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THE RIESZ APPROACH TO THE LEBESGUE INTEGRAL AND COMPLETE FUNCTION SPACES

Abstract

This paper is a step by step account of the Riesz approach to the Lebesgue integral. Besides, motivating the use of "almost everywhere" tools, we eliminate unnecessary equivalences, and we give a simple representation of a complete space of integrable functions, usually missing from classical treatises.

1 Introduction

1.1 Aims and Motivations

The aim of this paper is to give a step by step account of the Riesz approach [1] to the Lebesgue integral. This approach is particularly useful [2] for several reasons, both practical and theoretical:

- it allows us to define the Lebesgue integral directly by extending the Cauchy or Riemann integral, thus allowing calculus students to derive the Lebesgue integral by means of integrals that they are familiar with;

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[1]For the original expositions, see [7], [8].

[2]It is not the purpose of this paper to demonstrate that the Riesz approach is the main route to the integral, but rather to show that this approach has several useful characteristics compared with the classical measure-theoretic approach. Besides, there is a third approach: the Riemann-like definition of the Kurzweil-Henstock integral. The Kurzweil-Henstock integral is even more general than Lebesgue's, and because of its intuitive definition, an increasing number of authors adopt it in their textbooks (see [1] and [5]) as a basic notion of integral.

- it shows that the passage of limit under the integral sign may be used as the definition of the Lebesgue integral itself (see Extension 2.13 and 2.30);
- it provides different versions of the Monotone Convergence Theorem; from an elementary setting to the final one, so that the role played by newly introduced concepts can be seen at each step (see Monotone Convergence Theorems 2.6.1, 2.6.2 and 2.6.3);
- it strongly motivates the use of the "almost everywhere" concept (see Section 2.4, Remark 2.25, Lemma 2.32 and Section 3.3);
- it quickly reaches density and completeness results (see Section 2.7, 2.8 and 3.2);
- it can be set in an abstract framework that also provides L^p spaces, and Radon measures [3].

The Riesz approach to the Lebesgue integral rests upon

1. the concept of a Riesz sequence (see Definition 2.5);
2. the basic property of Daniell continuity (see Definitions 2.7, 2.28 and Theorem 2.29).

Those two ingredients are fundamental to the application of the Riesz approach to more abstract frameworks.

1.2 Domains and Functions Spaces

In vector spaces of regular functions we are accustomed to defining pointwise algebraic operations among functions all having the same domain of definition. This can occur because such functions attain finite values. However, when dealing with integrable functions, it may happen that they diverge at some point. Thus, pointwise vector operations are inconsistent for such functions (see Appendix 5). Nevertheless, we can notice that certain functions [4] diverge on sets that are reasonably small (see Theorem 3.2). The "smallness" of such sets (see Section 2.4) allows us to introduce the concept of "almost everywhere true" properties. Thanks to this concept one can define new vector operations that are then fully consistent and still act pointwise (but now "almost everywhere" and no more on a fixed domain). With respect to such operations the space $\mathcal{R}^I - \mathcal{R}^I$ (see Definitions 2.6 and 2.21) can be considered as a vector space.

[3] See [15], [13]. In a subsequent writing we will consider Sobolev spaces.

[4] Those belonging to $\mathcal{R}^I$ (see Section 2.1).

1.3 About Completion

Usual treatments [5] extend the integral from a Riesz space $\mathcal{R}$ (see Definition 2.1) directly to the class which we will denote as $\mathcal{R}^{I-}$ (see Definition 2.6 and 2.39), and then they show that the space of all Lebesgue integrable functions L^1 coincides with $\mathcal{R}^{I-} - \mathcal{R}^{I-}$ (see also Section 3.2). However, this construction misses important observations from the intermediate set $\mathcal{R}^I$. In fact, in order to get a completion of $\mathcal{R}$ it is sufficient to consider $\mathcal{R}^I - \mathcal{R}^I$ (see Definition 2.21, Theorem 2.37 and Section 3.2). L^1 and $\mathcal{R}^I - \mathcal{R}^I$ are isometrically isomorphic, but functions in the latter space are far more explicitly characterized than elements in L^1. Since completing the seminormed space $(\mathcal{R}, I(|\cdot|))$ is one of the main motivations for introducing the Lebesgue integral, completeness of $\mathcal{R}^I - \mathcal{R}^I$ is remarkable.

1.4 Structure of the Paper

This paper is divided into two main parts: an abstract setting and a motivating guideline. The aim of the abstract setting is twofold:

- to show in full generality how simply and quickly the Riesz method achieves powerful results and can be extended for applications to other situations;
- to provide a template any instructor can fill choosing his (or her) favorite concrete Riesz space $\mathcal{R}$ and integral I, depending on didactical needs.

The second part of this article attempts to describe the concrete background that lies behind the abstract approach. It briefly retraces some basic motivations in the Riesz approach, and can be considered as a technical introduction to the abstract setting. It follows the abstract setting only because there are particular notions and results related to the previous general theory. In this respect, one may prefer to read it first.

2 The Abstract Setting

2.1 Basic Pointwise Definitions

Definition 2.1. A class $\mathcal{R} = \mathcal{R}(\Omega)$ of real-valued functions all defined on a set Ω is called a **Riesz space on** Ω if,

[5] See [2], [3], [4], [6].

