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CONSTRUCTION OF BOREL INSEPARABLE COANALYTIC SETS

Abstract

We prove general results from which large families of pairwise disjoint, Borel inseparable, complete coanalytic sets can be obtained. The elements of such families are naturally indexed by $2^{\aleph_0}$, ω_1 or the classes of a Borel complete equivalence relation. We also give some concrete examples of such families in analysis, topology etc.

Introduction

A classical theorem of Lusin states that every two disjoint analytic sets in a Polish space can be separated by a Borel set, i.e. there is a Borel set which contains one but does not intersect the other. It is also a classical result that there exist two disjoint coanalytic sets which can not be separated by a Borel set. One such basic example in modern terminology is set WF, the set of all well-founded trees, and set UB, the set of all trees with a unique infinite branch. Sets WF and UB are complete coanalytic sets which can not be separated by a Borel set (see [Ke95]). We will say two coanalytic sets are Borel inseparable if these sets are disjoint and there is no Borel set which contains one but misses the other.

Several examples of pairs of inseparable coanalytic sets can be found in the literature. Some of the earliest examples were given by Novikov in [No31] and Sierpiński in [Si31]. Maitra in [Mai74] gives a pair of inseparable coanalytic

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sets using a game theoretic construction. Other families of Borel inseparable coanalytic sets have been studied by Becker in [Be86] and by Kechris and the first author in [CK00]. There are many natural examples of coanalytic complete sets in analysis, topology, etc. Becker presents a general procedure which shows how to obtain a pair of Borel inseparable coanalytic sets from one of these natural coanalytic complete sets. Kechris and the first author obtained a large and a natural collection of pairwise Borel inseparable coanalytic sets. This collection is large in the sense of cardinality as well as definability. We will discuss this result in more detail in the first section.

In this paper we prove three general combinatorial results each of which gives rise to a large class of pairwise disjoint Borel inseparable complete coanalytic sets. Two of our main tools are the facts that UB and WF are Borel inseparable complete coanalytic sets and the fact that $\mathrm{UB}\times\mathrm{WF}$ and $\mathrm{WF}\times\mathrm{UB}$ are Borel inseparable complete coanalytic sets. The first fact is more or less due to Becker [Be86] and its proof can be found in [Ke95]. The second fact is due Kechris and the first author, proof of which can be found in [CK00]. We use our combinatorial results to give examples of large collections of Borel inseparable coanalytic sets in analysis, topology, etc. This is done in the spirit of work of Becker [Be86]. The basic idea is along the following line. Suppose $\mathcal{G}$ is a family of pairwise Borel inseparable, complete coanalytic subsets of Tr. Assume $f : \mathrm{Tr} \to X$ is a Borel construction assigning to each tree T an object $f(T)$ of some Polish space X. Suppose further that, for each $G \in \mathcal{G}$, there is a significant coanalytic set $C_G \subseteq X$ such that $G = f^{-1}(C_G)$ and $C_G \cap C_{G'} = \emptyset$ for $G \neq G'$. Then $\{C_G\}_{G\in\mathcal{G}}$ is a family of pairwise Borel inseparable, complete coanalytic sets too.

Our first result in this paper is that if K, L are nonhomeomorphic compact subsets of $\mathbb{N}^{\mathbb{N}}$, then $\mathcal{T}_K$, $\mathcal{T}_L$, sets of trees whose body is homeomorphic to K and L respectively, are complete coanalytic sets which are Borel inseparable. Of course, if we let K be the empty set and L be a singleton set then our result reduces to the classical result.

Our second result is that if α, β are two distinct countable ordinals, then $\mathcal{V}_\alpha$, $\mathcal{V}_\beta$, sets of trees whose body is order isomorphic to α and β respectively, are complete coanalytic sets which are Borel inseparable. Again, this is a generalization of inseparability of WF and UB but in a somewhat different direction.

Our third main result is that if $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ are disjoint coanalytic sets each containing a closed copy of $\mathbb{N}^{\mathbb{N}}$, then UB_A and UB_B, sets of all trees in UB whose body is contained in A and B respectively, are complete coanalytic sets which are Borel inseparable and are Borel inseparable from WF. Again, this includes the classical result.

Let us remark here that each of these results give rise to a large family of pairwise disjoint complete coanalytic sets which are pairwise Borel inseparable. Let $\mathcal{G}_1$ be the set of all $\mathcal{T}_K$ where K ranges over all compact subsets of $\mathbb{N}^{\mathbb{N}}$. Then $\mathcal{G}_1$ is such a family of size continuum as there are continuum many nonhomeomorphic compact subsets of $\mathbb{N}^{\mathbb{N}}$. In addition, family $\mathcal{G}_1$ is large in another sense. It is indexed by the classes of a Borel complete equivalence relation, namely the equivalence relation on $\boldsymbol{K}(\mathbb{N}^{\mathbb{N}})$ formed by homeomorphism. This will be discussed in more detail in Section 1. Similarly, letting $\mathcal{G}_2$ be the set of $\mathcal{V}_\alpha$ where α ranges over all countable ordinals, we obtain such a family of size ω_1. As $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, we can write $\mathbb{N}^{\mathbb{N}}$ as the union of continuum many pairwise disjoint closed sets each one homeomorphic to $\mathbb{N}^{\mathbb{N}}$. If we let $\mathcal{I}$ be such a family, then $\mathcal{G}_3$, the set of all UB_A such that $A \in \mathcal{I}$, is another such family.

In Section 4 we show how to use our results to obtain large collections of pairwise Borel inseparable coanalytic sets in various spaces. This is done in the spirit of aforementioned work of Becker.

1 Families of trees with compact body

Our main result in this section is Theorem 1.4. From Theorem 1.4 we obtain the above mentioned collection $\mathcal{G}_1$. We shortly discuss definable cardinality before we proceed with proofs.

Let us go back for a moment to the example of [CK00]. The family $\{\mathcal{U}_G\}_G$ presented there is naturally indexed, in an injective way, by the isomorphism classes of countable groups; indeed, each member $\mathcal{U}_G$ of the family is the class of countable graphs (or countable structures for a suitable language L) whose automorphism group is isomorphic to the countable group G. Thus there are, of course, $2^{\aleph_0}$ many of them; in other words, there is a bijection between $\mathbb{R}$ and the index set, which is the quotient set of countable groups modulo isomorphism. However, if we look at the definable cardinality, that is allowing only restricted classes of functions — for example Borel ones — to compute the cardinality of a space, things change. Indeed, let Gr be the class of countable groups and $\simeq_{\mathrm{Gr}}$ be the relation of isomorphism in Gr. There is no injection $\mathrm{Gr}/\simeq_{\mathrm{Gr}} \to \mathbb{R}$ admitting a Borel lifting $\mathrm{Gr} \to \mathbb{R}$. Moreover, by [Me81], $\simeq_{\mathrm{Gr}}$ is *Borel complete*, or S_∞*-universal*. This means that, if $\simeq_{\mathcal{C}}$ is the equivalence relation of isomorphism on a class $\mathcal{C}$ of countable structures, there is a Borel function $f : \mathcal{C} \to \mathrm{Gr}$ such that

$$\forall C, C' \in \mathcal{C}\ (C \simeq_{\mathcal{C}} C' \Leftrightarrow f(C) \simeq_{\mathrm{Gr}} f(C'));$$

in symbols: $\simeq_{\mathcal{C}} \leq_B \simeq_{\mathrm{Gr}}$. For a treatment of this theory, see [Ke].

