-x}$ is between $\frac{f^*(a_s)-f(x)}{a_s-x}$ and $\frac{f^*(b_s)-f(x)}{b_s-x}$. \square

The second extension theorem we need follows immediately from [M]. Note that we cannot use results from [J], since Jarník in [J] extends functions from perfect sets only and does not discuss the ‘‘Lipschitz case’’.

Lemma 2.6. *Let $C \subset \mathbb{R}$ be closed and $h : C \rightarrow \mathbb{R}$ be a Lipschitz function which is bilaterally smooth w.r.t C at each point $a \in C$. Then there exists a Lipschitz function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g|_C = h$ and $g'(x)$ exists for all $x \in \mathbb{R}$.*

PROOF. Set $S := C$, $F := h$, $A := -\text{Lip}(h) - 1$ and $B := \text{Lip}(h) + 1$. It is easy to see that then we can use Theorem 8 of [M] to obtain a function g with the desired properties. \square

We will also need the following extension result, which can be proved by modifications of constructions from [J] or [M]. It is a special case of the main result of [NZ2] where we use the ideas of [J].

Lemma 2.7. *Let $A \subset \mathbb{R}$ be an arbitrary set and $h : A \rightarrow \mathbb{R}$ be a Lipschitz function which is bilaterally smooth w.r.t. A at each point $a \in A$. Then there exists a Lipschitz function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $g|_A = h$ and $g'(x)$ exists for all $x \in A$.*

Lemma 2.8. *Let $X = \mathbb{R}^2$. Then $\mathcal{A}_1^+(X) = \mathcal{B}_1^+(X)$. Moreover, each closed set from $\mathcal{A}_1^+(X)$ is contained in an unilaterally smooth Lipschitz curve.*

PROOF. Let $Z \in \mathcal{A}_1^+(X)$. By Definition 2.3 there exist two one-dimensional subspaces Y, W of X with $X = Y \oplus W$ and a Lipschitz function $g : Y \rightarrow W$ such that $Z \subset S := \{y + g(y) : y \in Y\}$ and g is unilaterally smooth w.r.t. $P(Z)$ at each point $a \in P(Z)$, where P is the projection of X on Y parallel to W . Since both Y and W are (linearly) isometric to \mathbb{R} , Lemma 2.5 implies that there exists a Lipschitz function $g^* : Y \rightarrow W$ such that $g|_{P(Z)} = g^*|_{P(Z)}$ and g^* is unilaterally smooth at all points of $P(Z) \cup (Y \setminus \overline{P(Z)})$. If Z is closed, then $P(Z)$ is closed (cf. Remark 2.4 (iii)) and therefore g^* is unilaterally smooth. Thus the assertion of the lemma follows immediately. \square

Lemma 2.9. *Let $X = \mathbb{R}^3$. Then $\mathcal{A}_2(X) = \mathcal{B}_2(X)$. Moreover, if $Z \in \mathcal{A}_2(X)$ is closed, then Z is contained in a smooth Lipschitz curve.*

PROOF. Let $Z \in \mathcal{A}_2(X)$. By Definition 2.3 there exist a closed one-dimensional subspace Y of X , a 2-dimensional topological complement W to Y and a Lipschitz function $g : Y \rightarrow W$ such that $Z \subset S := \{y + g(y) : y \in Y\}$ and g is bilaterally smooth w.r.t. $P(Z)$ at each point $a \in P(Z)$, where P is the projection of X on Y parallel to W . Let $W = \text{span}\{w_1, w_2\}$ and $g_1, g_2 : Y \rightarrow \mathbb{R}$ be the functions for which $g(y) = g_1(y)w_1 + g_2(y)w_2$. Since Y is (linearly) isometric to \mathbb{R} and g_1, g_2 are clearly bilaterally smooth w.r.t. $P(Z)$ at each point $a \in P(Z)$, by Lemma 2.7 there are Lipschitz extensions $g_1^* : Y \rightarrow \mathbb{R}, g_2^* : Y \rightarrow \mathbb{R}$ of $g_1|_{P(Z)}, g_2|_{P(Z)}$ such that $(g_1^*)'(x), (g_2^*)'(x)$ exist for all $x \in P(Z)$. Moreover, if Z is closed, Lemma 2.6 gives that we can even obtain that $(g_1^*)'(x), (g_2^*)'(x)$ exist for all $x \in Y$. If we now put $g^*(y) := g_1^*(y)w_1 + g_2^*(y)w_2$ and consider the Lipschitz curve $S^* := \{y + g^*(y) : y \in Y\}$, the inclusion $Z \subset S^*$ proves the lemma. \square

We do not know whether each closed set from \mathcal{A}_1^+ belongs to \mathcal{B}_1^+ , if $\dim X > 2$. We also do not know whether each closed set from \mathcal{A}_2 belongs to \mathcal{B}_2 , if $\dim X > 3$. The following lemma is well-known in Euclidean spaces (cf. [Sa]) and was proved and used in [Z1] in the case $\dim W = 1$.

Lemma 2.10. *Let X be a Banach space and W a finite-dimensional subspace of X with $\dim W \geq 1$. Let $A \subset X$ and let $\text{contg}(A, a) \cap W = \emptyset$ for each $a \in A$. Then A can be covered by countably many Lipschitz surfaces associated with W .*

PROOF. Let V be a topological complement of W . Let P_w and P_v be projections of X onto W and V parallel to W and V , respectively. Then for each $x \in A$ there exists $n \in \mathbb{N}$ such that

$$\left(y \in A, \|P_w(y - x)\| \leq \frac{1}{n} \right) \Rightarrow \|P_v(y - x)\| \geq \frac{1}{n} \|P_w(y - x)\|. \quad (2.1)$$

Assume for the moment the contrary. Then there exist $x \in A$ and a sequence $y_n \in A$ such that

$$\|P_w(y_n - x)\| \leq \frac{1}{n} \text{ and } \|P_v(y_n - x)\| < \frac{1}{n} \|P_w(y_n - x)\|. \quad (2.2)$$

Put $z_n := y_n - x$. By (2.2) we have $P_w(z_n) \neq 0$ and $P_w(z_n) \rightarrow 0$. Since the unit sphere of W is compact, we can find $u \in W, \|u\| = 1$, and a subsequence $\{P_w(z_{n_k})\}$ of $\{P_w(z_n)\}$ such that $\|P_w(z_{n_k})\|^{-1} P_w(z_{n_k}) \rightarrow u$. Putting $t_k := \|P_w z_{n_k}\|$, we obtain

$$t_k \rightarrow 0 \text{ and } \frac{1}{t_k} \|t_k u - P_w(z_{n_k})\| \rightarrow 0. \quad (2.3)$$

Since $z_{n_k} = P_w(z_{n_k}) + P_v(z_{n_k})$ and $t_k = \|P_w(z_{n_k})\|$, (2.2) and (2.3) give

$$\frac{1}{t_k} \|t_k u - z_{n_k}\| \leq \frac{1}{t_k} \|t_k u - P_w(z_{n_k})\| + \frac{1}{t_k} \|P_v(z_{n_k})\| \rightarrow 0.$$

Thus $u \in \text{contg}(M, x)$ which is a contradiction.

Let A_n be the set of all points $x \in A$, for which (2.1) holds. We have proved $A = \bigcup_{n=1}^\infty A_n$. Fix $n \in \mathbb{N}$. Since $\dim W < \infty$ we can clearly write $W = \bigcup_{m=1}^\infty B_m$ where $\text{diam } B_m < \frac{1}{n}$ for each $m \in \mathbb{N}$. Putting

$$A_{n,m} = \{x \in A_n : P_w(x) \in B_m\},$$

we obtain $A = \bigcup_{n,m=1}^\infty A_{n,m}$. If $x, y \in A_{n,m}$, then we have $\|P_w(y-x)\| < \frac{1}{n}$ and consequently, by the definition of A_n , $\|P_w(y-x)\| \leq n\|P_v(y-x)\|$. Therefore the set $\{(P_v(x), P_w(x)) : x \in A_{n,m}\}$ is a graph of a Lipschitz function $f_{n,m} : P_v(A_{n,m}) \rightarrow W$. There exists a Lipschitz function $\tilde{f}_{n,m} : V \rightarrow W$ which extends $f_{n,m}$; it easily follows from the well-known results (see for instance [Sn]) on extensions of real-valued functions. Thus $A_{n,m}$ is a subset of a Lipschitz surface associated with W , which finishes the proof. \square

Lemma 2.11. *Let X be a Banach space, Y be a closed linear subspace of X of codimension n , W be a topological complement of Y in X and $h : Y \rightarrow W$ be a Lipschitz mapping. Let $S = \{y + h(y) : y \in Y\}$ and $Z \subset S$ be given. Then the following statements hold.*

- (i) *If $(\text{contg}(Z, z) - \text{contg}(Z, z)) \cap W \subset \{0\}$ for each $z \in Z$, then $Z \in \mathcal{A}_n^+$.*
- (ii) *If $(\text{contg}(Z, z) - \text{contg}(Z, z)) \cap W \subset \{0\}$ and $(\text{contg}(Z, z) + \text{contg}(Z, z)) \cap W \subset \{0\}$ for each $z \in Z$, then $Z \in \mathcal{A}_n$.*

PROOF. Denote by π_Y the projection of X on Y parallel to W and put $A := \pi_Y(Z)$. To prove (i), suppose that $Z \notin \mathcal{A}_n^+$. Then there exist $a \in A, a = \pi_Y(z)$, and $v \in Y$ such that $v \in \text{contg}(A, a)$ and $h'_{A,+}(a, v)$ does not exist. Consequently there exist sequences $x_i \in A, \lambda_i > 0$ such that

$$\lambda_i \rightarrow 0 \text{ and } (1/\lambda_i)\|a + \lambda_i v - x_i\| \rightarrow 0 \tag{2.4}$$

but $\lim_{i \rightarrow \infty} (1/\lambda_i)(h(x_i) - h(a))$ does not exist. Since

$$q_i := \left\| \frac{h(x_i) - h(a)}{\lambda_i} \right\| \leq \text{Lip}(h) \frac{\|x_i - a\|}{\lambda_i} \leq \text{Lip}(h) \frac{\|\lambda_i v\| + \|a + \lambda_i v - x_i\|}{\lambda_i},$$