- it is a real vector space with respect to pointwise vector operations in $(\mathbb{R}, +, \cdot)$:

$$\begin{array}{c} (f+g)(x) := f(x) + g(x) \\ (r \cdot f)(x) := r \cdot f(x) \end{array} \quad \forall f, g \in \mathcal{R},\ \forall x \in \Omega,\ \forall r \in \mathbb{R}$$

- when f and g are in $\mathcal{R}$, then also $f \vee g$ is in $\mathcal{R}$,

$$\text{where } \big(f \vee g\big)(x) := \max\{f(x), g(x)\} \quad \forall x \in \Omega.$$

Example 2.2. Step functions and continuous functions with compact support are typical Riesz spaces [6].

Definition 2.3. A functional $J : \mathcal{A} \to \mathbb{R}$ defined on a class of real valued functions $\mathcal{A}$ will be called **pointwise increasing** if, given $f, g \in \mathcal{A}$ such that

$$f(x) \leq g(x) \quad \forall x \in \Omega, \text{ then } J(f) \leq J(g).$$

Definition 2.4. A function $f : \Omega \to \mathbb{R} \cup \{+\infty\}$ will be called a **positively extended real function**.

Definition 2.5. Let $J : \mathcal{E} \to \mathbb{R}$ be a functional defined on a class $\mathcal{E} = \mathcal{E}(\Omega)$ of positively extended real functions all defined on the same set Ω. A sequence (f_n) in $\mathcal{E}$ will be called a **Riesz sequence** for $(J, \mathcal{E})$ if it satisfies the two following conditions:

$$i) f_n(x) \leq f_{n+1}(x) \qquad \forall n \in \mathbb{N}, \quad \forall x \in \Omega \tag{1}$$

$$ii) J(f_n) \leq C \qquad C \in \mathbb{R},\ \text{independent of } n.$$

Here relation "$\leq$" in i) extends the usual order relation on $\mathbb{R}$ in the following way: $\forall x \in \mathbb{R} \cup \{+\infty\} \quad x \leq +\infty$.

Condition (i) implies that any Riesz sequence has always pointwise limit (possibly $+\infty$).

Definition 2.6. Let $J : \mathcal{R} \to \mathbb{R}$ be a functional on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$; Define the following class of positively extended real functions:

$$\mathcal{R}^J := \Big\{ f : \Omega \to \mathbb{R} \cup \{+\infty\};\ f \text{ is the pointwise limit of a Riesz sequence for } (J, \mathcal{R}) \Big\}.$$

[6] See Sections 3.1.2 and 3.1.3.

Definition 2.7. A functional $J : \mathcal{R} \to \mathbb{R}$ on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ is called **pointwise Daniell continuous** if, given a Riesz sequence (δ_n) for $(J, \mathcal{R})$, such that

$$\lim_n \delta_n(x) = 0 \quad \forall x \in \Omega, \qquad \text{then} \qquad \lim_n J(\delta_n) = 0.$$

Definition 2.8. By an **integral** on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ we mean a pointwise increasing linear functional $I : \mathcal{R} \to \mathbb{R}$ that is pointwise Daniell continuous.

Remark 2.9. integral over the Riesz space $\mathcal{K}(\Omega)$ of continuous functions with compact support is also called a **Radon measure**[7].

If I is an integral on a Riesz space $\mathcal{R}$, then $I(|\cdot|)$ is a seminorm and a Riesz sequence is simply a particular Cauchy sequence with respect to that seminorm:

Proposition 2.10. Let I be an integral on a Riesz space $\mathcal{R}$. Let (ϕ_n) be an increasing sequence of functions in $\mathcal{R}$:

$$\begin{array}{ccl} (\phi_n) \text{ is a Cauchy sequence} & \iff & I(\phi_n) \leq C, \\ \text{with respect to } I(|\cdot|) & & (C \in \mathbb{R} \text{ independent of } n). \end{array}$$

Remark 2.11. Contrary to Riesz sequences, Cauchy sequences may not converge at any point.

In the next section we will extend an integral I to each function in $\mathcal{R}^I$ thanks to the Daniell Continuity and the following

Proposition 2.12. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$.*

Let (δ_n) be a Riesz sequence for $(I, \mathcal{R})$ such that $\lim_n \delta_n(x) \geq 0$ $\forall x \in \Omega$, then $\lim_n I(\delta_n) \geq 0$.

PROOF OF PROPOSITION 2.12. Consider $\beta_n(x) = \min\{\delta_n(x), 0\} \leq \delta_n(x)\ \forall x \in \Omega$. (β_n) is a Riesz sequence for $(I, \mathcal{R})$ and $\lim_n \beta_n(x) = 0$. By Daniell Continuity and monotonicity of I we have that $\lim_n I(\delta_n) \geq \lim_n I(\beta_n) = 0$. $\square$

We want to stress that, in the foregoing proposition, the pointwise limit of (δ_n) may be infinite at some point in Ω.

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[7] See [13].