So, we can say that the quotient space $\mathrm{Gr}/\simeq_{\mathrm{Gr}}$ has the biggest possible *Borel cardinality* among classes of countable structures up to isomorphism. Consequently, we should regard the family $\{\mathcal{U}_G\}_G$ as very complicated, since the natural index set is.

The relation of homeomorphism $\cong_{\mathbb{N}^\mathbb{N}}$ between compact subsets of the Baire space is also S_∞-universal. Indeed, in [CG01] it is shown that the homeomorphism relation $\cong_{2^\mathbb{N}}$ between compact subsets of the Cantor space is S_∞-universal. The natural inclusion $2^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N}$ shows that $\cong_{2^\mathbb{N}} \leq_B \cong_{\mathbb{N}^\mathbb{N}}$, while, embedding $\mathbb{N}^\mathbb{N}$ in $2^\mathbb{N}$, we get $\cong_{\mathbb{N}^\mathbb{N}} \leq_B \cong_{2^\mathbb{N}}$. Moreover, this shows that the equivalence classes of $\cong_{\mathbb{N}^\mathbb{N}}$ are Borel, since the equivalence classes of $\cong_{2^\mathbb{N}}$ are. Hence, we have that $\mathcal{G}_1$ is complicated in definable sense as well as being large in the cardinality sense. Now we proceed with proofs.

We denote by $\cong$ the relation of homeomorphism between topological spaces (so $\cong_{\mathbb{N}^\mathbb{N}}$ and $\cong_{2^\mathbb{N}}$ are restrictions of it). We use $\boldsymbol{F}(X)$ and $\boldsymbol{K}(X)$ to denote all the closed and compact subsets of a Polish space X, respectively. $\boldsymbol{K}(X)$ endowed with the Hausdorff metric is a Polish space; $\boldsymbol{F}(X)$ endowed with the Effros Borel structure is a standard Borel space. If X is a standard Borel space, $\mathbf{B}(X)$ is the family of Borel subsets of X and $\mathbf{\Pi}^1_1(X)$ is the family of coanalytic subsets of X. We denote by Tr the Polish space of trees on $\mathbb{N}$. For T a tree and n a natural number, $[T]$ is the body of T (that is the set of infinite branches of T) and $\mathrm{Lev}_n(T)$ is the n-th level in T.

For $K \in \boldsymbol{K}(\mathbb{N}^\mathbb{N})$ let

$$\mathcal{T}_K = \{T \in \mathrm{Tr} \mid [T] \cong_{\mathbb{N}^\mathbb{N}} K\}$$

.

The first thing we want to check is that each $\mathcal{T}_K$ is coanalytic. We have indeed the following fact.

Theorem 1.1. *For any non-empty Borel $B \subseteq \boldsymbol{K}(\mathbb{N}^\mathbb{N})$, the set $\mathcal{R}_B = \{T \in \mathrm{Tr} \mid [T] \in B\}$ is a complete coanalytic subset of* Tr.

The part of Theorem 1.1 asserting coanalyticity of $\mathcal{R}_B$ is in fact a special case of the following known result, which follows using effective descriptive set theory or game theoretic arguments. We thank A.S. Kechris for pointing this out to us.

Theorem 1.2. *Let X be a standard Borel space, Y a Polish space and let $A \in \mathbf{B}(X \times Y)$. Fix also $B \in \mathbf{B}(\boldsymbol{K}(Y))$. Then $\{x \in X \mid A_x \in B\}$ is a coanalytic subset of X, where $A_x = \{y \in Y \mid (x, y) \in A\}$.*

However, we give here an independent self-contained proof for Theorem 1.1 using classical descriptive set theory.

Lemma 1.1. *Let X be a standard Borel space, Y a Polish space and $A \subseteq X \times Y$ be Borel. Then $\{x \in X \mid A_x \in \boldsymbol{F}(Y)\}$ and $\{x \in X \mid A_x \in \boldsymbol{K}(Y)\}$ are coanalytic.*

Proof. Fix a countable basis $\mathcal{B} = \{U_n\}_{n\in\mathbb{N}}$ of Y. For $x \in X$, we have

$$A_x \in \boldsymbol{F}(Y) \Leftrightarrow \forall y \in Y\ (\forall n \in \mathbb{N}\ (y \in U_n \Rightarrow U_n \cap A_x \neq \emptyset) \Rightarrow y \in A_x).$$

Now fix also a compatible complete metric in Y. For $x \in X$,

$$\begin{aligned} &A_x \in \boldsymbol{K}(Y) \Leftrightarrow A_x \in \boldsymbol{F}(Y)\wedge \\ &\wedge \forall \varepsilon \in \mathbb{Q}^+\ \exists U_{j_0}, \dots, U_{j_n} \in \mathcal{B}\ (\forall h \leq n\ \mathrm{diam}(U_{j_h}) < \varepsilon \wedge A_x \subseteq \bigcup_{k=0}^{n} U_{j_k}). \end{aligned}$$

□

Lemma 1.2. *Let*

$$C_0 = \{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}}) \setminus \{\emptyset\}\}$$

be the set of trees with compact non-empty body. Then $C_0 \in \mathbf{\Pi}^1_1(\mathrm{Tr})$.

Proof. We use a game theoretic approach similar to the one used in the proof of coanalyticity of UB in [Ke95, theorem 18.11].

Given $h \in \mathbb{N}$ and finite sets $A \subseteq \mathbb{N}^h, B \subseteq \mathbb{N}^{h+1}$, we say that A is *below* B, or B is *above* A, and we write $A \prec B$, if every element of A has some extension in B and every element of B is the extension of some element in A.

For any tree T on $\mathbb{N}$, let G_T be the following game:

$$\begin{array}{lcccccccccc} \mathrm{I} & n_0 & & x(0) & & x(1) & & x(2) & & \\ \mathrm{II} & & y(0) & & y(1) & & y(2) & & y(3) & \cdots \end{array}$$

where $n_0 \in \mathbb{N}$ and, for $h \in \mathbb{N}$, $x(h), y(h)$ are non-empty finite subsets of $\mathbb{N}^{h+1}$. Of course, with a suitable coding, each move can be thought of as a natural number.

Player I wins this run of the game if and only if either $y(h+1)$ is not above $y(h)$ for some $h \in \mathbb{N}$ or the following conditions are all satisfied:

(1) $\forall h \in \mathbb{N}\ x(h) \prec x(h+1)$;

(2) $\forall h \in \mathbb{N}\ (y(h) \subseteq T \Rightarrow x(h) \subseteq T)$;

(3) $\exists h < n_0\ x(h) \not\subseteq y(h)$.

Let L_∞ be the set of trees containing sequences of arbitrarily long length (this is a Borel subset of Tr).