(2.4) gives that the sequence (q_i) is bounded. Since W is finite-dimensional, we can find increasing sequences of natural numbers $i(k), i^*(k)$ and elements $d \neq d^*$ of W such that

$$\lim_{k \rightarrow \infty} \frac{h(x_{i(k)}) - h(a)}{\lambda_{i(k)}} = d, \quad \lim_{k \rightarrow \infty} \frac{h(x_{i^*(k)}) - h(a)}{\lambda_{i^*(k)}} = d^*.$$

By the definition of A we have

$$z_k := x_{i(k)} + h(x_{i(k)}) \in Z, \quad z_k^* := x_{i^*(k)} + h(x_{i^*(k)}) \in Z.$$

Since clearly $z_k \rightarrow z$ and

$$\begin{aligned} \frac{1}{\lambda_{i(k)}} \|z + \lambda_{i(k)}(v + d) - z_k\| &\leq \frac{1}{\lambda_{i(k)}} \|a - x_{i(k)} - \lambda_{i(k)}v\| \\ &\quad + \frac{1}{\lambda_{i(k)}} \|h(a) - h(x_{i(k)}) - \lambda_{i(k)}d\| \rightarrow 0, \end{aligned}$$

we obtain $v + d \in \text{contg}(Z, z)$. Quite analogously we obtain $v + d^* \in \text{contg}(Z, z)$. Consequently $0 \neq d - d^* \in (\text{contg}(Z, z) - \text{contg}(Z, z)) \cap W$ which is a contradiction.

To prove (ii), we suppose that $Z \notin \mathcal{A}_n$ and obtain $a \in A, a = \pi_Y(z)$, and $v \in Y$ such that $v \in \text{contg}(A, a) \cup (-\text{contg}(A, a))$ and $h'_A(a, v)$ does not exist. Further we proceed quite similarly to the proof of (i). We obtain $x_i \in A$ and λ_i as above, with the only difference that now $\lambda_i \neq 0$ (instead of $\lambda_i > 0$). Passing to the subsequences $\lambda_{i(k)}, \lambda_{i^*(k)}$, we can clearly suppose that their signs are constant, but possibly different. If $\text{sign}(\lambda_{i(k)}) = \text{sign}(\lambda_{i^*(k)}) = 1$, we obtain a contradiction in the same way as above. If $\text{sign}(\lambda_{i(k)}) = \text{sign}(\lambda_{i^*(k)}) = -1$, then we easily obtain $-v - d \in \text{contg}(Z, z), -v - d^* \in \text{contg}(Z, z)$,

$$0 \neq d - d^* \in (\text{contg}(Z, z) - \text{contg}(Z, z)) \cap W,$$

which is a contradiction. If $\text{sign}(\lambda_{i(k)}) = 1$ and $\text{sign}(\lambda_{i^*(k)}) = -1$, then $v + d \in \text{contg}(Z, z), -v - d^* \in \text{contg}(Z, z)$,

$$0 \neq d - d^* \in (\text{contg}(Z, z) + \text{contg}(Z, z)) \cap W,$$

which is a contradiction. □

Remark 2.12. It is not difficult to prove that the opposite implications in (i) and (ii) also hold.

3 Properties of Sets of Singular Points

In the following we suppose that X, Y are Banach spaces, $G \subset X$ is an open set and $f : G \rightarrow Y$ is a Lipschitz mapping.

Lemma 3.1. *Let $A \subset G$, $x_0 \in G$, $u, c \in X$, $U \in Y$, $\delta > 0$, $c \in \text{contg}(A, x_0)$ and $\varepsilon > 0$ be given. Then the following assertions hold.*

- (i) *If the inequality $\| \frac{f(x+tu)-f(x)}{t} - U \| \leq \varepsilon$ holds for each point $x \in A$ and $0 < t < \delta$, then*

$$\|f'_+(x_0, c + \lambda u) - f'_+(x_0, c) - \lambda U\| \leq \lambda \varepsilon$$

whenever $\lambda > 0$ and the one-sided directional derivatives $f'_+(x_0, c)$, $f'_+(x_0, c + \lambda u)$ exist.

- (ii) *If the inequality $\| \frac{f(x+tu)-f(x)}{t} - U \| < \varepsilon$ holds for each $x \in A$ and $0 < |t| < \delta$, then*

$$\|f'_+(x_0, c + \lambda u) - f'_+(x_0, c) - \lambda U\| \leq |\lambda| \varepsilon$$

whenever $\lambda \in \mathbb{R}$ and the one-sided directional derivatives $f'_+(x_0, c)$, $f'_+(x_0, c + \lambda u)$ exist.

PROOF. By the definition of the contingent, we can choose a sequence of numbers $t_n > 0, t_n \rightarrow 0$ and a sequence of points $x_n \in A$ such that

$$d_n := \|x_n - (x_0 + t_n c)\| = o(t_n), \quad n \rightarrow \infty.$$

Suppose that the assumptions of (i) are satisfied and a $\lambda > 0$ is given. We have

$$\begin{aligned} V_n &:= \left\| \frac{f(x_0 + t_n(c + \lambda u)) - f(x_0)}{t_n} - \frac{f(x_0 + t_n c) - f(x_0)}{t_n} - \lambda U \right\| \\ &\leq \left\| \lambda \left(\frac{f(x_n + t_n \lambda u) - f(x_n)}{\lambda t_n} - U \right) \right\| \\ &\quad + \left\| \frac{f(x_0 + t_n(c + \lambda u)) - f(x_n + t_n \lambda u)}{t_n} \right\| \\ &\quad + \left\| \frac{f(x_0 + t_n c) - f(x_n)}{t_n} \right\| := \alpha_n + \beta_n + \gamma_n. \end{aligned}$$

By our assumptions $\alpha_n \leq \lambda \varepsilon$ for all sufficiently large n . Since clearly $\beta_n \leq \text{Lip}(f) \frac{d_n}{t_n}$ and $\gamma_n \leq \text{Lip}(f) \frac{d_n}{t_n}$, we obtain

$$\begin{aligned} \|f'_+(x_0, c + \lambda u) - f'_+(x_0, c) - \lambda U\| &= \lim_{n \rightarrow \infty} V_n \\ &\leq \lambda \varepsilon + 0 + 0 = \lambda \varepsilon, \end{aligned}$$

which proves (i). To prove (ii), consider a negative λ . Put $\tilde{\lambda} := -\lambda, \tilde{u} := -u, \tilde{U} := -U$. Since $\|\frac{f(x+t\tilde{u})-f(x)}{t} - \tilde{U}\| < \varepsilon$ for $0 < t < \delta$, by (i) we obtain

$$\begin{aligned} \|f'_+(x_0, c + \lambda u) - f'_+(x_0, c) - \lambda U\| &= \|f'_+(x_0, c + \tilde{\lambda}\tilde{u}) - f'_+(x_0, c) - \tilde{\lambda}\tilde{U}\| \\ &\leq \tilde{\lambda}\varepsilon = |\lambda|\varepsilon. \end{aligned} \quad \square$$

Lemma 3.2. *Let $M \subset G, x_0 \in G, u, v, w, s_1, s_2 \in X, \varepsilon, \delta > 0, U, V, W \in Y$ be given such that $w = u + v, \|W - (U + V)\| > 3\varepsilon, s_1, s_2 \in \text{contg}(M, x_0)$ and $s_1 - s_2 \in \text{span}\{u\}$. Further let $f'_+(x_0, q)$ exist for each $q \in X$ and*

$$\begin{aligned} \left| \frac{f(x + tu) - f(x)}{t} - U \right| &\leq \varepsilon, \quad \left| \frac{f(x + tv) - f(x)}{t} - V \right| \leq \varepsilon, \\ \left| \frac{f(x + tw) - f(x)}{t} - W \right| &\leq \varepsilon \end{aligned}$$

whenever $x \in M$ and $0 < t < \delta$. Then $s_1 = s_2$.

PROOF. Suppose on the contrary that $s_1 \neq s_2$. Then we can suppose without any loss of generality that $s_1 - s_2 = \alpha u$, where $\alpha > 0$. By Lemma 3.1 (i) there exist vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in Y, \|\varepsilon_i\| \leq \varepsilon, i = 1, 2, 3$, such that

$$\begin{aligned} f'_+(x_0, s_1 + \alpha v) &= f'_+(x_0, s_1) + \alpha V + \alpha\varepsilon_1, \\ f'_+(x_0, s_2 + \alpha w) &= f'_+(x_0, s_2) + \alpha W + \alpha\varepsilon_2, \\ f'_+(x_0, s_1) &= f'_+(x_0, s_2 + \alpha u) = f'_+(x_0, s_2) + \alpha U + \alpha\varepsilon_3. \end{aligned}$$

Since $s_1 + \alpha v = s_2 + \alpha w$, the above equalities imply

$$\begin{aligned} f'_+(x_0, s_1) + \alpha V + \alpha\varepsilon_1 &= f'_+(x_0, s_2) + \alpha W + \alpha\varepsilon_2, \\ f'_+(x_0, s_1) &= f'_+(x_0, s_2) + \alpha U + \alpha\varepsilon_3. \end{aligned}$$

Subtracting these equalities, we obtain

$$\begin{aligned} \alpha V + \alpha\varepsilon_1 &= \alpha W + \alpha\varepsilon_2 - \alpha U - \alpha\varepsilon_3, \\ W - (U + V) &= \varepsilon_1 + \varepsilon_3 - \varepsilon_2, \\ \|W - (U + V)\| &\leq 3\varepsilon, \end{aligned}$$

which is a contradiction. □

Theorem 3.3. *Let X be a separable Banach space, $G \subset X$ an open set, Y a Banach space and $f : G \rightarrow Y$ a Lipschitz mapping. Let $S_+(f)$ be the set of all points $x \in G$ at which there exist all one-sided directional derivatives*

$f'_+(x, v)$, $v \in X$, but f is not Gâteaux differentiable at x . Then $S_+(f)$ can be covered by countably many Lipschitz surfaces of codimension 1. More precisely, $S_+(f)$ is a countable union of sets from the class \mathcal{A}_1^+ .