2.2 The First Extension

Extension 2.13. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. Corresponding to any $f \in \mathcal{R}^I$, there exists a Riesz sequence (ϕ_n) for $(I, \mathcal{R})$ such that $f(x) = \lim_n \phi_n(x)\ \forall x \in \Omega$; define

$$\bar{I}(f) := \lim_n I(\phi_n).$$

Proposition 2.14. If I is an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, then $\bar{I}$ is a well defined functional on the class of positively extended real functions $\mathcal{R}^I$.

The well definiteness of $\bar{I}$ follows directly from the following

Proposition 2.15. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If (ϕ_n) and (ψ_n) are Riesz sequences for $(I, \mathcal{R})$ such that*

$$\lim_n \phi_n(x) \le \lim_n \psi_n(x) \quad \forall x \in \Omega, \qquad \textit{then} \qquad \lim_n I(\phi_n) \le \lim_n I(\psi_n).$$

PROOF OF PROPOSITION 2.15. For all fixed m the sequence

$$\psi_1 - \phi_m\ ,\ \psi_2 - \phi_m\ ,\ \ldots\ ,\ \psi_n - \phi_m\ ,\ \ldots$$

is a Riesz sequence for $(I, \mathcal{R})$. Besides, its pointwise limit is positive: let $x \in \Omega$, then

$$\lim_n \Big(\psi_n(x) - \phi_m(x) \Big) = \Big(\lim_n \psi_n(x) \Big) - \phi_m(x) \ge \Big(\lim_n \phi_n(x) \Big) - \phi_m(x) \ge 0.$$

The thesis then follows on applying Proposition 2.12. □

2.3 Algebraic Operation within $\mathcal{R}^I$

The foregoing propositions have shown that the elements in $\mathcal{R}^I$ may be considered as new integrable functions. Let us consider these objects closely.

Among elements $f, g \in \mathcal{R}^I$ we can perform certain pointwise operations such as

$$f \vee g = \max\{f, g\}, \quad f \wedge g = \min\{f, g\}, \quad f + g \quad \text{and} \quad \rho \cdot f \quad (\text{where } \rho \ge 0)\ ,$$

still obtaining elements in $\mathcal{R}^I$; moreover the following holds

Proposition 2.16. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, then $\bar{I}$ is additive on $\mathcal{R}^I$; that is, if $f, g \in \mathcal{R}^I$, then

$$f + g \in \mathcal{R}^I \quad \text{and} \quad \bar{I}(f+g) = \bar{I}(f) + \bar{I}(g).$$

In particular, if $f \in \mathcal{R}^I$ and $\phi \in \mathcal{R}$ we have $f - \phi \in \mathcal{R}^I$ and $\bar{I}(f-\phi) = \bar{I}(f) - I(\phi)$. However, if $f \in \mathcal{R}^I$, $-f$ does not necessarily belong to $\mathcal{R}^I$ (see Example 4.1). Nevertheless, one would like to also consider $-f$ as integrable, and $-\bar{I}(f)$ as its integral. Trying to perform pointwise the difference $f - g$ a problem arises, as this produces contradictions [8]; that occurs since functions in $\mathcal{R}^I$ may attain the extended real value $+\infty$. Such problems would be resolved if one could exclusively manage real values. In this regard, let us associate to each element $f \in \mathcal{R}^I$ its domain as a real function

$$\Omega_f := \big\{x \in \Omega \ : \ f(x) \in \mathbb{R}\big\}.$$

In the following section we will study such domains.

2.4 Full and Null Sets

If I is an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega,)$ then, of course, $\Omega_f = \Omega$ for each $f \in \mathcal{R}$, but if $f \in \mathcal{R}^I \setminus \mathcal{R}$, then Ω_f could trivially be empty. However, there is a simple condition that prevents any Ω_f from ever being empty.

Proposition 2.17. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If there exists a function $\psi \in \mathcal{R}$ such that $I(\psi) \neq 0$, then for all $f \in \mathcal{R}^I$ we have that $\Omega_f \neq \emptyset$.*

Proof of Proposition 2.17. To reach a contradiction, suppose that there exists a particular $g \in \mathcal{R}^I \setminus \mathcal{R}$ such that $\Omega_g = \emptyset$. That is to say that $g(x) = +\infty$ $\forall x \in \Omega$ and $\bar{I}(g) < +\infty$. In particular, this would imply that $g + \psi = g$.

Since $\bar{I}$ is well defined, additive and it attains only real values, we would have the contradiction $\bar{I}(\psi) = I(\psi) = 0$. $\square$

If I is not trivially zero, the domains of the new integrable functions share much stronger properties than nonemptyness.

Proposition 2.18. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If there exists a function $\psi \in \mathcal{R}$ such that $I(\psi) \neq 0$, then*

i) $\forall f, g \in \mathcal{R}^I$, $\qquad \Omega_f \cap \Omega_g \neq \emptyset$

ii) *given any sequence (f_n) in $\mathcal{R}^I$, then* $\qquad \displaystyle\bigcap_{n=1}^{\infty} \Omega_{f_n} \neq \emptyset.$

[8]See Section 5.