The claim is that, for $T \in L_\infty$,

$$T \in \mathrm{WF} \vee [T] \notin \boldsymbol{K}(\mathbb{N}^{\mathbb{N}}) \Leftrightarrow \text{I has a winning strategy in } G_T.$$

Assuming this, let $W \subseteq \mathrm{Tr}^2$ be defined by

$$(\sigma, T) \in W \Leftrightarrow \sigma \text{ is a winning strategy for I in } G_T.$$

Then $W \in \mathbf{B}(\mathrm{Tr}^2)$ and, for $T \in \mathrm{Tr}$,

$$[T] \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}}) \setminus \{\emptyset\} \Leftrightarrow T \in L_\infty \wedge \neg \exists \sigma \in \mathrm{Tr}\ (\sigma, T) \in W.$$

To prove the claim, suppose first $[T] \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}}) \setminus \{\emptyset\}$. Let T^* be a pruned tree, with $[T^*] = [T]$. Then player II wins by playing the levels of T^*, which are finite since $[T^*]$ is compact.

Assume now $[T]$ is not compact and let again T^* be pruned, with $[T^*] = [T]$. Let n_0 be least such that $\mathrm{Lev}_{n_0}(T^*)$ is infinite. Player I starts by playing n_0. Then, independently of what player II does, he plays $x(0) = \mathrm{Lev}_1(T^*), \ldots, x(n_0-2) = \mathrm{Lev}_{n_0-1}(T^*)$. For any finite subset $y(n_0-1) \subseteq \mathbb{N}^{n_0}$, there is a finite subset $x(n_0-1) \subseteq \mathrm{Lev}_{n_0}(T^*)$ — which is above $x(n_0-2)$ if $n_0 > 1$ — such that $x(n_0-1) \not\subseteq y(n_0-1)$. So, if II plays $y(n_0-1)$, player I plays this $x(n_0-1)$ and then continues by playing finite subsets $x(n_0) \subseteq \mathrm{Lev}_{n_0+1}(T^*), x(n_0+1) \subseteq \mathrm{Lev}_{n_0+2}(T^*), \ldots$ each one above the preceding one.

Finally, suppose $T \in \mathrm{WF}$. Let ρ be the rank function associated to T. Since $T \in L_\infty$, we have $\rho(\emptyset) \geq \omega$. So there are a limit ordinal λ and a natural number n such that $\rho(\emptyset) = \lambda + n$. Player I starts by playing $n_0 = n+1$.

A position $(n_0, y(0), x(0), \ldots, y(k), x(k))$, with $k < n_0$, is *decisive* if, for all $h \leq k$, $x(h) = \{p_h\}$ is a singleton, $1 \leq h \leq k \Rightarrow x(h-1) \prec x(h)$, and one of the following holds:

(A) $k \geq 1$ and $y(k-1) \not\prec y(k)$;

(B) (A) does not hold and $y(k-1), x(k-1) \subseteq T$ (if $k > 0$), $x(k) \not\subseteq y(k) \not\subseteq T$;

(C) (A) does not hold and $y(k), x(k) \subseteq T$, $\max \rho(y(k)) < \rho(p_k)$ (in particular, $x(k) \not\subseteq y(k)$).

If player I can reach a decisive position then he can win the run of the game as follows:

- in cases (A) and (B) he plays arbitrarily singletons $x(k+1) = \{p_{k+1}\} \prec x(k+2) = \{p_{k+2}\} \prec \ldots$, with $x(k) \prec x(k+1)$ too;

- in case (C) he plays singletons $x(k+1) = \{p_{k+1}\} \prec x(k+2) = \{p_{k+2}\} \prec \ldots$ in such a way that $x(k) \prec x(k+1)$ and $\forall m > k\ (y(k) \prec y(k+1) \prec \ldots \prec y(m) \wedge y(m) \subseteq T \Rightarrow p_m \in T \wedge \max \rho(y(m)) \leq \rho(p_m))$.

So it is enough to show that I can reach a decisive position. Let $y(0)$ be the first move of II. If $y(0) \not\subseteq T$ then, playing any singleton $x(0) \not\subseteq y(0)$, I reaches a decisive position. So suppose $y(0) \subseteq T$. If there is $p_0 \in \mathbb{N}$ such that $p_0 \in T$ and $\rho(p_0) > \max \rho(y(0))$, then I reaches a decisive position playing $x(0) = \{p_0\}$. Otherwise it must be that $n > 0$ and $\max \rho(y(0)) = \lambda + n - 1$; player I plays any singleton $x(0) = \{p_0\}$ with $p_0 \in T, \rho(p_0) = \lambda + n - 1$. Let $y(1)$ be II's next move. If $y(1)$ is not above $y(0)$ or $y(1) \not\subseteq T$, then any singleton $x(1)$ above $x(0)$, with $x(1) \not\subseteq y(1)$ provides a decisive position for player I. If $y(1)$ is above $y(0)$ and $y(1) \subseteq T$, there are again two cases. If there exists $x(1) = \{p_1\} \subseteq T$ above $x(0)$, with $\rho(p_1) > \max \rho(y(1))$, then we are done. Otherwise $n > 1$ and $\max \rho(y(1)) = \lambda + n - 2$. Again, player I plays some singleton $x(1) = \{p_1\}$, with $p_0 \subseteq p_1 \in T, \rho(p_1) = \lambda + n - 2$.

Continuing this way, if by level $n-1$ player I has not reached a decisive position — the current position being $(n_0, y(0), x(0), y(1), \ldots, y(n-1), x(n-1))$ — we have $x(n-1) = \{p_{n-1}\}$, $y(n-1) \subseteq T$ and $\max \rho(y(n-1)) = \rho(p_{n-1}) = \lambda$. Then, if $y(n)$ is above $y(n-1)$ and $y(n) \subseteq T$, we have $\max \rho(y(n)) < \lambda$, so there exists $p_n \in T, p_{n-1} \subseteq p_n$, with $\max \rho(y(n)) < \rho(p_n) < \lambda$ and we are done. $\square$

In the next lemma we introduce a construction that will also come handy later.

Lemma 1.3. *Let M be a closed subset of the Baire space $\mathbb{N}^{\mathbb{N}}$. Then there is a continuous function $F : \mathrm{Tr} \to \mathrm{Tr}$ such that:*

(1) *if T is a well founded tree then $F(T)$ is well founded;*

(2) *if T is a tree with a unique infinite branch then $[F(T)]$ and M are homeomorphic and order isomorphic under the lexicographical ordering;*

(3) *if T has more than one infinite branch, then $[F(T)]$ contains a closed subset homeomorphic to M.*

Proof. Fix a tree V such that $[V] = M$ and let $T \in \mathrm{Tr}$ with the aim to define $F(T)$.

Of course, $\mathrm{Lev}_0(F(T)) = \{\emptyset\}$. Let $\mathrm{Lev}_1(F(T)) = \mathrm{Lev}_1(T)$ and, for each $u \in \mathrm{Lev}_1(F(T))$, put $\tilde{u} = u \in \mathrm{Lev}_1(T)$. To define $\mathrm{Lev}_2(F(T))$, for every $u \in \mathrm{Lev}_1(F(T))$ and every $n \in \mathbb{N}$ let $u^\frown n \in \mathrm{Lev}_2(F(T))$ if and only if $(n) \in \mathrm{Lev}_1(V)$; if $u^\frown n \in \mathrm{Lev}_2(F(T))$ let also $(u^\frown n)\tilde{} = (n) \in \mathrm{Lev}_1(V)$.