If, moreover, f has all one-sided directional derivatives at all points of G , then $S_+(f)$ is a countable union of closed sets from the class \mathcal{A}_1^+ .

PROOF. Since the closed linear span of the range $f(G)$ is clearly separable, we may suppose without any loss of generality that Y is separable. Let D_X and D_Y be countable dense subsets of X and Y , respectively, and let Q^+ be the set of all positive rational numbers. Put

$$P = \{(u, v, w, U, V, W, \varepsilon, \delta) : u \in D_X, v \in D_X, w = u + v, U \in D_Y,$$

$$V \in D_Y, W \in D_Y, \varepsilon \in Q^+, \delta \in Q^+, \|W - (U + V)\| > 3\varepsilon\}.$$

Note that P is countable. For each $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$ denote by $M(p)$ the set of all $x \in G$ for which $\text{dist}(x, X \setminus G) \geq \delta$ and

$$\begin{aligned} \left\| \frac{f(x + tu) - f(x)}{t} - U \right\| &\leq \varepsilon, \quad \left\| \frac{f(x + tv) - f(x)}{t} - V \right\| \leq \varepsilon, \\ \left\| \frac{f(x + tw) - f(x)}{t} - W \right\| &\leq \varepsilon \text{ for each } 0 < t < \delta. \end{aligned} \quad (3.1)$$

It is easy to see that each $M(p)$, $p \in P$, is a closed set. We have also

$$S_+(f) \subset \bigcup_{p \in P} M(p). \quad (3.2)$$

Indeed, if $x \in S_+(f)$, then there exist $\tilde{u}, \tilde{v} \in X$ such that $f'_+(x, \tilde{u}) + f'_+(x, \tilde{v}) \neq f'_+(x, \tilde{u} + \tilde{v})$. (Otherwise the mapping $v \rightarrow f'_+(x, v)$ would be linear and also Lipschitz (cf. Lemma 1.3 (i)); consequently it would be a Gâteaux derivative of f at x .) Since the function $v \rightarrow f'_+(x, v)$ is continuous (Lemma 1.3 (i)), we can choose $u \in D_X$, $v \in D_X$, $w := u + v$ and $\varepsilon \in Q^+$ such that $\|f'_+(x, u) + f'_+(x, v) - f'_+(x, w)\| > 6\varepsilon$. Now choose $U, V, W \in D_Y$ for which

$$\|f'_+(x, u) - U\| < \varepsilon, \quad \|f'_+(x, v) - V\| < \varepsilon, \quad \|f'_+(x, w) - W\| < \varepsilon.$$

Then clearly $\|W - (U + V)\| > 3\varepsilon$ and we can choose $\delta \in Q^+$ such that $\text{dist}(x, X \setminus G) \geq \delta$ and (3.1) holds. Thus $x \in M(u, v, w, U, V, W, \varepsilon, \delta)$ and (3.2) holds. Consequently $S_+(f) = \bigcup_{p \in P} (S_+(f) \cap M(p))$. Now fix $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$ and consider $M := S_+(f) \cap M(p)$. Lemma 3.2 implies that $(\text{contg}(M, x) - \text{contg}(M, x)) \cap \text{span}\{u\} \subset \{0\}$ and therefore $\text{contg}(M, x) \cap \text{span}\{u\} = \emptyset$ for each $x \in M$. By Lemma 2.10 there exists a sequence $(S_n)_{n=1}^\infty$

of Lipschitz surfaces of codimension 1 associated with $\text{span}\{u\}$ such that $M \subset \bigcup_{n=1}^{\infty} S_n$ and therefore $M = \bigcup_{n=1}^{\infty} (S_n \cap M)$. Fix $n \in \mathbb{N}$ and put $M^* = S_n \cap M$. By Lemma 3.2 we have $(\text{contg}(M^*, x) - \text{contg}(M^*, x)) \cap \text{span}\{u\} \subset \{0\}$ for each $x \in M^*$. Consequently Lemma 2.11 (i) implies $M^* \in \mathcal{A}_1^+$ and therefore $S_+(f)$ is a countable union of sets from the class \mathcal{A}_1^+ . Now suppose that f has all one-sided derivatives at each point $x \in G$. Then we even have $S_+(f) = \bigcup_{p \in P} M(p)$. Indeed, if $x \in M(p)$ for some $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$, we clearly obtain $\|f'_+(x, u) - U\| \leq \varepsilon$, $\|f'_+(x, v) - V\| \leq \varepsilon$, $\|f'_+(x, w) - W\| \leq \varepsilon$. Therefore $\|f'_+(x, u) + f'_+(x, v) - f'_+(x, w)\| > 0$ and thus $x \in S_+(f)$. Consequently the sets $M = S_+(f) \cap M(p) = M(p)$ and $M^* = S_n \cap M$ considered above are closed. Therefore $S_+(f)$ is a countable union of closed sets from the class \mathcal{A}_1^+ . \square

Remark 3.4. Note that we have proved the above theorem using Lemma 2.11 (i). In particular the above proof gives (without using Remark 2.12) that $S_+(f)$ can be written in the form $S_+(f) = \bigcup_{n=1}^{\infty} M_n$ where M_n are sets for which

$$(\text{contg}(M_n, x) - \text{contg}(M_n, x)) \cap \text{span}\{u\} \subset \{0\}, \quad x \in M_n. \tag{3.3}$$

Moreover, if f has all one-sided derivatives at each point $x \in G$, then M_n can be chosen to be closed. Further note that (3.3) clearly implies $\text{int}(\text{contg}(M_n, x)) = \emptyset$, $x \in M_n$.

Lemma 3.5. *Let $M \subset X, x_0, u, v, w, s_1, s_2 \in X, \varepsilon, \delta > 0, U, V, W \in Y$ be given such that $w = u + v, \|W - (U + V)\| > 5\varepsilon, s_1, s_2 \in \text{contg}(M, x_0)$. Further let $f'(x_0, q)$ exist for each $q \in X$ and*

$$\begin{aligned} \left| \frac{f(x + tu) - f(x)}{t} - U \right| \leq \varepsilon, \quad \left| \frac{f(x + tv) - f(x)}{t} - V \right| \leq \varepsilon, \\ \left| \frac{f(x + tw) - f(x)}{t} - W \right| \leq \varepsilon, \end{aligned}$$

whenever $x \in M$ and $0 < |t| < \delta$. Then the following assertions hold.

- (i) If $s_1 - s_2 \in \text{span}\{u, v\}$, then $s_1 = s_2$.
- (ii) If $s_1 + s_2 \in \text{span}\{u, v\}$, then $s_1 = -s_2$.

PROOF. To prove (i), suppose on the contrary that $s_1 - s_2 \in \text{span}\{u, v\}$ and $s_1 \neq s_2$. Without any loss of generality we can suppose that $s_1 - s_2 =$

$\alpha u + \beta v$, where $|\alpha| > 0$ and $|\alpha| \geq |\beta|$. By Lemma 3.1 (ii) there exist vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in Y$, $\|\varepsilon_i\| \leq \varepsilon$, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} f'(x_0, s_1 - \beta v) &= f'(x_0, s_1) - \beta V + |\beta|\varepsilon_1, \\ f'(x_0, s_2 + \alpha u) &= f'(x_0, s_2) + \alpha U + |\alpha|\varepsilon_2, \\ f'(x_0, s_1 - \alpha w) &= f'(x_0, s_1) - \alpha W + |\alpha|\varepsilon_3, \\ f'(x_0, s_2 + (\beta - \alpha)v) &= f'(x_0, s_2) + (\beta - \alpha)V + |\beta - \alpha|\varepsilon_4. \end{aligned}$$

Since $s_1 - \beta v = s_2 + \alpha u$ and $s_1 - \alpha w = s_2 + (\beta - \alpha)v$, the above equalities imply

$$\begin{aligned} f'(x_0, s_1) - \beta V + |\beta|\varepsilon_1 &= f'(x_0, s_2) + \alpha U + |\alpha|\varepsilon_2, \\ f'(x_0, s_1) - \alpha W + |\alpha|\varepsilon_3 &= f'(x_0, s_2) + (\beta - \alpha)V + |\beta - \alpha|\varepsilon_4. \end{aligned}$$

Subtracting these equalities, we obtain

$$\begin{aligned} \alpha W - \beta V + |\beta|\varepsilon_1 - |\alpha|\varepsilon_3 &= \alpha U - (\beta - \alpha)V + |\alpha|\varepsilon_2 - |\beta - \alpha|\varepsilon_4, \\ W - (U + V) &= \frac{1}{\alpha}(|\alpha|\varepsilon_2 - |\beta - \alpha|\varepsilon_4 - |\beta|\varepsilon_1 + |\alpha|\varepsilon_3), \\ \|W - (U + V)\| &\leq \varepsilon + 2\varepsilon + \varepsilon + \varepsilon = 5\varepsilon, \end{aligned}$$

which is a contradiction.