PROOF. Part i) follows immediately observing that, given any $f, g \in \mathcal{R}^I$, $\Omega_f \cap \Omega_g$ is the domain of $f+g$ that is still in $\mathcal{R}^I$ by Proposition 2.16, and thus it has a nonempty domain.

We show that $\bigcap_{n=1}^{\infty} \Omega_{f_n}$ contains the domain of a function $h \in \mathcal{R}^I$. By hypothesis, we know that for each $n \in \mathbb{N}$ there exists a Riesz sequence $(\phi_k^{(n)})$ for $(I, \mathcal{R})$ such that $\lim_k \phi_k^{(n)}(x) = f_n(x)$ for each $x \in \Omega_{f_n}$ and for each $n \in \mathbb{N}$; in particular we have that

$$\forall k, n \in \mathbb{N}, \quad \forall x \in \Omega \qquad \phi_k^{(n)}(x) \leq \phi_{k+1}^{(n)}(x)$$

$$\forall n \in \mathbb{N} \;\; \exists C_n > 0 \;\; \forall k \in \mathbb{N} \qquad I(\phi_k^{(n)}) \leq C_n.$$

Fix a convergent series $\displaystyle\sum_{k=1}^{\infty} \epsilon_k$ (having each $\epsilon_k > 0$), and define

$$\Phi_n(x) := \frac{\epsilon_1}{C_1}\phi_n^{(1)}(x) + \frac{\epsilon_2}{C_2}\phi_n^{(2)}(x) + \cdots + \frac{\epsilon_n}{C_n}\phi_n^{(n)}(x) = \sum_{k=1}^{n} \frac{\epsilon_k}{C_k}\; \phi_n^{(k)}(x).$$

(Φ_n) is a Riesz sequence for $(I, \mathcal{R})$, indeed each $\Phi_n \in \mathcal{R}$, $\Phi_n \leq \Phi_{n+1}$ and

$$I(\Phi_n) = \sum_{k=1}^{n} \frac{\epsilon_k}{C_k} I(\phi_n^{(k)}) \leq \sum_{k=1}^{n} \epsilon_k < C.$$

Thus $\lim_n \Phi_n =: h \in \mathcal{R}^I$ and, by Proposition 2.17, $\Omega_h \neq \emptyset$.

Now we simply note that, if $\lim_n \Phi_n(x) < +\infty$, then necessarily $x \in \displaystyle\bigcap_{n=1}^{\infty} \Omega_{f_n}$, that is

$$\Omega_h \subset \bigcap_{n=1}^{\infty} \Omega_{f_n}.$$

□

The sets that are domains of some element in $\mathcal{R}^I$ deserve the following

Definition 2.19. Let $J : \mathcal{R} \to \mathbb{R}$ be a functional on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. A set $F \subset \Omega$ will be called a **full set** [9] for $(J, \mathcal{R})$ if there exists some $f \in \mathcal{R}^J$ such that $F \supset \Omega_f$. The complement of any full set (relative to Ω) will be called a **null set** [10].

[9] To our knowledge the term 'set of full measure' originates from [10].

[10] This definition originates from Theorem 3.2 characterizing Lebesgue's negligible sets on $\mathbb{R}$.

Thus, Proposition 2.17 states that if I is a nontrivial integral on a Riesz space $\mathcal{R}$, then full sets for $(I, \mathcal{R})$ are not empty, and Proposition 2.18 states that countable intersections of such full sets are full sets.

Definition 2.20. Let $J : \mathcal{R} \to \mathbb{R}$ be a functional on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. A property $P(x)$ dealing with $x \in \Omega$ is said to be true **almost everywhere** for $(J, \mathcal{R})$ (a.e. for short), if the set $\{ \ x \in \Omega \ : \ P(x) \text{ is true} \ \}$ is a full set for $(J, \mathcal{R})$.

In the following we will see that, if the above functional J is an integral, then properties holding almost everywhere play a central role in extending it. In this regard, we show how certain sets of functions behave as vector spaces with respect to slightly modified pointwise operations.

Definition 2.21. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. By $\mathcal{R}^I - \mathcal{R}^I$ we denote the set of functions $l : \Lambda \to \mathbb{R}$ such that

$$\exists f,\ g \in \mathcal{R}^I \ : \ \begin{cases} \Lambda = \Omega_f \cap \Omega_g \\ l(x) = f(x) - g(x) \quad \forall x \in \Lambda. \end{cases}$$

We will indicate Λ also as Ω_l.

Now define pointwise operations among functions l, l_1 and l_2 belonging to $\mathcal{R}^I - \mathcal{R}^I$:

$$\begin{aligned} \big(l_1 + l_2\big)(x) &:= l_1(x) + l_2(x) && \forall x \in \Omega_{l_1} \cap \Omega_{l_2} \\ \big(\alpha \cdot l\big)(x) &:= \alpha \cdot l(x) && \forall x \in \Omega_l \qquad \forall \alpha \in \mathbb{R}; \end{aligned} \tag{2}$$

$$\left.\begin{aligned} \big(l_1 \vee l_2\big)(x) &:= \max\{l_1(x), l_2(x)\} \\ \big(l_1 \wedge l_2\big)(x) &:= \min\{l_1(x), l_2(x)\} \end{aligned}\right. \ \forall x \in \Omega_{l_1} \cap \Omega_{l_2}. \tag{3}$$

It is now easy to verify the following

Proposition 2.22. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If $l_1, l_2 \in \mathcal{R}^I - \mathcal{R}^I$, then $\forall \alpha \in \mathbb{R}$ the following real functions*

$$l_1 + \alpha l_2, \qquad l_1 \vee l_2, \qquad l_1 \wedge l_2$$

still belong to $\mathcal{R}^I - \mathcal{R}^I$.