For $h \in \mathbb{N}$, assume $\mathrm{Lev}_{2h+1}(F(T))$ and $\mathrm{Lev}_{2h+2}(F(T))$ have been constructed. Moreover suppose that for each $u \in \mathrm{Lev}_{2h+1}(F(T))$ an element $\tilde{u} \in \mathrm{Lev}_{h+1}(T)$ has been defined and, similarly, to each $v \in \mathrm{Lev}_{2h+2}(F(T))$ we have associated an element $\tilde{v} \in \mathrm{Lev}_{h+1}(V)$. The aim is to define $\mathrm{Lev}_{2h+3}(F(T))$ and $\mathrm{Lev}_{2h+4}(F(T))$. Given any element $u^\frown n \in \mathrm{Lev}_{2h+2}(F(T))$ and any $m \in \mathbb{N}$, we let $u^\frown n^\frown m \in \mathrm{Lev}_{2h+3}(F(T))$ if and only if $\tilde{u}^\frown m \in T$; in such a case, put $(u^\frown n^\frown m)\tilde{} = \tilde{u}^\frown m$. Similarly, for $v^\frown n \in \mathrm{Lev}_{2h+3}(F(T)), m \in \mathbb{N}$, let $v^\frown n^\frown m \in \mathrm{Lev}_{2h+4}(F(T))$ if and only if $\tilde{v}^\frown m \in V$; then define $(v^\frown n^\frown m)\tilde{} = \tilde{v}^\frown m$.

The assignment $T \mapsto F(T)$ is continuous; moreover it has the desired properties. $\square$

We are now ready to give the proof of Theorem 1.1.

Proof of theorem 1.1. First we prove that $\mathcal{R}_B$ is coanalytic. We begin by checking the assertion for B a sub-basic open set of $\boldsymbol{K}(\mathbb{N}^\mathbb{N})$, that is a set of the form $B = \{K \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \mid K \subseteq U\}$ or $B = \{K \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \mid K \cap U \neq \emptyset\}$, where U is an open set in $\mathbb{N}^\mathbb{N}$. If $B = \{K \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \mid K \subseteq U\}$ we have, for $T \in \mathrm{Tr}$,

$$T \in \mathcal{R}_B \Leftrightarrow [T] \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \wedge \forall x \in \mathbb{N}^\mathbb{N}\ (x \in [T] \Rightarrow x \in U).$$

For the case $B = \{K \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \mid K \cap U \neq \emptyset\}$, let us consider first a basic open set $U = N_t = \{x \in \mathbb{N}^\mathbb{N} \mid t \subseteq x\}$, where $t \in \mathbb{N}^{<\omega}$. Letting then $T_t = \{u \in \mathbb{N}^{<\omega} \mid u \subseteq t \vee t \subseteq u\}$, the continuous function $\mathrm{Tr} \to \mathrm{Tr}, T \mapsto T \cap T_t$ reduces $\{T \in \mathrm{Tr} \mid [T] \cap U \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \setminus \{\emptyset\}\}$ to the set C_0 of trees with compact non-empty body and witnesses, by Lemma 1.2, the coanalyticity of $\{T \in \mathrm{Tr} \mid [T] \cap U \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \setminus \{\emptyset\}\}$. Since $\mathcal{R}_B = \{T \in \mathrm{Tr} \mid [T] \cap U \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \setminus \{\emptyset\}\} \cap \{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^\mathbb{N})\}$ and $\{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^\mathbb{N})\}$ is coanalytic by Lemma 1.1, $\mathcal{R}_B$ is coanalytic as well. If $U = \bigcup_{n \in \mathbb{N}} N_{t_n}$ is a countable union of basic open sets of $\mathbb{N}^\mathbb{N}$, then $\mathcal{R}_B = \{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \wedge [T] \cap U \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} \{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^\mathbb{N}) \wedge [T] \cap N_{t_n} \neq \emptyset\} \in \mathbf{\Pi}^1_1(\mathrm{Tr})$.

If $B = \bigcap_{h=1}^n B_h$ and each B_h is as above, then $\mathcal{R}_B = \bigcap_{h=1}^n \mathcal{R}_{B_h} \in \mathbf{\Pi}^1_1(\mathrm{Tr})$ and this takes care of B member of the basis generated by the above mentioned sub-basic sets.

Finally, if $B = \bigcup_{j \in J} B_j$, with J countable and each B_j a basic open set, then $\mathcal{R}_B = \bigcup_{j \in J} \mathcal{R}_{B_j} \in \mathbf{\Pi}^1_1(\mathrm{Tr})$ and this takes care of B an arbitrary open set.

If B is closed, then $B = \bigcap_{j \in J} B_j$, with J countable and each B_j open; so $\mathcal{R}_B = \bigcap_{j \in J} \mathcal{R}_{B_j} \in \mathbf{\Pi}^1_1(\mathrm{Tr})$.

Proceed now by induction on the Borel hierarchy.

To prove that $\mathcal{R}_B$ is $\mathbf{\Pi}^1_1$-hard, first apply Lemma 1.3 with $M = \mathbb{N}^{\mathbb{N}}$ to get a continuous function $F : \mathrm{Tr} \to \mathrm{Tr}$ such that if T is well founded then $F(T)$ is well founded and if T is ill founded then $[F(T)]$ contains a closed subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Fix now a tree T_B such that $[T_B] \in B$. For $T \in \mathrm{Tr}$ let $\Phi(T) = T_B \cup F(T)$. Then $\Phi : \mathrm{Tr} \to \mathrm{Tr}$ is a continuous function reducing WF to $\{T \in \mathrm{Tr} \mid [T] \in B\}$. $\square$

Now we continue towards the main goal of this section.

Theorem 1.3. *Let C and C' be non homeomorphic closed subsets of the Baire space. Then $\{T \in \mathrm{Tr} \mid [T] \cong C\}$ and $\{T \in \mathrm{Tr} \mid [T] \cong C'\}$ are Borel inseparable subsets of* Tr*.*

Proof. Apply Lemma 1.3 twice, with $M = C$ and $M = C'$ respectively, to get continuous functions $F_C, F_{C'} : \mathrm{Tr} \to \mathrm{Tr}$. For $T, T' \in \mathrm{Tr}$, let $\Psi(T, T')$ be obtained by joining $F_C(T)$ and $F_{C'}(T')$ to a common root. Then $\Psi : \mathrm{Tr}^2 \to \mathrm{Tr}$ is continuous. If $(T, T') \in \mathrm{WF} \times \mathrm{UB}$, then $[\Psi(T, T')]$ is homeomorphic with C'; if $(T, T') \in \mathrm{UB} \times \mathrm{WF}$, then $[\Psi(T, T')]$ is homeomorphic with C. Since $\mathrm{UB} \times \mathrm{WF}$ and $\mathrm{WF} \times \mathrm{UB}$ are Borel inseparable by [CK00, theorem 3], $\{T \in \mathrm{Tr} \mid [T] \cong C\}$ and $\{T \in \mathrm{Tr} \mid [T] \cong C'\}$ are Borel inseparable as well. $\square$

The results of this section allow us to state the following.