To prove (ii), suppose on the contrary that $s_1 + s_2 \in \text{span}\{u, v\}$ and $s_1 \neq -s_2$. Without any loss of generality we can suppose that $s_1 + s_2 = \alpha u + \beta v$, where $|\alpha| > 0$ and $|\alpha| \geq |\beta|$. By Lemma 3.1 there exist vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in Y$, $\|\varepsilon_i\| \leq \varepsilon$, $i = 1, 2, 3, 4$, such that

$$\begin{aligned} f'(x_0, s_1 - \alpha u) &= f'(x_0, s_1) - \alpha U + |\alpha|\varepsilon_1, \\ f'(x_0, s_2 - \beta v) &= f'(x_0, s_2) - \beta V + |\beta|\varepsilon_2, \\ f'(x_0, s_1 - \alpha w) &= f'(x_0, s_1) - \alpha W + |\alpha|\varepsilon_3, \\ f'(x_0, s_2 + (\alpha - \beta)v) &= f'(x_0, s_2) + (\alpha - \beta)V + |\alpha - \beta|\varepsilon_4. \end{aligned}$$

Since $s_1 - \alpha u = -(s_2 - \beta v)$ and $s_1 - \alpha w = -(s_2 + (\alpha - \beta)v)$, the above equalities imply

$$\begin{aligned} f'(x_0, s_1) - \alpha U + |\alpha|\varepsilon_1 &= -(f'(x_0, s_2) - \beta V + |\beta|\varepsilon_2), \\ f'(x_0, s_1) - \alpha W + |\alpha|\varepsilon_3 &= -(f'(x_0, s_2) + (\alpha - \beta)V + |\alpha - \beta|\varepsilon_4). \end{aligned}$$

Subtracting these equalities, we obtain

$$\alpha W - \alpha U + |\alpha|\varepsilon_1 - |\alpha|\varepsilon_3 = \alpha V + |\alpha - \beta|\varepsilon_4 - |\beta|\varepsilon_2,$$

$$W - (U + V) = \frac{1}{\alpha}(-|\alpha|\varepsilon_1 + |\alpha|\varepsilon_3 + |\alpha - \beta|\varepsilon_4 - |\beta|\varepsilon_2),$$

$$\|W - (U + V)\| \leq \varepsilon + \varepsilon + 2\varepsilon + \varepsilon = 5\varepsilon,$$

which is a contradiction. □

Theorem 3.6. *Let X be a separable Banach space, $G \subset X$ an open set, Y a Banach space and $f : G \rightarrow Y$ a Lipschitz mapping. Let $S(f)$ be the set of all points $x \in G$ at which there exist all (two-sided) directional derivatives $f'(x, v)$, $v \in X$, but f is not Gâteaux differentiable at x . Then $S(f)$ can be covered by countably many Lipschitz surfaces of codimension 2. More precisely, $S(f)$ is a countable union of sets from the class \mathcal{A}_2 . If, moreover, f has all (two-sided) directional derivatives at all points of G , then $S(f)$ is a countable union of closed sets from the class \mathcal{A}_2 .*

PROOF. Since the closed linear span of the range $f(G)$ is clearly separable, we may suppose without any loss of generality that Y is separable. Let D_X and D_Y be countable dense subsets of X and Y , respectively, and let Q^+ be the set of all positive rational numbers. Put

$$P = \{(u, v, w, U, V, W, \varepsilon, \delta) : u \in D_X, v \in D_X, w = u + v, U \in D_Y, \\ V \in D_Y, W \in D_Y, \varepsilon \in Q^+, \delta \in Q^+, \|W - (U + V)\| > 5\varepsilon\}.$$

Note that P is countable. For each $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$ denote by $M(p)$ the set of all $x \in G$ for which $\text{dist}(x, X \setminus G) \geq \delta$ and

$$\left\| \frac{f(x + tu) - f(x)}{t} - U \right\| \leq \varepsilon, \quad \left\| \frac{f(x + tv) - f(x)}{t} - V \right\| \leq \varepsilon, \\ \left\| \frac{f(x + tw) - f(x)}{t} - W \right\| \leq \varepsilon \quad \text{for each } 0 < |t| < \delta. \tag{3.4}$$

It is easy to see that each $M(p), p \in P$, is a closed set. We have also

$$S(f) \subset \bigcup_{p \in P} M(p). \tag{3.5}$$

Indeed, if $x \in S(f)$, then there exist $\tilde{u}, \tilde{v} \in X$ such that $f'(x, \tilde{u}) + f'(x, \tilde{v}) \neq f'(x, \tilde{u} + \tilde{v})$. (Otherwise the mapping $v \rightarrow f'(x, v)$ would be linear and also Lipschitz (cf. Lemma 1.3 (i)); consequently it would be a Gâteaux derivative of f at x .) Since the function $v \rightarrow f'(x, v)$ is continuous (Lemma 1.3 (i)), we can choose $u \in D_X, v \in D_X, w := u + v$ and $\varepsilon \in Q^+$ such that $\|f'(x, u) + f'(x, v) - f'(x, w)\| > 8\varepsilon$. Now choose $U, V, W \in D_Y$ for which

$$\|f'(x, u) - U\| < \varepsilon, \quad \|f'(x, v) - V\| < \varepsilon, \quad \|f'(x, w) - W\| < \varepsilon.$$

Then clearly $\|W - (U + V)\| > 5\varepsilon$ and we can choose $\delta \in Q^+$ such that $\text{dist}(x, X \setminus G) \geq \delta$ and (3.4) holds. Therefore $x \in M(u, v, w, U, V, W, \varepsilon, \delta)$ and (3.5) holds. Consequently $S(f) = \bigcup_{p \in P} (S(f) \cap M(p))$. Now fix $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$ and consider $M := S(f) \cap M(p)$. Lemma 3.5 immediately implies that $(\text{contg}(M, x) - \text{contg}(M, x)) \cap \text{span}\{u, v\} \subset \{0\}$ and therefore $\text{contg}(M, x) \cap \text{span}\{u, v\} = \emptyset$ for each $x \in M$. By Lemma 2.10 there exists a sequence $(S_n)_{n=1}^\infty$ of Lipschitz surfaces of codimension 2 associated with $\text{span}\{u, v\}$ such that $M \subset \bigcup_{n=1}^\infty S_n$ and therefore $M = \bigcup_{n=1}^\infty (S_n \cap M)$. Fix $n \in \mathbb{N}$ and put $M^* = S_n \cap M$. By Lemma 3.5 we have $(\text{contg}(M^*, x) - \text{contg}(M^*, x)) \cap \text{span}\{u, v\} \subset \{0\}$, $(\text{contg}(M^*, x) + \text{contg}(M^*, x)) \cap \text{span}\{u, v\} \subset \{0\}$ for all $x \in M^*$. Consequently Lemma 2.11 (ii) implies $M^* \in \mathcal{A}_2$ and therefore $S(f)$ is a countable union of sets from the class \mathcal{A}_2 . Now suppose that f has all two-sided derivatives at each point $x \in G$. Then we have even $S(f) = \bigcup_{p \in P} M(p)$. Indeed, if $x \in M(p)$ for some $p = (u, v, w, U, V, W, \varepsilon, \delta) \in P$, we clearly obtain $\|f'(x, u) - U\| \leq \varepsilon$, $\|f'(x, v) - V\| \leq \varepsilon$, $\|f'(x, w) - W\| \leq \varepsilon$. Therefore $\|f'(x, u) + f'(x, v) - f'(x, w)\| > 2\varepsilon$ and thus $x \in S(f)$. Consequently the sets $M = S(f) \cap M(p) = M(p)$ and $M^* = S_n \cap M$ considered above are closed. Therefore $S(f)$ is a countable union of closed sets from the class \mathcal{A}_2 . \square

4 Constructions

In the following lemma we use the method of B. Kirchheim's proof of Proposition 14 from [PZ]. Recall the notation $B(x, r)$ for the open ball with a center x and radius r .

Lemma 4.1. *Let X be a separable Banach space and f a non-negative bounded Lipschitz function defined on X . Let $F := \{x : f(x) = 0\} \neq \emptyset$. Then there exists a function g such that*

- (i) $g|_F = 0$, $g|_{(X \setminus F)} > 0$;
- (ii) g is Gâteaux differentiable on $X \setminus F$;
- (iii) g is Lipschitz ;
- (iv) the Fréchet derivative of $f - g$ is 0 at each point of F .

PROOF. Assume first $0 \leq f \leq 1$, $\text{Lip}(f) \leq 1$. Set

$$f_k(x) = \min(\max(f(x) - 2^{-k}, 0), 2^{-k}), k \in \mathbb{N}.$$

It is easy to see that $f = \sum_{k=1}^{\infty} f_k$, $0 \leq f_k \leq 2^{-k}$, $\text{Lip}(f_k) \leq 1$ and

$$f_k(x) = 0 \text{ if } f(x) \leq 2^{-k}; \quad f_k(x) = 2^{-k} \text{ if } f(x) \geq 2^{-k+1}. \quad (4.1)$$

For each $k \in \mathbb{N}$, let μ_k be a non-degenerate probability cube measure (see [BL], p. 142) with

$$\text{supp } \mu_k \subset B(0, 4^{-1-k}) \quad (4.2)$$

and $g_k = f_k \star \mu_k$ (i.e. $g_k(x) = \int f_k(x-t)d\mu_k(t)$). Since f_k is Lipschitz, it is Gâteaux differentiable at μ_k -almost all points of X (see [BL], 6.25, 6.27, 6.42) and therefore Lebesgue's dominated convergence theorem easily gives (as in [BL], 6.43) that $g'_k(x, v) = (f'_k(\cdot, v) \star \mu_k)(x)$ for every $x, v \in X$ and therefore g_k is an everywhere Gâteaux differentiable function. It is easy to verify (using $0 \leq f_k \leq 2^{-k}$, $\text{Lip}(f_k) \leq 1$ and (4.2)) that

$$0 \leq g_k \leq 2^{-k}, \text{Lip}(g_k) \leq 1 \text{ and } |g_k - f_k| \leq 4^{-1-k}. \quad (4.3)$$

Using (4.1), (4.2) and $\text{Lip}(f) \leq 1$, we easily obtain that

$$g_k(x) = f_k(x) = 0 \text{ if } f(x) < 2^{-k} - 4^{-1-k} \quad (4.4)$$

and

$$g_k(x) = f_k(x) = 2^{-k} \text{ if } f(x) > 2^{-k+1} + 4^{-1-k}. \quad (4.5)$$

Set $g := \sum_{k=1}^{\infty} g_k$. From (4.4), (4.3) and (4.5) we immediately obtain (i).

Now, fix $n \in \mathbb{N}$ and consider g on the set $D_n = \{x : 2^{-n} < f(x) < 2^{-n+2}\}$. By (4.4) and (4.5) we have that $g_k(x) = f_k(x) = 0$ for $x \in D_n$ and $1 \leq k \leq n-3$; $g_k(x) = f_k(x) = 2^{-k}$ for $x \in D_n$ and $k \geq n+2$. Consequently,

$$g(x) = \sum_{k \geq n+2} 2^{-k} + \sum_{\max(1, n-2) \leq k \leq n+1} g_k(x), \quad x \in D_n.$$

Therefore g is Gâteaux differentiable on D_n and $\text{Lip}(g, x) \leq 4$ for each $x \in D_n$.