Let us point out a remark as simple as it is crucial:

Remark 2.23. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ and define the following two relations among elements $f, g \in \mathcal{R}^I$:

$$\begin{aligned}
&\forall x \in \Omega_f \cap \Omega_g \\
&\qquad f \preceq g \quad \iff \quad \{x :\ f(x) \le g(x)\} = \Omega_f \cap \Omega_g \\
&\qquad f \ll g \quad \iff \quad f \le g \ \text{ a.e.} \\
&\qquad\qquad\quad \iff \quad \{x :\ f(x) \le g(x)\} \qquad \text{is a full set}
\end{aligned}$$

We observe that only relation $\ll$ is transitive.

Notations 2.24. If $f \ll g$ and $g \ll f$, then we will write $f \equiv g$. That is,

$$f \equiv g \quad \iff \quad f(x) = g(x) \ \text{ a.e}$$

Remark 2.25. In this exposition we intentionally put relation $\equiv$ in the background. We want to stress it is relation $\ll$ that plays a fundamental role, not $\equiv$. In fact, $\ll$ will be useful in order to weaken the hypothesis in Proposition 2.15. Moreover, we will never use $\equiv$ to identify functions, as we want to deal with functions and not with equivalence classes. We will see in the following how such resolution do not cause any harm; actually, it allows us to better see where a.e. properties really play their role.

The operation $+$ $\big($as defined in (2)$\big)$ is associative, commutative and the zero function $0 \in \mathcal{R}$ (everywhere defined on Ω) is a neutral element: $\forall l \in \mathcal{R}^I - \mathcal{R}^I \ \ l + 0 = l$. Less obvious is the question about an opposite element of l. In fact, we should find a function $\tilde{l}$ such that $l + \tilde{l} = 0$. The first difficulty concerns domains: $\Omega_{l+\tilde{l}} \subset \Omega_l \subset \Omega$; indeed, it may well happen that $\Omega_l \neq \Omega$. In that case, identity $l + \tilde{l} = 0$ could never hold pointwise on Ω. Nevertheless each $l \in \mathcal{R}^I$ has a natural unique opposite function: $-l : \Omega_l \to \mathbb{R}$. We assume it as the definition of opposite element. We make this "inelegant" choice as we prefer to work with functions rather than equivalence classes of functions [11]. We have just to be careful dealing with the cancellation rule; indeed, if l_1 and l_2 are in $\mathcal{R}^I - \mathcal{R}^I$

$$l_1 = l_1 + l_2 - l_2 \quad \text{may be false, while}$$

$$l_1 \equiv l_1 + l_2 - l_2 \quad \text{is always true, whatever is } l_2 \ .$$

Many properties resting on conditions holding everywhere on Ω still hold if one requires just a.e. pointwise conditions. An example is given by the following fundamental

[11] See also [14] p.xiii.

Theorem 2.26. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, and let $f \in \mathcal{R}^I$:*

$$\text{if} \quad f \geq 0 \quad \text{a.e.,} \quad \text{then} \quad \bar{I}(f) \geq 0.$$

PROOF. Let (ϕ_n) be a Riesz sequence for $(I, \mathcal{R})$ such that $f(x) = \lim_n \phi_n(x)$ $\forall x \in \Omega$. By hypothesis $f \geq 0$ a.e., so there exists $g \in \mathcal{R}^I$ such that

$$\{x \ : \ f(x) \geq 0\} \supset \Omega_g. \tag{4}$$

Let (ψ_n) be a Riesz sequence for $(I, \mathcal{R})$ such that $g(x) = \lim_n \psi_n(x)$ $\forall x \in \Omega$. So we have that, $\forall x \in \Omega$

$$\lim_n (\phi_n(x) + \psi_n(x)) = f(x) + g(x) \geq g(x) = \lim_n \psi_n(x)$$

By Proposition 2.15 we have that

$$\lim_n I(\phi_n + \psi_n) \geq \lim_n I(\psi_n) = \bar{I}(g)$$

; i.e., $\bar{I}(f+g) = \bar{I}(f) + \bar{I}(g) \geq \bar{I}(g)$. As $\bar{I}(g)$ is finite, we obtain the thesis. □

By simply remembering that if $f \in \mathcal{R}^I$ and $\phi \in \mathcal{R}$, then $f - \phi \in \mathcal{R}^I$ and $\bar{I}(f - \phi) = \bar{I}(f) - I(\phi)$, we can reach the following general result:

Corollary 2.27. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, and let $f, g \in \mathcal{R}^I$:*

$$\text{if} \quad f \leq g \quad \text{a.e.,} \quad \text{then} \quad \bar{I}(f) \leq \bar{I}(g).$$

PROOF. Let (ϕ_n) and (ψ_n) be two Riesz sequences for $(I, \mathcal{R})$ converging to f and g respectively. If we consider m as fixed, then $(\psi_n - \phi_m)$ is a Riesz sequence for $(I, \mathcal{R})$ and

$$\lim_n (\psi_n - \phi_m) = g - \phi_m \geq f - \phi_m \ \text{ a.e.}$$

Since $f(x) - \phi_m(x) \geq 0$ for each $m \in \mathbb{N}$ and each $x \in \Omega$, we have that $g - \phi_m \geq 0$ a.e., so

$$\bar{I}(g - \phi_m) = \bar{I}(g) - I(\phi_m) \geq 0,$$

and that implies the thesis. □

The foregoing results lead to an extension of pointwise Daniell continuity.