Theorem 1.4. *Let $K \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}})$. Then $\mathcal{T}_K$ is complete coanalytic and if $L \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}})$ is not homeomorphic to K, then $\mathcal{T}_K$ and $\mathcal{T}_L$ are Borel inseparable.*

Note that $\{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}})\}$ is $\mathbf{\Pi}^1_1$-hard, by applying Lemma 1.3 with $M = \mathbb{N}^{\mathbb{N}}$. From Theorem 1.4 we get the announced family $\mathcal{G}_1$.

Corollary 1.5. *The set $\{\mathcal{T}_K\}_{K \in \mathcal{S}}$ — where $\mathcal{S}$ is a transversal for the relation of homeomorphism on $\boldsymbol{K}(\mathbb{N}^{\mathbb{N}})$, that is $\mathcal{S}$ intersects each homeomorphism class of $\boldsymbol{K}(\mathbb{N}^{\mathbb{N}})$ in exactly one point — is a family of pairwise Borel inseparable, complete coanalytic subsets of* Tr *partitioning the complete coanalytic set $\{T \in \mathrm{Tr} \mid [T] \in \boldsymbol{K}(\mathbb{N}^{\mathbb{N}})\}$.*

We observe that the Borel inseparability statement of Theorem 1.3 holds for any closed sets. However, the following theorem shows that Theorem 1.4 and Corollary 1.5 do not extend to arbitrary closed sets since $\mathbf{\Sigma}^1_1$-hard sets cannot be coanalytic.

Theorem 1.6. *The set* $\{T \in \mathrm{Tr} \mid [T] \cong \mathbb{N}^{\mathbb{N}}\}$ *is a* $\mathbf{\Sigma}^1_1$*-hard subset of* Tr.

Proof. It is enough to reduce the set IF of ill founded trees to $\{T \in \mathrm{Tr} \mid [T] \cong \mathbb{N}^{\mathbb{N}}\}$. We do this with a more detailed analysis of the construction given in Lemma 1.3. Let $F : \mathrm{Tr} \to \mathrm{Tr}$ be the function built in Lemma 1.3 for $M = \mathbb{N}^{\mathbb{N}}$; we want to show $F^{-1}(\{T \in \mathrm{Tr} \mid [T] \cong \mathbb{N}^{\mathbb{N}}\}) = \mathrm{IF}$.

If $T \in \mathrm{WF}$ then $F(T) \in \mathrm{WF}$. So assume $T \in \mathrm{IF}$, in order to show that $[F(T)]$ is homeomorphic with $\mathbb{N}^{\mathbb{N}}$. By Lemma 1.3, $[F(T)] \neq \emptyset$, so it is enough to show that each non-empty basic clopen subset of $[F(T)]$ contains a closed subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and then apply Alexandrov-Urysohn theorem ([Ke95, theorem 7.7]). Let $t \in \mathbb{N}^{<\omega}$ be such that $N_t \cap [F(T)] \neq \emptyset$ and let $x \in [F(T)]$ with $t \subseteq x$. If $\forall n \in \mathbb{N}\ y(n) = x(2n)$, then $y \in [T]$. Since the tree V in the proof of Lemma 1.3 is now the complete tree $\mathbb{N}^{<\omega}$, we have that, for each $z \in \mathbb{N}^{\mathbb{N}}$, the sequence $(y(0), z(0), y(1), z(1), y(2), \ldots)$ is in $[F(T)]$. The set of such infinite sequences, for z ranging over all elements of $\mathbb{N}^{\mathbb{N}}$ such that $(y(0), z(0), y(1), z(1), y(2), \ldots)$ extends t, is a closed subspace of $N_t \cap [F(T)]$ homeomorphic to $\mathbb{N}^{\mathbb{N}}$. □

2 Families of trees with countable well ordered body

In this section we describe an example of a family of complete coanalytic, pairwise Borel inseparable sets whose natural index set is ω_1. Thus this family carries a natural well ordering.

For ξ a countable order type, let $\mathcal{V}_\xi$ be the set of trees whose body is countable and ordered in type ξ by the lexicographical ordering $<_{\mathrm{lex}}$ of $\mathbb{N}^{\mathbb{N}}$.

Lemma 2.1. *Every* $\mathcal{V}_\xi$ *is a coanalytic subset of* Tr.

Proof. For clarity we split the proof into two parts — for ξ finite and ξ infinite — giving a separate argument for ξ finite, though this is not strictly necessary.

So, assume first ξ is finite. If ξ is 0 or 1, then $\mathcal{V}_\xi$ is WF or UB. Thus, let $\xi \geq 2$. For $T \in \mathrm{Tr}$,

$$T \in \mathcal{V}_\xi \Leftrightarrow \exists! x \in (\mathbb{N}^{\mathbb{N}})^\xi\ \forall m < \xi\ (x(m) \in [T] \wedge (m+1 < \xi \Rightarrow x(m) <_{\mathrm{lex}} x(m+1))).$$

For ξ infinite, we use the same argument as in [CK00, theorem 9]. Let

$$B_{\aleph_0} = \{T \in \mathrm{Tr} \mid 0 \neq \mathrm{card}([T]) \leq \aleph_0\} \in \mathbf{\Pi}^1_1(\mathrm{Tr}).$$

Let $f : B_{\aleph_0} \to (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ be a $\mathbf{\Pi}^1_1$-measurable function such that $\forall T \in B_{\aleph_0}\ [T] = \{f(T)(n)\}_{n \in \mathbb{N}}$.

Let $L = \{<\}$ be a language consisting of one binary relation symbol and let $X_L = 2^{\mathbb{N}^2}$ be the space of (codes for) L-structures with universe $\mathbb{N}$. There is a Borel function $g : (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to X_L$ such that, if $z \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ has infinite range, then the structure $\mathcal{A}_{g(z)}$ coded by $g(z)$ is order isomorphic to $\{z(n)\}_{n\in\mathbb{N}}$ under the lexicographical ordering. Indeed, for $z \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, define $g(z)(k,l) = 1$ if and only if, after deleting repetitions, the k-th value enumerated by z is $<_{\text{lex}}$ the l-th value.

Then $gf : B_{\aleph_0} \to X_L$ is $\mathbf{\Pi}^1_1$-measurable and, for each $T \in B_{\aleph_0}$, if $[T]$ is infinite then $\mathcal{A}_{gf(T)}$ is order isomorphic to $[T]$. Since the subset of X_L of those structures that are total orderings of type ξ is Borel (being an isomorphism class), $\mathcal{V}_\xi$ is coanalytic as required. □

Now notice that $\forall\alpha \in \omega_1\ \mathcal{V}_\alpha \neq \emptyset$. In other words, each countable ordinal can be order preservingly embedded as a closed subset of $\mathbb{N}^{\mathbb{N}}$. This can be seen by induction. It is immediate for $\alpha = 0$. Assume the statement for some $\alpha \in \omega_1$; embedding α as a closed subset of the open basic set $N_{(0)} = \{x \in \mathbb{N}^{\mathbb{N}} \mid x(0) = 0\}$ and adding a point in $N_{(1)} = \{x \in \mathbb{N}^{\mathbb{N}} \mid x(0) = 1\}$, we get the assertion for $\alpha + 1$. Finally, let α be a limit ordinal. Write it as an infinite sum $\alpha = \sum_{n\in\mathbb{N}} \alpha_n$, with $\forall n \in \mathbb{N}\ \alpha_n < \alpha$; embed each α_n as a closed subset of $N_{(n)} = \{x \in \mathbb{N}^{\mathbb{N}} \mid x(0) = n\}$.