Since clearly $X \setminus F = \bigcup_{n=1}^{\infty} D_n$, we obtain (ii) and

$$\text{Lip}(g, x) \leq 4 \text{ for each } x \in X \setminus F. \quad (4.6)$$

Using (4.3) we obtain, for each $x \in D_n$,

$$|f(x) - g(x)| \leq \sum_{\max(1, n-2) \leq k \leq n+1} |f_k(x) - g_k(x)| \leq 4 \cdot 4^{-n+1}. \quad (4.7)$$

Fix $y \in F$ and consider $x \in X$, $x \neq y$. Consider for $x \in X \setminus F$ the least $n = n(x)$ for which $x \in D_n$. Since $\text{Lip}(f) \leq 1$, we have $\|y - x\| > 2^{-n}$. By (4.7) we obtain

$$\frac{|f(x) - g(x)|}{\|y - x\|} \leq \frac{4 \cdot 4^{-n+1}}{2^{-n}}.$$

Clearly $\lim_{\substack{x \rightarrow y \\ x \notin F}} n(x) = \infty$, which proves (iv).

Now, (iv) and $\text{Lip}(f) \leq 1$ gives $\text{Lip}(g, x) \leq 4$ for each $x \in F$. Using (4.6) and Lemma 1.3 (ii), we obtain (iii).

In the general case find $\varepsilon > 0$ such that $\varepsilon f \leq 1$ and $\text{Lip}(\varepsilon f) \leq 1$. Applying the construction above to $\tilde{f} := \varepsilon f$ we obtain a function \tilde{g} and $g := \frac{1}{\varepsilon} \tilde{g}$ satisfies the properties (i)-(iv). \square

The next lemma generalizes Lemma 4.1 in the sense that F is any closed subset of $\{x \in X : f(x) = 0\}$ but need not be equal to this set and f need not be non-negative.

Lemma 4.2. *Let X be a separable Banach space and f be a bounded Lipschitz real-valued function. Let $\emptyset \neq F \subset X$ be closed and $f|_F = 0$. Then there exists a function g such that g is bounded Lipschitz, g is Gâteaux differentiable on $X \setminus F$, $g|_F = 0$ and $f - g$ has zero Fréchet derivative at each point $x \in F$.*

PROOF. Set $d(x) = \min(\text{dist}(x, F), 1)$. Let f^+ , f^- be the positive and negative parts of f . Put $h_1(x) = f^+(x) + d(x)$, $h_2(x) = f^-(x) + d(x)$. Clearly h_1, h_2 are non-negative bounded Lipschitz functions and

$$F = \{x : h_1(x) = 0\} = \{x : h_2(x) = 0\}.$$

By Lemma 4.1 we can find bounded Lipschitz functions g_1, g_2 which are Gâteaux differentiable on $X \setminus F$, $g_i|_F = 0$ and $h_i - g_i$ has the Fréchet derivative 0 at each point $x \in F$, $i = 1, 2$. Set $g = g_1 - g_2$. Then $g|_F = 0$ and g is Gâteaux differentiable on $X \setminus F$. Let $x \in F$. Since $f = f^+ - f^- = h_1 - h_2$, we have

$$\begin{aligned} & \frac{|(f - g)(y) - (f - g)(x)|}{\|y - x\|} \\ & \leq \frac{|(h_1 - g_1)(y) - (h_1 - g_1)(x)|}{\|y - x\|} + \frac{|(h_2 - g_2)(y) - (h_2 - g_2)(x)|}{\|y - x\|}, \end{aligned}$$

which with the fact that $h_i - g_i$ has zero Fréchet derivative at all points of F finishes the proof. \square

The following easy lemma is well-known (cf. Proposition 3.6. (i) from [Sp]).

Lemma 4.3. *Let X, Y, Z be Banach spaces, $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two mappings and let g be Lipschitz. Let $x, u \in X$ and let there exist $v := f'_+(x, u)$, $g'_+(f(x), v)$. Then*

$$(g \circ f)'_+(x, u) = g'_+(f(x), v). \quad (4.8)$$

Remark 4.4. The preceding lemma clearly holds also in the trivial cases when at least one of the spaces X, Y, Z is the trivial null-dimensional space $\{0\}$. Of course, then the directional derivatives involved in (4.8) equal to zero.

Lemma 4.5. *Let X be a separable Banach space and $\emptyset \neq A \in \mathcal{B}_1^+(X)$. Then there exists a bounded Lipschitz function $f : X \rightarrow \mathbb{R}$ such that*

- (i) $f'_+(x, u)$ exists for all $x \in A$ and $u \in X$;
- (ii) f is Gâteaux differentiable on $X \setminus \bar{A}$;
- (iii) there is $w \in X$, $\|w\| = 1$, such that for all $x \in A$, $f'_+(x, w) = f'_+(x, -w) = 1$.

In particular, each closed set from $\mathcal{B}_1^+(X)$ belongs to $\tilde{\mathcal{S}}_+(X)$.

PROOF. According to Definition 2.3 there exist a subspace Y of codimension 1, a topological complement W of Y and a Lipschitz function $h : Y \rightarrow W$ such that $A \subset S := \{y + h(y) : y \in Y\}$ and

$$h'_+(y, v) \text{ exists whenever } v \in Y \text{ and } y \in P(A), \quad (4.9)$$

where P is the projection of X on Y parallel to W . For $x \in X$, put

$$g(x) := \|x - P(x) - h(P(x))\|.$$

Clearly $g|_S = 0$. Since the sum and the composition of two Lipschitz functions is Lipschitz and the functions $P, h, \|\cdot\|$ are Lipschitz, we obtain that g is Lipschitz. Since P is linear and $\|\cdot\|$ is convex, (4.9) and Lemma 4.3 imply that

$$g'_+(x, u) \text{ exists for each } x \in A \text{ and } u \in X. \quad (4.10)$$

Choose a unit vector $w \in W$ and suppose that $x \in A$ is given. Since clearly $P(x) = P(x + tw)$ for each $t \in \mathbb{R}$, we have

$$g(x + tw) = \|x + tw - P(x) - h(P(x))\| = \|tw\| = |t|.$$

Thus

$$g'_+(x, w) = 1 \text{ and } g'_+(x, -w) = 1. \quad (4.11)$$

Set $g_1(x) := \min(g(x), 1)$. Then g_1 satisfies the assumptions of Lemma 4.2 with $F = \bar{A}$ and so we can find a bounded Lipschitz function f such that $f|_{\bar{A}} = 0$, f is Gâteaux differentiable on $X \setminus \bar{A}$ and $f - g_1$ has zero Fréchet derivative at any point $x \in \bar{A}$. These properties of f , (4.10) and (4.11) immediately give (i) and (iii). \square

Lemma 4.6. *Let X be a separable Banach space, $\{a_n\}, \{b_n\}$ sequences of positive numbers, $\sum_{n=1}^{\infty} a_n < \infty, \sum_{n=1}^{\infty} b_n < \infty$. Let $f_n : X \rightarrow \mathbb{R}$ satisfy $|f_n| \leq a_n$ and $\text{Lip}(f_n) \leq b_n$. Let $(f_n)'_+(x, u)$ exist for some $x, u \in X$. Set $f = \sum_{n=1}^{\infty} f_n$. Then f is a Lipschitz function and*

$$f'_+(x, u) = \sum_{n=1}^{\infty} (f_n)'_+(x, u). \quad (4.12)$$

Moreover, if f_n are Gâteaux differentiable at x , then f is Gâteaux differentiable at x .

PROOF. It is easy to see that f is Lipschitz. For $t \in (0, \infty)$, put

$$g(t) = \frac{1}{t}(f(x + tu) - f(x)), \quad g_n(t) = \frac{1}{t}(f_n(x + tu) - f_n(x)).$$

Clearly, $g(t) = \sum_{n=1}^{\infty} g_n(t)$ for $t > 0$, and $|g_n(t)| \leq b_n \|u\|$ because $\text{Lip}(f_n) \leq b_n$.

Consequently $\sum_{n=1}^{\infty} g_n(t)$ converges uniformly on $(0, \infty)$. Therefore

$$\sum_{n=1}^{\infty} (f_n)'_+(x, u) = \sum_{n=1}^{\infty} \lim_{t \rightarrow 0_+} g_n(t) = \lim_{t \rightarrow 0_+} g(t) = f'_+(x, u),$$

which proves (4.12).

If f_n are Gâteaux differentiable at x , then the mappings $u \rightarrow (f_n)'_+(x, u)$ are linear on X . By (4.12) we have that $v \rightarrow f'_+(x, v)$ is linear and it is also continuous by Lemma 1.3 (i). \square

Theorem 4.7. *Let X be a separable Banach space, $\dim(X) \geq 1$, and $A \subset X$ be a countable union of closed sets from \mathcal{B}_1^+ . Then there exists a Lipschitz function $f : X \rightarrow \mathbb{R}$ such that $f'_+(x, u)$ exists for all $x, u \in X$, f is Gâteaux differentiable at all points of $X \setminus A$ and f is Gâteaux differentiable at no point of A (i.e. $A \in \tilde{\mathcal{S}}_+(X)$).*

PROOF. The case $A = \emptyset$ is trivial. Thus suppose that $A = \bigcup_{n=1}^{\infty} A_n$, where $\emptyset \neq A_n \in \mathcal{B}_1^+$ are closed. According to Lemma 4.5 there are vectors $w_n \in X$, $\|w_n\| = 1$, functions $f_n : X \rightarrow \mathbb{R}$ and numbers L_n, K_n with

$$|f_n| \leq L_n, \text{Lip}(f_n) \leq K_n; \quad (4.13)$$

$$(f_n)'_+(x, u) \text{ exists for all } x, u \in X; \quad (4.14)$$

$$f_n \text{ is Gâteaux differentiable on } X \setminus A_n; \quad (4.15)$$

$$(f_n)'_+(x, w_n) = (f_n)'_+(x, -w_n) = 1 \text{ for all } x \in A_n. \quad (4.16)$$