Definition 2.28. A functional $J : \mathcal{R} \to \mathbb{R}$ defined on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ is called **Daniell continuous** if, given a Riesz sequence (δ_n) for $(J, \mathcal{R})$, such that

$$\lim_n \delta_n(x) = 0 \quad \text{a.e.,} \qquad \text{then} \qquad \lim_n J(\delta_n) = 0.$$

The following crucial theorem states that Definition 2.7 and Definition 2.28 are, in a certain sense, equivalent.

Theorem 2.29. *Let J be a pointwise increasing linear functional on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, then*

$$J \text{ is pointwise Daniell continuous} \iff J \text{ is Daniell continuous.}$$

PROOF. Part $[\Longleftarrow]$ is immediate. Consider part $[\Longrightarrow]$. Let (δ_n) be a Riesz sequence for $(J, \mathcal{R})$ converging to zero on a full set (that is a.e.). Denoting $f = \lim_n \delta_n$, we have $f \in \mathcal{R}^J$ and $f = 0$ a.e. By hypothesis, J is an integral on $\mathcal{R}$, thus we can apply Corollary 2.27 and obtain our thesis. $\square$

The above theorem is fundamental; indeed it shows that pointwise Daniell continuity of integrals is far more general than it appears. This theorem is the main key for extending an integral to its proper setting. As a matter of fact by using Theorem 2.29 we will accomplish the extension of $\bar{I}$ over all of $\mathcal{R}^I - \mathcal{R}^I$. Indeed, as pointwise Daniell continuity has been used in conjunction with Proposition 2.15 to extend integral I to $\bar{I}$, we will proceed in a similar manner to further extend $\bar{I}$.

2.5 Second Extension

Extension 2.30. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If $l \in \mathcal{R}^I - \mathcal{R}^I$, there exist $f, g \in \mathcal{R}^I$ such that $l = f - g$;

$$\bar{\bar{I}}(l) := \bar{I}(f) - \bar{I}(g).$$

By using the foregoing propositions, one can verify the following:

Proposition 2.31. If I is an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, then $\bar{\bar{I}}$ is a well defined linear functional on the real vector space $\mathcal{R}^I - \mathcal{R}^I$ such that, given l_1 and l_2 in $\mathcal{R}^I - \mathcal{R}^I$

$$\text{if} \;\; l_1 \ll l_2, \quad \text{then} \quad \bar{\bar{I}}(l_1) \leq \bar{\bar{I}}(l_2).$$

2.6 Limit Theorems

Thus far we have obtained a space of real valued functions not necessarily defined everywhere on Ω, and a linear increasing functional on such a space.

One could wonder, applying the same procedure that extended $\mathcal{R}$ and I, if it could be possible to obtain new integrable functions from $\mathcal{R}^I - \mathcal{R}^I$ and $\bar{\bar{I}}$.

The next sections will show that this is not the case, by providing a series of fundamental convergence theorems.

2.6.1 Monotone Convergence Theorems

Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. As we have already noticed in Section 2.3, $\mathcal{R}^I$ is closed with respect to many pointwise algebraic operations; in this section we will see closure properties for limiting processes.

We know that $\bar{I}$ extends I, so every Riesz sequence for $(I, \mathcal{R})$ is also a Riesz sequence for $(\bar{I}, \mathcal{R})$; in short, that means $\mathcal{R}^{\bar{I}} = \mathcal{R}^I$. As we have already seen, elements in $\mathcal{R}^I$ are positively extended real functions all defined on the same set Ω, and $\bar{I}$ is a functional on $\mathcal{R}^I$; we can thus define Riesz sequences for $(\bar{I}, \mathcal{R}^I)$ and, consequently, $\left(\mathcal{R}^I\right)^{\bar{I}}$. The following theorem essentially states that $\left(\mathcal{R}^I\right)^{\bar{I}} = \mathcal{R}^I$. This result will prove to be the first simplified version of the Monotone Convergence Theorem; that is why we treat it in detail.