Lemma 2.2. *Let $\alpha \in \omega_1$. Then $\mathcal{V}_\alpha$ is a $\mathbf{\Pi}^1_1$-hard subset of* Tr.

Proof. Fix $T_\alpha \in \mathcal{V}_\alpha$. We shall define a continuous function $* : \mathrm{Tr} \to \mathrm{Tr}, T \mapsto T^*$ reducing WF to $\mathcal{V}_\alpha$. To this aim, simply join a copy of T_α and a copy of T to a common root, T_α to the left and T to the right. □

Using now Lemma 1.3 we get the desired inseparability result.

Lemma 2.3. *Let α, β be countable ordinals, with $\alpha \neq \beta$. Then $\mathcal{V}_\alpha$ and $\mathcal{V}_\beta$ are Borel inseparable.*

Proof. Assume $\alpha < \beta$ and let γ be such that $\alpha + \gamma = \beta$. It is enough to define a continuous function $* : \mathrm{Tr} \to \mathrm{Tr}, T \mapsto T^*$ such that

$$\begin{aligned} T \in \mathrm{WF} &\quad\Rightarrow\quad T^* \in \mathcal{V}_\alpha \\ T \in \mathrm{UB} &\quad\Rightarrow\quad T^* \in \mathcal{V}_\beta. \end{aligned}$$

Let M be a closed subset of $\mathbb{N}^{\mathbb{N}}$ order isomorphic to γ and let $F : \mathrm{Tr} \to \mathrm{Tr}$ be the function constructed in Lemma 1.3 for this M. Also, fix $T_\alpha \in \mathcal{V}_\alpha$.

Now, for $T \in \mathrm{Tr}$, let T^* be the tree defined by joining to a common root a copy of T_α to the left and a copy of $F(T)$ to the right. Thus if T is well founded $[T^*]$ has order type α; if T has a unique infinite branch $[T^*]$ has order type $\alpha + \gamma = \beta$. □

The results of this section are summarised in the following statements, from which we get family $\mathcal{G}_2$.

Theorem 2.1. *Let $\alpha, \beta \in \omega_1$ be two distinct ordinals. Then $\mathcal{V}_\alpha$ is complete coanalytic and $\mathcal{V}_\alpha, \mathcal{V}_\beta$ are Borel inseparable.*

Corollary 2.2. *The set $\{\mathcal{V}_\alpha\}_{\alpha\in\omega_1}$ is a family of pairwise Borel inseparable, complete conalytic subsets of* Tr.

3 Families of trees with one infinite branch

In this section we build a partition of the $\mathbf{\Pi}^1_1$-complete set UB of trees on $\mathbb{N}$ with a unique branch into $2^{\aleph_0}$ many complete coanalytic sets. The elements of this partition will be pairwise Borel inseparable.

For $A \subseteq \mathbb{N}^\mathbb{N}$, let

$$\mathrm{UB}_A = \{T \in \mathrm{Tr} \mid T \in \mathrm{UB} \wedge [T] \subseteq A\}.$$

Theorem 3.1. *Let $A \subseteq \mathbb{N}^\mathbb{N}$ be a non-empty coanalytic set; then UB_A is complete coanalytic. If $A, B \subseteq \mathbb{N}^\mathbb{N}$ are two disjoint coanalytic sets, each containing a closed subset of $\mathbb{N}^\mathbb{N}$ homeomorphic to $\mathbb{N}^\mathbb{N}$, then UB_A and UB_B are Borel inseparable complete coanalytic sets; in addition, each is Borel inseparable from* WF.

Proof. For $T \in \mathrm{Tr}$,

$$\begin{aligned} T \in \mathrm{UB}_A &\Leftrightarrow T \in \mathrm{UB} \wedge [T] \subseteq A \Leftrightarrow \\ &\Leftrightarrow T \in \mathrm{UB} \wedge \forall x \in \mathbb{N}^\mathbb{N}\ (x \in [T] \Rightarrow x \in A). \end{aligned}$$

Since A is coanalytic, this shows that UB_A is coanalytic as well.

To show completeness, let $V \in \mathrm{UB}_A$. Let $f : \mathrm{Tr} \to \mathrm{Tr}$ be the function granted by Lemma 1.3 for M a non-singleton closed subset of $\mathbb{N}^\mathbb{N}$. For $T \in \mathrm{Tr}$ define $\Phi(T) = V \cup f(T)$. Then $\Phi^{-1}(\mathrm{UB}_A) = \mathrm{WF}$.

Now suppose that there is $F \in \boldsymbol{F}(\mathbb{N}^\mathbb{N})$, $F \subseteq A$, with F homeomorphic to $\mathbb{N}^\mathbb{N}$ and let T_F be the tree of F.

First note that, for every $u \in T_F$, there is $n > \mathrm{length}(u)$ such that u has infinitely many extensions in T_F of length n. Otherwise, $N_u \cap F$ is compact clopen in F, contradicting Alexandrov-Urysohn theorem. This defines an embedding $g : \mathbb{N}^{<\omega} \to T_F$ such that, for every $v \in \mathbb{N}^{<\omega}$, all $g(v^\frown k)$ have the same length in T_F. To this aim, let $g(\emptyset) = \emptyset$. Then assume that, for some $v \in \mathbb{N}^{<\omega}$, $g(v)$ has been defined. Let $n_v > \mathrm{length}(g(v))$ be least such that $g(v)$ has infinitely many extensions of length n_v in T_F and, for $k \in \mathbb{N}$, let $g(v^\frown k)$ be the k-th of such extensions in the lexicographic order.

For $T \in \mathrm{Tr}$, let $f_A(T) \in \mathrm{Tr}$ be the subtree of T_F generated by $g(T)$, that is the smallest subtree of T_F containing $g(T)$. The function $f_A : \mathrm{Tr} \to \mathrm{Tr}$ is Borel: for $T \in \mathrm{Tr}$ we have

$$\forall u \in \mathbb{N}^{<\omega} \ (u \in f_A(T) \Leftrightarrow \exists v \in \mathbb{N}^{<\omega} \ (v \in T \wedge u \subseteq g(v))).$$

Moreover $f_A^{-1}(\mathrm{UB}_A) = \mathrm{UB}$ and $f_A^{-1}(\mathrm{WF}) = \mathrm{WF}$, showing that UB_A is Borel inseparable from WF.

Finally, if A, B are disjoint subsets of the Baire space, containing a closed subset homeomeorphic to $\mathbb{N}^{\mathbb{N}}$, let $\Phi : \mathrm{Tr}^2 \to \mathrm{Tr}, (T, T') \mapsto f_A(T) \cup f_B(T')$. Then

$$\begin{array}{l}(T, T') \in \mathrm{UB} \times \mathrm{WF} \Rightarrow \Phi(T, T') \in \mathrm{UB}_A \\ (T, T') \in \mathrm{WF} \times \mathrm{UB} \Rightarrow \Phi(T, T') \in \mathrm{UB}_B.\end{array}$$

By the inseparability of $\mathrm{UB} \times \mathrm{WF}$ and $\mathrm{WF} \times \mathrm{UB}$ we get the inseparability of UB_A and UB_B. □

Remark. Note that the hypothesis that A, B contain a closed subset of $\mathbb{N}^{\mathbb{N}}$ homeomorphic to $\mathbb{N}^{\mathbb{N}}$ cannot be dropped in the inseparability statement of theorem 3.1. If, for instance, A is σ-compact, then $\{T \in \mathrm{Tr} \mid [T] \cap A \neq \emptyset\} = \{T \in \mathrm{Tr} \mid \exists z \in A \ \forall n \in \mathbb{N} \ z \restriction n \in T\}$ is Borel and separates $\mathrm{UB}_A, \mathrm{UB}_B$.