Without loss of generality we can suppose $L_n \geq 1, K_n \geq 1$. Define a sequence $\{\varepsilon_n\}$ by

$$\varepsilon_1 = \frac{1}{L_1 K_1}, \varepsilon_{n+1} = \frac{\varepsilon_n}{3L_{n+1} K_{n+1}}. \quad (4.17)$$

Since $L_n \geq 1, K_n \geq 1$, we have

$$\varepsilon_n \leq 3^{l-n} \varepsilon_l \text{ for } 1 \leq l \leq n \quad (4.18)$$

and consequently, for each $n > 1$, $L_n \varepsilon_n = (3K_n)^{-1} \varepsilon_{n-1} \leq 3^{-1} \varepsilon_{n-1} \leq 3^{1-n} \varepsilon_1$. Analogously, $K_n \varepsilon_n \leq 3^{1-n} \varepsilon_1$. Set $g_n = \varepsilon_n f_n$, $n \in \mathbb{N}$. Then by (4.13) we have $|g_n| \leq 3^{1-n} \varepsilon_1$ and $|\text{Lip}(g_n)| \leq 3^{1-n} \varepsilon_1$. Put $f = \sum_{n=1}^{\infty} g_n$. By Lemma 4.6, (4.14) and (4.15) we have that f is Lipschitz, $f'_+(x, u)$ exists for all $x, u \in X$,

$$f'_+(x, u) = \sum_{n=1}^{\infty} (g_n)'_+(x, u) \text{ for all } x, u \in X \quad (4.19)$$

and f is Gâteaux differentiable at all points of $X \setminus A$. Let $x \in A$ be given. Put $m := \min\{n : x \in A_n\}$. Then by (4.15), for each $1 \leq l < m$, f_l is Gâteaux differentiable at x and consequently $(f_l)'_+(x, w_m) + (f_l)'_+(x, -w_m) = 0$, which gives with (4.19) and (4.16)

$$\begin{aligned} f'_+(x, w_m) + f'_+(x, -w_m) &= \sum_{n=1}^{\infty} \varepsilon_n ((f_n)'_+(x, w_m) + (f_n)'_+(x, -w_m)) \\ &= 2\varepsilon_m + \sum_{n=m+1}^{\infty} \varepsilon_n ((f_n)'_+(x, w_m) + (f_n)'_+(x, -w_m)). \end{aligned}$$

By (4.13) we have $|(f_n)'_+(x, u)| \leq K_n$ for each $u \in X$ which gives with (4.17)

and (4.18)

$$\begin{aligned} \left| \sum_{n=m+1}^{\infty} \varepsilon_n ((f_n)'_+(x, w_m) + (f_n)'_+(x, -w_m)) \right| &\leq \left| \sum_{n=m+1}^{\infty} \varepsilon_n 2K_n \right| \\ &= 2 \sum_{n=m+1}^{\infty} \frac{\varepsilon_{n-1}}{3L_n} \leq 2 \sum_{n=m}^{\infty} \frac{\varepsilon_n}{3} \leq \frac{2}{3} \sum_{n=m}^{\infty} \varepsilon_n 3^{m-n} = \varepsilon_m. \end{aligned}$$

Consequently, $f'_+(x, w_m) + f'_+(x, -w_m) \geq 2\varepsilon_m - \varepsilon_m > 0$, which proves the theorem. \square

Lemma 4.8. *Let $W = \text{span}\{w_1, w_2\}$ be a 2-dimensional Banach space. Then there exists a bounded Lipschitz function $q : W \rightarrow \mathbb{R}$ such that $q(0) = 0$, $q'(0, w)$ exists for each $w \in W$, $q'(0, w_1) = q'(0, w_2) = 0$ and $q'(0, w_1 + w_2) = 1$.*

PROOF. Let $L : W \rightarrow \mathbb{R}^2$ be the linear mapping for which $L(w_1) = (1, 0)$ and $L(w_2) = (0, 1)$. For $(x, y) \in \mathbb{R}^2$ put

$$p(x, y) = \begin{cases} \min(x, y, 1) & x > 0, y > 0 \\ \max(x, y, -1) & x < 0, y < 0 \\ 0 & xy \leq 0. \end{cases}$$

It is easy to verify that p is a bounded Lipschitz function, $p'((0, 0), w)$ exists for each $w \in \mathbb{R}^2$, $p'((0, 0), (1, 0)) = p'((0, 0), (0, 1)) = 0$ and $p'((0, 0), (1, 1)) = 1$. The function $q := p \circ L$ has clearly all desired properties. \square

Lemma 4.9. *Let X be a separable Banach space and $A \in \mathcal{B}_2(X)$. Then there exists a bounded Lipschitz function $f : X \rightarrow \mathbb{R}$ such that*

- (i) $f'(x, u)$ exists for all $x \in A$ and $u \in X$;
- (ii) f is Gâteaux differentiable on $X \setminus \overline{A}$;
- (iii) there are unit vectors $w_1, w_2 \in X$ such that, for each $x \in A$, $f'(x, w_1) = f'(x, w_2) = 0$, $f'(x, w_1 + w_2) = 1$.

In particular, each closed set from $\mathcal{B}_2(X)$ belongs to $\tilde{\mathcal{S}}(X)$.

PROOF. According to Definition 2.3 there exists a closed subspace Y of X , its topological complement W of dimension 2 and a Lipschitz mapping $h : Y \rightarrow W$ such that $A \subset S := \{y + h(y) : y \in Y\}$ and

$$h'(y, v) \text{ exists whenever } y \in P(A) \text{ and } v \in Y, \quad (4.20)$$

where P is the projection of X onto Y parallel to W . Let $W = \text{span}\{w_1, w_2\}$, $\|w_1\| = \|w_2\| = 1$ and $q : W \rightarrow \mathbb{R}$ be the Lipschitz function from Lemma 4.8. For $x \in X$, put $g(x) := q(x - P(x) - h(P(x)))$. It is easy to see that g is a bounded Lipschitz function. Since clearly $x - P(x) - h(P(x)) = 0$ for $x \in A$, (4.20), properties of q and Lemma 4.3 easily gives that $g'(x, v)$ exists for each $x \in A$ and $v \in X$. Let $x \in A$ and $w \in W$. Then clearly $P(x + w) = P(x)$ and thus $g(x + w) = q(w)$. This fact immediately implies $g'(x, w_1) = g'(x, w_2) = 0$, $g'(x, w_1 + w_2) = 1$. By Lemma 4.2 there exists a bounded Lipschitz function f on X such that $f/\bar{A} = 0$, f is Gâteaux differentiable on $X \setminus \bar{A}$ and $f'(x, u) = g'(x, u)$ for all $x \in A, u \in X$. Consequently f has all desired properties. \square

Theorem 4.10. *Let X , $\dim(X) \geq 2$, be a separable Banach space and $A \subset X$ be a countable union of closed sets from $\mathcal{B}_2(X)$. Then there exists a Lipschitz function $f : X \rightarrow \mathbb{R}$ such that $f'(x, u)$ exists for all $x, u \in X$, f is Gâteaux differentiable at all points of $X \setminus A$ and f is Gâteaux differentiable at no point of A (i.e. $A \in \tilde{\mathcal{S}}(X)$).*

PROOF. The case $A = \emptyset$ is trivial. Thus suppose that $A = \bigcup_{n=1}^{\infty} A_n$, where $\emptyset \neq A_n \in \mathcal{B}_2$ are closed sets. According to Lemma 4.9 there are unit vectors $w_{1n} \in X$, $w_{2n} \in X$, functions $f_n : X \rightarrow \mathbb{R}$ and numbers L_n, K_n with

$$|f_n| \leq L_n, \text{Lip}(f_n) \leq K_n; \quad (4.21)$$

$$(f_n)'(x, u) \text{ exists for all } x, u \in X; \quad (4.22)$$

$$f_n \text{ is Gâteaux differentiable on } X \setminus A_n; \quad (4.23)$$

$$(f_n)'(x, w_{1n}) = (f_n)'(x, w_{2n}) = 0, \quad (4.24)$$

$$f'(x, w_{1n} + w_{2n}) = 1 \text{ for all } x \in A_n.$$

Without any loss of generality we can assume $L_n \geq 1$, $K_n \geq 1$. Define the sequence $\{\varepsilon_n\}$ by

$$\varepsilon_1 = \frac{1}{L_1 K_1}, \quad \varepsilon_{n+1} = \frac{\varepsilon_n}{5L_{n+1}K_{n+1}}. \quad (4.25)$$

Since $L_n \geq 1$, $K_n \geq 1$, we have

$$\varepsilon_n \leq 5^{l-n} \varepsilon_l \text{ for } 1 \leq l \leq n \quad (4.26)$$

and consequently, for $n > 1$, $L_n \varepsilon_n = (5K_n)^{-1} \varepsilon_{n-1} \leq 5^{-1} \varepsilon_{n-1} \leq 5^{1-n} \varepsilon_1$. Analogously, $K_n \varepsilon_n \leq 5^{1-n} \varepsilon_1$. Put $g_n = \varepsilon_n f_n$. Then by (4.21) we have $|g_n| \leq$

$5^{1-n}\varepsilon_1$ and $|\text{Lip}(g_n)| \leq 5^{1-n}\varepsilon_1$. Put $f = \sum_{n=1}^{\infty} g_n$. By Lemma 4.6, (4.22) and (4.23) we obtain that f is Lipschitz, $f'(x, u)$ exists for all $x, u \in X$,

$$f'(x, u) = \sum_{n=1}^{\infty} (g_n)'(x, u) \quad \text{for all } x, u \in X \quad (4.27)$$

and f is Gâteaux differentiable at all points of $X \setminus A$. Let $x \in A$ be given. Put $m := \min\{n : x \in A_n\}$. Then by (4.23), for each $1 \leq l < m$, f_l is Gâteaux differentiable at x and thus $(f_l)'(x, w_{1m} + w_{2m}) = (f_l)'(x, w_{1m}) + (f_l)'(x, w_{2m})$, which gives by (4.27)

$$\begin{aligned} & f'(x, w_{1m} + w_{2m}) - f'(x, w_{1m}) - f'(x, w_{2m}) \\ &= \sum_{n=1}^{\infty} \varepsilon_n ((f_n)'(x, w_{1m} + w_{2m}) - (f_n)'(x, w_{1m}) - (f_n)'(x, w_{2m})) \\ &= \sum_{n=m}^{\infty} \varepsilon_n ((f_n)'(x, w_{1m} + w_{2m}) - (f_n)'(x, w_{1m}) - (f_n)'(x, w_{2m})) \\ &= \varepsilon_m ((f_m)'(x, w_{1m} + w_{2m}) - (f_m)'(x, w_{1m}) - (f_m)'(x, w_{2m})) \\ &\quad + \sum_{n=m+1}^{\infty} \varepsilon_n ((f_n)'(x, w_{1m} + w_{2m}) - (f_n)'(x, w_{1m}) - (f_n)'(x, w_{2m})). \end{aligned}$$