Monotone Convergence Theorem 2.6.1. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. Given a Riesz sequence (f_n) for $(\bar{I}, \mathcal{R}^I)$, then*

- *its pointwise limit $f = \lim_n f_n$ still belongs to $\mathcal{R}^I$;*
- *the passage of limit under "integral" [12] sign holds; i.e.,*

$$\bar{I}(f) = \lim_n \bar{I}(f_n).$$

PROOF OF MONOTONE CONVERGENCE THEOREM 2.6.1. Each f_n is in $\mathcal{R}^I$, so it is the limit of some Riesz sequence $(\phi_{n,k})$ of functions in $\mathcal{R}$ (increasing with respect to k). Thus we can write $f = \lim_n \lim_k \phi_{n,k}$. We want to find a Riesz sequence for $(I, \mathcal{R})$ pointwise converging to f. As we adopt a "diagonal"

[12] We put the word *integral* in quotation marks as $\mathcal{R}^I$ is not a vector space.

method, the following diagram will be useful:

$$\begin{array}{ccccccccccccc}
 & \phi_{1,1} & \leq & \phi_{1,2} & \leq & \phi_{1,3} & \leq & \phi_{1,4} & \leq & \cdots & \leq & f_1 \\
\leq & \phi_{2,1} & \leq & \phi_{2,2} & \leq & \phi_{2,3} & \leq & \phi_{2,4} & \leq & \cdots & \leq & f_2 \\
\leq & \phi_{3,1} & \leq & \phi_{3,2} & \leq & \phi_{3,3} & \leq & \phi_{3,4} & \leq & \cdots & \leq & f_3 \\
\leq & \phi_{4,1} & \leq & \phi_{4,2} & \leq & \phi_{4,3} & \leq & \phi_{4,4} & \leq & \cdots & \leq & f_4 \\
\vdots & & \vdots & & \vdots & & \vdots & & & & \vdots &
\end{array}$$

Denote $\psi_n := \max_{1\leq i,j\leq n}\{\phi_{i,j}\} = \max_{1\leq i\leq n}\{\phi_{i,n}\}$. Since $\mathcal{R}$ is a Riesz space, each ψ_n is in $\mathcal{R}$; besides (ψ_n) is increasing and

$$\forall n \in \mathbb{N},\ \forall i(1 \leq i \leq n) \text{ and } \forall x \in \Omega,\ \phi_{i,n}(x) \leq \psi_n(x) \leq f_n(x). \tag{5}$$

So, (ψ_n) is a Riesz sequence for $(I, \mathcal{R})$. Letting n tend to infinity in (5), we obtain

$$\forall i \in \mathbb{N} \text{ and } \forall x \in \Omega,\ \lim_n \phi_{i,n}(x) = f_i(x) \leq \lim_n \psi_n(x) \leq f(x). \tag{6}$$

Relations (5) and (6) imply $f(x) = \lim_n \psi_n(x)$ for each $x \in \Omega$. From that we get $f \in \mathcal{R}^I$ and $\bar{I}(f) = \lim_n I(\psi_n)$. Also, (5) implies

$$\lim_n I(\psi_n) \leq \lim_n \bar{I}(f_n) \quad \text{and} \quad I(\phi_{i,n}) \leq I(\psi_n). \tag{7}$$

Finally, $\lim_n I(\phi_{i,n}) = \bar{I}(f_i) \leq \lim_n I(\psi_n)$ and (7) give the final result. $\square$

The above Theorem implies that the pointwise increasing, additive, homogeneous functional $\bar{I}$ on $\mathcal{R}^I$ cannot be further extended via Riesz sequences: convergence of Riesz sequences is stable within $\mathcal{R}^I$. By slightly modifying the above arguments, it is now easy to prove the following

Monotone Convergence Theorem 2.6.2. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. Let (f_n) be a sequence in $\mathcal{R}^I$ such that*

i) $f_n \ll f_{n+1} \quad \forall n \in \mathbb{N}$

ii) $\bar{I}(f_n) \leq C \quad C \in \mathbb{R},$ *independent of* n,

then,

1. *the sequence converges almost everywhere;*
2. *there exists an element f in $\mathcal{R}^I$ such that $f \equiv \lim_n f_n$*

3. $\bar{I}(f) = \lim_n \bar{I}(f_n)$.

Sketch of the proof. Let $(\phi_{n,k})$ and (ψ_n) be as in the foregoing proof. Relation (5) becomes

$$\phi_{i,n} \leq \psi_n \ll f_n.$$

This still implies that (ψ_n) is a Riesz sequence for $(I, \mathcal{R})$ and that $\lim_n \psi_n \ll \lim_n f_n$. Relation (6) is reduced to

$$\forall x \in \Omega \ \lim_n \phi_{i,n}(x) = f_i(x) \leq \lim_n \psi_n(x).$$

Thus $\lim_n f_n \equiv \lim_n \psi_n =: f \in \mathcal{R}^I$. Part 3. of the thesis follows on applying Corollary 2.27. □

More delicate is to prove the corresponding Theorem for $\mathcal{R}^I - \mathcal{R}^I$:

Monotone Convergence Theorem 2.6.3. *Let I an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ and let (l_n) be a sequence in $\mathcal{R}^I - \mathcal{R}^I$ such that*

$$\begin{aligned} &i) \quad l_n \ll l_{n+1} \quad \forall n \in \mathbb{N} \\ &ii) \quad \bar{\bar{I}}(l_n) \leq C \quad C \in \mathbb{R}, \ \textit{independent of } n, \end{aligned} \tag{8}$$

then we have

1. *the sequence converges almost everywhere;*
2. *there exists an element l in $\mathcal{R}^I - \mathcal{R}^I$ such that $l \equiv \lim_n l_n$*
3. *$\bar{\bar{I}}(l) = \lim_n \bar{\bar{I}}(l_n)$.*