To get family $\mathcal{G}_3$, note that it is possible to partition the Baire space into continuum many closed subspaces homeomorphic to itself. For example, recall that the Baire space is homeomorphic to its square and consider the family $\mathcal{F} = \{\{x\} \times \mathbb{N}^{\mathbb{N}}\}_{x \in \mathbb{N}^{\mathbb{N}}}$.

Corollary 3.2. *If $\mathcal{F}$ is a partition of the Baire space into continuum many closed subspaces homeomorphic to the Baire space, $\{\mathrm{UB}_F\}_{F \in \mathcal{F}}$ constitutes a partition of the $\mathbf{\Pi}^1_1$-complete set* UB *into $2^{\aleph_0}$ many complete coanalytic, Borel inseparable subsets.*

4 Applications

We show now how to apply the constructions given in the preceding sections to obtain several other examples of families of pairwise Borel inseparable, complete coanalytic sets. Suppose $\mathcal{G}$ is a family of pairwise Borel inseparable, complete coanalytic subsets of Tr — like the ones built so far. Assume $f : \mathrm{Tr} \to X$ is a Borel construction assigning to each tree T an object $f(T)$ of some Polish space X. Suppose further that, for each $G \in \mathcal{G}$, there is a significant coanalytic set $C_G \subseteq X$ such that $G = f^{-1}(C_G)$ and $C_G \cap C_{G'} = \emptyset$ for $G \neq G'$. Then $\{C_G\}_{G \in \mathcal{G}}$ is a family of pairwise Borel inseparable, complete coanalytic sets too. To illustrate this kind of argument we say a few more words about the results we are going to get using the family $\mathcal{G}_1 = \{\mathcal{T}_K\}_{K\in\mathcal{S}}$ obtained with Corollary 1.5. Analogous remarks apply also to the examples we can get using $\mathcal{G}_2$ and $\mathcal{G}_3$. In [Be86] the author provides a general procedure for producing pairs of disjoint Borel inseparable complete coanalytic sets, given natural examples of complete coanalytic sets. Such pairs take the form

$$\begin{aligned} A &= \{\text{points with no singularities}\} \\ B &= \{\text{points with one singularity}\}. \end{aligned}$$

These sets A and B are subsets of some Polish space X and are such that $\mathrm{WF} = f^{-1}(A), \mathrm{UB} = f^{-1}(B)$, where $f : \mathrm{Tr} \to X$ is a Borel function. Using $\mathcal{G}_1$, here we develop a similar procedure, which produces families of continuum many pairwise Borel inseparable, complete coanalytic sets. Such sets have the form

$$\begin{aligned} A_K \quad = \quad & \{\text{points whose set of singularities} \\ & \text{is compact and homeomorphic with } K\} \end{aligned}$$

where K ranges over compact subsets of $\mathbb{N}^{\mathbb{N}}$ up to homeomorphism. Our first application will produce topological examples. Let $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$ and Tr^* be the space of trees on $\mathbb{N}^*$. Let also $\varphi : \mathbb{N}^{*<\omega} \to \mathbb{Q}\cap[0,1]$ be the function assigning to each finite sequence $(a_0, a_1, \ldots, a_n)$ of positive integers its terminating continued fraction

$$\frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_n}}}}$$

(where $\varphi(\emptyset) = 0$) and $\gamma : \mathbb{N}^{*\mathbb{N}} \to [0,1]\setminus\mathbb{Q}$ be the homeomorphism mapping each infinite sequence of positive integers to its continued fraction. Now define $f : \mathrm{Tr}^* \to \boldsymbol{K}([0,1])$ by letting

$$\forall T \in \mathrm{Tr}^* \ f(T) = \overline{\varphi(T)} = \varphi(T) \cup \gamma([T]).$$

The function f is Borel. We need now to establish the following fact.

Lemma 4.1. *Let X be a Polish space containing a subspace Y homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Then, for each $B \in \mathbf{B}(\boldsymbol{K}(Y))$,*

$$\{F \in \boldsymbol{F}(X) \mid F\cap Y \in B\} \in \mathbf{\Pi}^1_1(\boldsymbol{F}(X)).$$

In particular,

$$\{H \in \boldsymbol{K}(X) \mid H\cap Y \in B\} \in \mathbf{\Pi}^1_1(\boldsymbol{K}(X)).$$

Proof. Fix a homeomorphism $\psi : \mathbb{N}^{\mathbb{N}} \to Y$ and define $A \subseteq \boldsymbol{F}(X) \times \mathbb{N}^{\mathbb{N}}$ by

$$(F, x) \in A \Leftrightarrow \psi(x) \in F.$$

Then A is Borel and has closed sections A_F, for each $F \in \boldsymbol{F}(X)$. So, by [Ke95, exercise 28.9], there is a Borel function $F \in \boldsymbol{F}(X) \mapsto T_F \in \mathrm{Tr}$ such that $\forall F \in \boldsymbol{F}(X)\ [T_F] = A_F = \psi^{-1}(F \cap Y)$. Now apply Theorem 1.1. □

We can now apply our function f to the family $\mathcal{G}_1$. Let

$$\mathcal{C} = \{H \in \boldsymbol{K}([0,1]) \mid H \setminus \mathbb{Q} \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})\}$$

and for $K \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})$,

$$\mathcal{C}_K = \{H \in \boldsymbol{K}([0,1]) \mid H \setminus \mathbb{Q} \cong K\}.$$

Theorem 4.1. *Family $\{\mathcal{C}_K\}_{K \in \boldsymbol{K}([0,1]\setminus\mathbb{Q})}$ is a partition of $\mathcal{C}$ consisting of pairwise Borel inseparable complete coanalytic sets.*

Proof. First, by Lemma 4.1, each $\mathcal{C}_K$ is coanalytic. Moreover, $f^{-1}(\mathcal{C}_K) = \{T \in \mathrm{Tr}^* \mid [T] \cong K\}$. □

We give now an abstract formulation which allows to produce several other examples of big families of pairwise Borel inseparable complete coanalytic sets. For this we introduce the following definition, similar to the one given in [Ke85, section 6]. Note however that our definition is the complement of the one given there.

Definition. Let Z be a separable Fréchet space. A *notion of singularity* of Z in $[0,1]$ is any relation $N \subseteq Z \times [0,1]$. If $N(\vec{f}, x)$ holds, x is a *point of N-singularity* for $\vec{f}$. The notion of singularity N is *linear* if the two following conditions hold:

(1) $N(\vec{f}, x) \wedge N(\vec{g}, x) \Rightarrow N(\vec{f} + \vec{g}, x)$;

(2) if $N(\vec{f} + \vec{g}, x)$ holds, then either $N(\vec{f}, x), N(\vec{g}, x)$ both hold or none of $N(\vec{f}, x), N(\vec{g}, x)$ holds.