According to (4.24), (4.21), (4.25) and (4.26) we have

$$\begin{aligned} & |f'(x, w_{1m} + w_{2m}) - f'(x, w_{1m}) - f'(x, w_{2m})| \\ &\geq \varepsilon_m - 3 \sum_{n=m+1}^{\infty} \varepsilon_n K_n \geq \varepsilon_m - 3 \sum_{n=m+1}^{\infty} \frac{\varepsilon_{n-1}}{5L_n} \\ &\geq \varepsilon_m - \frac{3}{5} \sum_{n=m}^{\infty} \varepsilon_n \geq \varepsilon_m - \frac{3}{5} \sum_{n=m}^{\infty} \frac{\varepsilon_m}{5^{n-m}} = \frac{\varepsilon_m}{4} > 0. \end{aligned}$$

Thus, f is not Gâteaux differentiable at x which finishes the proof. \square

5 Consequences of the Theorems

In this section we present some consequences of the above four theorems. Remember that we consider Lipschitz curves of special kind – see Definition 2.3.

Proposition 5.1. *Let $A \subset \mathbb{R}^2$. Then the following conditions are equivalent.*

- (i) *There exists a Lipschitz real function f on \mathbb{R}^2 such that $f'_+(x, u)$ exists for all $x, u \in \mathbb{R}^2$ and A is the set of all Gâteaux non-differentiability points of f .*
- (ii) *There exists a Banach space Y and a Lipschitz function $f : \mathbb{R}^2 \rightarrow Y$ such that $f'_+(x, u)$ exists for all $x, u \in \mathbb{R}^2$ and A is the set of all Gâteaux non-differentiability points of f .*
- (iii) *$A = \bigcup_{n=1}^{\infty} F_n$, where each F_n is a closed subset of a unilaterally smooth Lipschitz curve L_n .*
- (iv) *A is an F_σ set which can be covered by countably many unilaterally smooth Lipschitz curves.*

PROOF. The implication (i) \Rightarrow (ii) is trivial. The implication (ii) \Rightarrow (iii) immediately follows from Theorem 3.3 and Lemma 2.8. The implication (iii) \Rightarrow (i) immediately follows from Theorem 4.7. The conditions (iii) and (iv) are clearly equivalent. \square

Remark 5.2. Theorem 3.3 and Theorem 4.7 easily imply that $\tilde{\mathcal{S}}_+(\mathbb{R}) = \mathcal{S}_+(\mathbb{R})$ is the system of all countable subsets of \mathbb{R} , which is an easy and well-known fact.

Proposition 5.3. *Let $X = \mathbb{R}^2$ or $X = \mathbb{R}^3$ and $A \subset X$. Then the following conditions are equivalent.*

- (i) *There exists a Lipschitz real function f on X such that $f'(x, u)$ exists for all $x, u \in X$ and A is the set of all Gâteaux non-differentiability points of f .*
- (ii) *There exists a Banach space Y and a Lipschitz function $f : X \rightarrow Y$ such that $f'(x, u)$ exists for all $x, u \in X$ and A is the set of all Gâteaux non-differentiability points of f .*
- (iii) *A is a countable union of closed sets from $\mathcal{B}_2(X)$.*

In the case $X = \mathbb{R}^2$ these conditions are equivalent to

- (iv)₁ *A is a countable subset of \mathbb{R}^2*

and in the case $X = \mathbb{R}^3$ to the condition

- (iv)₂ *A is an F_σ set which can be covered by countably many smooth Lipschitz curves.*

PROOF. The implication (i) \Rightarrow (ii) is trivial. The implication (ii) \Rightarrow (iii) immediately follows from Theorem 3.6, Lemma 2.9 and Remark 2.4 (i). The implication (iii) \Rightarrow (i) immediately follows from Theorem 4.10. The equivalence of (iii) and (iv)₁ in the case $X = \mathbb{R}^2$ is obvious. The equivalence of (iii) and (iv)₂ easily follows from Lemma 2.9. \square

The above propositions give immediately the following partial answers to Problems A, B (see Remark 5.2 for the case $X = \mathbb{R}$).

Proposition 5.4. $\tilde{\mathcal{S}}_+(\mathbb{R}^2)$ is the system of all F_σ subsets of \mathbb{R}^2 which can be covered by countably many unilaterally smooth Lipschitz curves. $\tilde{\mathcal{S}}(\mathbb{R}^3)$ is the system of all F_σ subsets of \mathbb{R}^3 which can be covered by countably many smooth Lipschitz curves. $\tilde{\mathcal{S}}(\mathbb{R}^2) = \mathcal{S}(\mathbb{R}^2)$ is the system of all countable subsets of \mathbb{R}^2 .

For a class of sets \mathcal{C} , denote by \mathcal{C}_H the smallest hereditary class which contains all sets from \mathcal{C} (i.e. $\mathcal{C}_H = \{A : A \subset C \text{ for some } C \in \mathcal{C}\}$) and by \mathcal{C}_I the σ -ideal generated by \mathcal{C} .

Since we are not able to solve completely Problems A and B, it is natural to consider the following problem, which could be easier.

Problem C. Let X be a Banach space.

- (i) Characterize the classes $(\tilde{\mathcal{S}}_+(X))_H, (\mathcal{S}_+(X))_H, (\tilde{\mathcal{S}}(X))_H, (\mathcal{S}(X))_H$.
- (ii) Characterize the σ -ideals $(\tilde{\mathcal{S}}_+(X))_I, (\mathcal{S}_+(X))_I, (\tilde{\mathcal{S}}(X))_I, (\mathcal{S}(X))_I$.

Of course, Proposition 5.4 immediately implies that $(\tilde{\mathcal{S}}_+(\mathbb{R}^2))_H = (\tilde{\mathcal{S}}_+(\mathbb{R}^2))_I$ is the system of all subsets of \mathbb{R}^2 which can be covered by countably many unilaterally smooth Lipschitz curves and $(\tilde{\mathcal{S}}(\mathbb{R}^3))_H = (\tilde{\mathcal{S}}(\mathbb{R}^3))_I$ is the system of all subsets of \mathbb{R}^3 which can be covered by countably many smooth Lipschitz curves. (Moreover, we have that $(\tilde{\mathcal{S}}_+(\mathbb{R}))_H = (\tilde{\mathcal{S}}_+(\mathbb{R}))_I = (\mathcal{S}_+(\mathbb{R}))_H = (\mathcal{S}_+(\mathbb{R}))_I$ is the system of all countable subsets of \mathbb{R} and $(\tilde{\mathcal{S}}(\mathbb{R}^2))_H = (\tilde{\mathcal{S}}(\mathbb{R}^2))_I = (\mathcal{S}(\mathbb{R}^2))_H = (\mathcal{S}(\mathbb{R}^2))_I$ is the system of all countable subsets of \mathbb{R}^2 .)

We are not able to give a satisfactory characterization of classes $\mathcal{S}_+(\mathbb{R}^2), \mathcal{S}(\mathbb{R}^3), (\mathcal{S}_+(\mathbb{R}^2))_H, (\mathcal{S}(\mathbb{R}^3))_H$. However, we have the following result which characterizes the σ -ideals $(\mathcal{S}_+(\mathbb{R}^2))_I, (\mathcal{S}(\mathbb{R}^3))_I$.

Proposition 5.5. (i) $(\mathcal{S}_+(\mathbb{R}^2))_I = (\mathcal{A}_1^+(\mathbb{R}^2))_I = (\mathcal{B}_1^+(\mathbb{R}^2))_I$.

- (ii) $(\mathcal{S}(\mathbb{R}^3))_I = (\mathcal{A}_2(\mathbb{R}^3))_I = (\mathcal{B}_2(\mathbb{R}^3))_I$.

PROOF. The statement (i) immediately follows from Theorem 3.3, Lemma 2.8 (which shows that $\mathcal{A}_1^+(\mathbb{R}^2) = \mathcal{B}_1^+(\mathbb{R}^2)$) and Lemma 4.5 (which implies that for each $M \in \mathcal{B}_1^+(X)$ there exists a Lipschitz function on \mathbb{R}^2 such that $M \subset S_+(f)$).

Similarly (ii) follows immediately from Theorem 3.6, Lemma 2.9 and Lemma 4.9. \square

We have a complete answer to Problem C in no other case. However, our results yield some partial answers which are based on the following proposition.

Proposition 5.6. *Let X be a separable Banach space and $M \subset X$ be a G_δ set. Then the following assertions hold.*

- (i) *If there exists a set $D \subset \overline{M}$ dense in \overline{M} such that $\text{int}(\text{contg}(M, x)) \neq \emptyset$ for each $x \in D$, then $M \notin (\tilde{\mathcal{S}}_+(X))_I$.*
- (ii) *If $\text{int}(\text{contg}(M, x)) \neq \emptyset$ for each $x \in M$, then $M \notin (\mathcal{S}_+(X))_I$.*

PROOF. To prove (i), suppose on the contrary that $M \in (\tilde{\mathcal{S}}_+(X))_I$. By Theorem 3.3 there exist closed sets $F_n \in \mathcal{A}_1^+$, $n = 1, 2, \dots$, such that $M \subset \bigcup_{n=1}^\infty F_n$. Using Remark 3.4 (cf. also Remark 2.12) we have also that $\text{int}(\text{contg}(F_n, x)) = \emptyset$ for each $x \in F_n$. Since M is a Baire space, there exists an open set $G \subset X$ and $n \in \mathbb{N}$ such that $\emptyset \neq G \cap M \subset F_n$. Choose a point $d \in G \cap D$. By the assumptions of (i) we have $\text{int}(\text{contg}(M, d)) \neq \emptyset$. Consequently also $\text{int}(\text{contg}(F_n, d)) \neq \emptyset$, which is a contradiction.

It seems that (ii) cannot be proved directly by the Baire category theorem, but a proof similar to the proof of the Baire category theorem can be given. This phenomenon is frequent in proofs which show that a “small” set is not σ -porous. Actually, we will show that (ii) is an easy consequence of well-known facts on “abstract porosity”.

We will say that $A \subset X$ is (c) -porous at $a \in X$ if $\text{int}(\text{contg}(A, a)) = \emptyset$. A set is said to be (c) -porous if it is (c) -porous at each its point. A set is said to be σ - (c) -porous if it is a countable union of (c) -porous sets.

Now we will use the generalized Foran lemma for G_δ -sets which is proved in [Z6]. It works with the notion of abstract porosity; we shall formulate this lemma here only in the special case of (c) -porosity. (It is clear that (c) -porosity satisfies axioms of “abstract porosity”. Namely: it is a local notion and if A is (c) -porous at x , then also each subset of \overline{A} is porous at x .)

We will say that a non-empty system \mathcal{F} of non-empty G_δ -subsets of a Banach space X is a generalized Foran system (for (c) -porosity) if the following condition holds.

If $A \in \mathcal{F}$, H is open and $A \cap H \neq \emptyset$, then there exists $A^* \in \mathcal{F}$ such that $A^* \subset A \cap H$ and $A \cap H$ is (c) -porous at no point of A^* .

The generalized Foran lemma (for (c) -porosity) is the following result.

Let X be a Banach space and let \mathcal{F} be a generalized Foran system of G_δ subsets of X . Then \mathcal{F} contains no σ - (c) -porous set.

Now suppose that (ii) does not hold. In the above terminology, M is (c) -porous at no of its point and $M \in (\mathcal{S}_+(X))_I$. Using Theorem 3.3 and Remark 3.4 (cf. also Remark 2.12) we obtain that M is σ - (c) -porous. It is easy to see that $\mathcal{F} := \{M \cap G : G \text{ is open and } M \cap G \neq \emptyset\}$ is a Foran system. Thus the generalized Foran lemma implies that M is not σ - (c) -porous which is a contradiction. \square

In the subsequent proposition we will use the following notation.

If X is a Banach space, then we denote by \mathcal{L} the system of all Lipschitz surfaces in X of codimension 1 and by \mathcal{L}_+ the system of all unilaterally smooth Lipschitz surfaces in X of codimension 1.

Proposition 5.7. *Let X be a separable Banach space, $\dim(X) \geq 2$. Then*

$$(\tilde{\mathcal{S}}_+(X))_I \subset (\mathcal{S}_+(X))_I, (\mathcal{L}_+)_I \subset (\mathcal{S}_+(X))_H, (\mathcal{S}_+(X))_I \subset (\mathcal{L})_I$$

and all these inclusions are proper.

PROOF. The inclusion $(\tilde{\mathcal{S}}_+(X))_I \subset (\mathcal{S}_+(X))_I$ is trivial. To prove that it is proper, by Proposition 5.6 (i) it is sufficient to find a G_δ set $M \in (\mathcal{S}_+(X))_H$ and a dense subset D of \overline{M} such that

$$\text{int}(\text{contg}(M, x)) \neq \emptyset \text{ for each } x \in D. \quad (5.1)$$

To this end find closed subspaces W, Y, Z, V of X such that $\dim W = 1$, $X = Y \oplus W$, $\dim Z = 1$ and $Y = Z \oplus V$. Choose $0 \neq w \in W$, $0 \neq z \in Z$ and a dense countable subset C of \mathbb{R} . It is not difficult to construct (see the proof of Proposition 5.8 below for a more general construction) a Lipschitz function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable at all points of $\mathbb{R} \setminus C$ and

$$\varphi'_+(x) \text{ exists at no point } x \in C. \quad (5.2)$$

If $y \in Y$ is of the form $y = \alpha z + v$, where $\alpha \in \mathbb{R}$ and $v \in V$, put $h(y) := \varphi(\alpha)w$. Further put

$$\begin{aligned} M &:= \{y + h(y) : y = \alpha z + v, v \in V, \alpha \in \mathbb{R} \setminus C\}, \text{ and} \\ D &:= \{y + h(y) : y = \alpha z + v, v \in V, \alpha \in C\}. \end{aligned}$$

Clearly M is a G_δ set, $M \in \mathcal{B}_1(X)$ and D is dense in \overline{M} . By Lemma 4.5 we have $M \in (\mathcal{S}_+(X))_H$. Since it is not difficult to see that (5.2) implies (5.1), we are done.

Theorem 4.7 immediately implies the inclusion $(\mathcal{L}_+)_I \subset (\widetilde{\mathcal{S}}_+(X))_H$. Since the inclusion $(\widetilde{\mathcal{S}}_+(X))_I \subset (\mathcal{S}_+(X))_I$ is proper, clearly the inclusion $(\widetilde{\mathcal{S}}_+(X))_H \subset (\mathcal{S}_+(X))_H$ is also proper and therefore the inclusion $(\mathcal{L}_+)_I \subset (\mathcal{S}_+(X))_H$ is proper as well.

The inclusion $(\mathcal{S}_+(X))_I \subset (\mathcal{L})_I$ follows from Theorem 3.3. To show that it is proper, we will need a function $\tilde{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\tilde{\varphi}$ is Lipschitz and there exists a dense G_δ set $P \subset \mathbb{R}$ such that $\tilde{\varphi}'_+(x)$ exists at no point $x \in P$.

The existence of such a function is well-known (for a well-known easy construction, see [Br]). Now let W, Y, Z, V, w, z, h be defined as above, but now using the function $\tilde{\varphi}$ instead of φ . Put $\widetilde{M} = \{y + h(y) : y = \alpha z + v, \alpha \in P, v \in V\}$. Of course, \widetilde{M} is a subset of a Lipschitz surface and thus $\widetilde{M} \in (\mathcal{L})_H \subset (\mathcal{L})_I$. On the other hand, \widetilde{M} is clearly a G_δ set and it is not difficult to verify that $\text{int}(\text{contg}(\widetilde{M}, x)) \neq \emptyset$ for each $x \in \widetilde{M}$. Consequently Proposition 5.6 (ii) implies $\widetilde{M} \notin (\mathcal{S}_+(X))_I$. \square

In the proof of the following proposition we will use the following notation. If W is a Banach space, $f : \mathbb{R} \rightarrow W$ and $a \in \mathbb{R}$, then we put

$$\mathcal{D}^+ f(a) := \{w \in W : \exists_{h_n \searrow a} \frac{f(a + h_n) - f(a)}{h_n} \rightarrow w\}.$$

It is easy to see that $\mathcal{D}^+ f(a)$ is always a closed set.

Proposition 5.8. *Let X be a separable Banach space, $\dim X \geq 2$, and let $X = Y \oplus W$, where $\dim Y = 1$ and W is a closed linear subspace of X . Then there exists a Lipschitz function $h : Y \rightarrow W$ such that the “Lipschitz curve” $L := \{y + h(y) : y \in Y\}$ does not belong to the σ -ideal $(\widetilde{\mathcal{S}}_+(X))_I$.*

PROOF. Choose $w_0 \in W$, $\|w_0\| = 1$ and a sequence $(w_n)_{n=1}^\infty$ of vectors from the ball $B(w_0, 1/2)$ which is dense in this ball. Further choose a sequence $a_n \searrow 0$ such that $a_{n+1}/a_n \rightarrow 0$. Let $p : \mathbb{R} \rightarrow W$ be the function with the following properties:

- (i) $p(t) = w_n$ for $t \in [a_{2n}, a_{2n-1}]$, $n \in \mathbb{N}$;
- (ii) p is affine on each interval $[a_{2n+1}, a_{2n}]$, $n \in \mathbb{N}$;
- (iii) $p(t) = 0$ for $t \in (-\infty, 0]$ and $p(t) = w_1$ for $t \in [a_1, \infty]$.

Clearly, p is bounded. Put $P(x) := \int_0^x p(t) dt$. It is easy to show that $\mathcal{D}^+P(0) = \overline{B(w_0, 1/2)}$. Now choose a sequence $(t_n)_{n=1}^\infty$ dense in \mathbb{R} and put $g(t) := \sum_{k=1}^\infty 2^{-k} p(t - t_k)$ and $G(x) := \int_0^x g(t) dt$. Thus, G is Lipschitz. For each $n \in \mathbb{N}$, put

$$G_n(x) := \int_0^x 2^{-n} p(t - t_n) dt, \quad \tilde{G}_n(x) := \int_0^x \sum_{k \neq n} 2^{-k} p(t - t_k) dt.$$

It is easy to see that $\tilde{G}'_n(t_n)$ exists,

$$(\mathcal{D}^+G_n(t_n)) = 2^{-n}\mathcal{D}^+P(0) = 2^{-n}\overline{B(w_0, 1/2)}$$

and consequently

$$\text{int}(\mathcal{D}^+G(t_n)) \neq \emptyset. \quad (5.3)$$

Now choose $y_0 \in Y$, $y_0 \neq 0$, define the mapping $h : Y \rightarrow W$ by $h(ty_0) := G(t)$, put $L := \{y + h(y) : y \in Y\}$ and $z_n := t_n y_0 + h(t_n y_0)$. Then h is Lipschitz. It is easy to show that (5.3) implies $\text{int}(\text{contg}(L, z_n)) \neq \emptyset$. Since $(z_n)_{n=1}^\infty$ is dense in L it is sufficient to apply Proposition 5.6 (i) (with $M := L$ and $D := \{z_n : n \in \mathbb{N}\}$) to see $L \notin (\tilde{\mathcal{S}}_+(X))_I$. \square

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