Theorem 4.2. *Let Z be a separable Fréchet space. Let N be a Borel linear notion of singularity for Z in $[0,1]$. Assume:*

(1) *there is a Borel function $H \in \boldsymbol{K}([0,1]) \mapsto \overrightarrow{f_H} \in Z$ such that $N(\overrightarrow{f_H}, x) \Leftrightarrow x \in H$;*

(2) *there is a Borel function* $P \in \mathcal{P}([0,1] \cap \mathbb{Q}) \mapsto \overrightarrow{g_P} \in Z$ *such that* $N(\overrightarrow{g_P}, x) \Leftrightarrow x \in [0,1] \setminus P$.

For each $K \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})$ *let*

$$N_K = \{\vec{f} \in Z \mid N_{\vec{f}} \subseteq [0,1] \setminus \mathbb{Q} \wedge N_{\vec{f}} \cong K\}.$$

Then N_K *is complete coanalytic and, for* K, K' *non homeomorphic compact subsets of the irrationals,* $N_K, N_{K'}$ *are Borel inseparable.*

Proof. Since N is Borel, each N_K is coanalytic by Theorem 1.2. Let $\psi : \boldsymbol{K}([0,1]) \to Z$ be defined by

$$\forall H \in \boldsymbol{K}([0,1]) \ \psi(H) = \overrightarrow{f_H} + \overrightarrow{g_{H \cap \mathbb{Q}}}.$$

Then ψ is Borel, as $H \in \boldsymbol{K}([0,1]) \mapsto H \cap \mathbb{Q} \in \mathcal{P}([0,1] \cap \mathbb{Q})$ is. Moreover, $N(\psi(H), x) \Leftrightarrow x \in H \setminus \mathbb{Q}$ and $\psi^{-1}(N_K) = \mathcal{C}_K$, where $\mathcal{C}_K$ is as in Theorem 4.1. $\square$

Corollary 4.3. *Let* Z *and* N *be as in Theorem 4.2. Then* $\{N_K\}_K$ *is a partition — indexed by compact subsets of the irrationals up to homeomorphism — of the complete coanalytic set*

$$\{\vec{f} \in Z \mid N_{\vec{f}} \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})\}$$

consisting of Borel inseparable, complete coanalytic subsets.

Now we turn to some examples from Real Analysis. Let $\mathcal{C}([0,1], \mathbb{R})$ be the space of continuous functions endowed with the supnorm. For $f \in \mathcal{C}([0,1], \mathbb{R})$, let $D(f)$ be the set of points where f is differentiable, and

$$\mathcal{D} = \{f \in \mathcal{C}([0,1], \mathbb{R}) \mid D(f) \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})\},$$

and for $K \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})$,

$$\mathcal{D}_K = \{f \in \mathcal{D} \mid D(f) \cong K\}.$$

Theorem 4.2 and its corollary allow us to apply our procedure to Mauldin theorem, which states that the class NDIFF of nowhere differentiable functions is $\boldsymbol{\Pi}^1_1$-complete (see [Mau79]). For this, take $Z = \mathcal{C}([0,1], \mathbb{R})$ and

$$N(f, x) \Leftrightarrow f \text{ is differentiable at } x.$$

Theorem 4.4. *Family* $\{\mathcal{D}_K\}_{K \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})}$ *is a partition of* $\mathcal{D}$ *consisting of pairwise Borel inseparable complete coanalytic sets.*

Similarly, by taking

$$N(\{f_n\}, x) \Leftrightarrow \{f_n(x)\} \text{ converges}$$

as a Borel linear notion of singularity for $(\mathcal{C}([0,1],\mathbb{R}))^{\mathbb{N}}$ in $[0,1]$, we get the following application. For $\{f_n\} \in (\mathcal{C}([0,1],\mathbb{R}))^{\mathbb{N}}$, let $L(\{f_n\})$ be the set of points where $\{f_n\}$ converges,

$$\mathcal{B} = \{\{f_n\} \in (\mathcal{C}([0,1],\mathbb{R}))^{\mathbb{N}} \mid L(\{f_n\}) \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})\},$$

and for $K \in \boldsymbol{K}([0,1] \setminus \mathbb{Q})$,

$$\mathcal{B}_K = \{\{f_n\} \in \mathcal{B} \mid L(\{f_n\}) \cong K\}.$$

Theorem 4.5. *Family $\{\mathcal{B}_K\}_{K \in \boldsymbol{K}([0,1]\setminus\mathbb{Q})}$ is a partition of $\mathcal{B}$ consisting of pairwise Borel inseparable complete coanalytic sets.*

The application of the function f above to family $\mathcal{G}_3$ produces other big families of pairwise Borel inseparable, complete coanalytic sets, which correspond to the ones obtained with Theorems 4.1 through 4.5 using $\mathcal{G}_1$. Of course the family $\{f(\mathcal{V}_\alpha)\}_{\alpha \in \omega_1}$, does also constitute a family of pairwise Borel inseparable, complete coanalytic subsets of $\boldsymbol{K}([0,1])$ and this translates to similar families for the other spaces considered above. However this family does not seem to have an intrinsic description and so does not seem to provide interesting examples. The reason is that continued fractions, used to get Theorem 4.1, provide a homeomorphism γ which is not order preserving. More elaborated constructions can be built so that also family $\mathcal{G}_2$ can provide meaningful examples. One of them is given by the coding of each point of the real line realisation of the Cantor space $E_{\frac{1}{3}}$ with a finite or infinite sequence of natural numbers as done in [CG01]. We omit the details here.

References

[Be86] H. Becker, *Some examples of Borel-inseparable pairs of coanalytic sets*, Mathematika 33 (1986), 72–79.

[CG01] R. Camerlo and S. Gao, *The completeness of the isomorphism relation for countable Boolean algebras*, Transactions of the American Mathematical Society 353 (2001), 491–518.

[CK00] R. Camerlo and A.S. Kechris, *Countable structures with a fixed group of automorphisms*, Israel Journal of Mathematics 117 (2000), 105–124.

[Ke85] A.S. Kechris, *Sets of everywhere singular functions*, in Recursion theory week, edited by H.-D. Ebbinghaus, G.H. Müller and G.E. Sacks, Springer-Verlag 1985, 233–244.

[Ke95] A.S. Kechris, *Classical descriptive set theory*, Springer-Verlag 1995.

[Ke] A.S. Kechris, *Actions of Polish groups and classification problems*, to appear, Cambridge University Press.

[Mai74] A. Maitra, *On the failure of the first principle of separation for co-analytic sets*, Proceedings of the American Mathematical Society 46 (1974), 299–301.

[Mau79] R.D. Mauldin, *The set of continuous nowhere differentiable functions*, Pacific Journal of Mathematics 83 (1979), 199–205.

[Me81] A.H. Mekler, *Stability of nilpotent groups of class 2 and prime exponent*, The Journal of Symbolic Logic 46 (1981), 781–788.

[No31] P. Novikov, *Sur les fonctions implicites mesurable B*, Fundamenta Mathematicae 17 (1931), 8–25.

[Si31] W. Sierpiński, *Sur deux complémentaires analytiques non séparable B*, Fundamenta Mathematicae 17 (1931), 296–297.