In order to prove Theorem 2.6.3, we give the following

Lemma 2.32. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$ and let l be in $\mathcal{R}^I - \mathcal{R}^I$. Then, for all $\epsilon > 0$ there exist two functions f^ϵ and g^ϵ belonging to $\mathcal{R}^I$ such that*

$$\begin{aligned} &a) \quad l = f^\epsilon - g^\epsilon \\ &b) \quad g^\epsilon(x) \geq 0 \qquad \forall x \in \Omega \\ &c) \quad \bar{I}(g^\epsilon) < \epsilon. \end{aligned}$$

Proof of Lemma 2.32. We know that if $l \in \mathcal{R}^I - \mathcal{R}^I$, then there exist two elements f and g in $\mathcal{R}^I$ such that $l = f - g$. Let (ϕ_n) and (ψ_n) be two Riesz sequences in $(\mathcal{R}, I)$ such that

$$\lim\nolimits_n \psi_n(x) \quad \forall x \in \Omega_g.$$

Thus we have $l(x) = \lim_n \phi_n(x) - \lim_n \psi_n(x) \quad \forall x \in \Omega_l := \Omega_f \cap \Omega_g$. In particular, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that, if $n \geq N_\epsilon$, then

$$\bar{I}(g - \psi_n) = \bar{I}(g) - I(\psi_n) < \epsilon.$$

Now, it suffices to put $f^\epsilon := f - \psi_{N_\epsilon}$ and $g^\epsilon := g - \psi_{N_\epsilon}$; as a matter of fact

$$l(x) = f^\epsilon(x) - g^\epsilon(x) \ \forall x \in \Omega_l, \quad g^\epsilon(x) \geq 0 \ \forall x \in \Omega, \quad \text{and } \bar{I}(g^\epsilon) < \epsilon.$$

□

PROOF OF MONOTONE CONVERGENCE THEOREM 2.6.3. For all $n \in \mathbb{N}$ consider two functions f_n and g_n in $\mathcal{R}^I$ such that

$$l_n(x) = f_n(x) - g_n(x) \qquad \forall x \in \Omega_{l_n} = \Omega_{f_n} \cap \Omega_{g_n}.$$

In general,

- the sequences (f_n) and (g_n) may not be monotone;
- the sequences $\bar{I}(f_n)$, $\bar{I}(g_n)$ may not be bounded.

We observe that

$$l_n \equiv l_1 + (l_2 - l_1) + \cdots + (l_n - l_{n-1}).$$

As each $l_k - l_{k-1}$ is in $\mathcal{R}^I - \mathcal{R}^I$, the preceding Lemma allow us to find $\tilde{f}_k$ and $\tilde{g}_k$ in $\mathcal{R}^I$ such that

$$\begin{array}{ll} \tilde{g}_k(x) \geq 0 & \forall x \in \Omega \\ \bar{I}(\tilde{g}_k) < \frac{1}{2^{k+1}}. & \end{array}$$

Define

$$\begin{array}{ll} F_n := f_1 + \tilde{f}_2 + \cdots + \tilde{f}_n & \in \mathcal{R}^I \\ G_n := g_1 + \tilde{g}_2 + \cdots + \tilde{g}_n & \in \mathcal{R}^I. \end{array}$$

We notice that

$$G_n(x) \leq G_{n+1}(x) \qquad \forall n \in \mathbb{N} \ \ \forall x \in \Omega.$$

Since $\bar{I}$ is additive on $\mathcal{R}^I$, then

$$\bar{I}(G_n) = \bar{I}(g_1) + \bar{I}(\tilde{g}_2) + \cdots + \bar{I}(\tilde{g}_n) \leq \bar{I}(g_1) + 1.$$

Thus (G_n) is a Riesz sequence for $(\bar{I}, \mathcal{R}^I)$. By Monotone Convergence Theorems 2.6.1 we have that there exists G in $\mathcal{R}^I$ such that

$$G = \lim_n G_n \quad \text{and} \quad \bar{I}(G) = \lim_n \bar{I}(G_n) < +\infty.$$

Now observe that $\tilde{f}_k \equiv (l_k - l_{k-1}) + \tilde{g}_k \gg 0$, so we have

$$F_n \equiv l_n + G_n \text{ and } F_n \ll F_{n+1} \ \forall n \in \mathbb{N}.$$

By Corollary 2.27, we have that

$$\bar{I}(F_n) \leq \bar{I}(F_{n+1}) \leq C + \bar{I}(G) \qquad \forall n \in \mathbb{N}.$$

We can then apply Monotone Convergence Theorem 2.6.2 to sequence (F_n), thus there exists F in $\mathcal{R}^I$ such that

$$F \equiv \lim_n F_n \quad \text{ and } \quad \bar{I}(F) = \lim_n \bar{I}(F_n) < +\infty.$$

If we define $l := F - G$, then $l \in \mathcal{R}^I - \mathcal{R}^I$ and $l \equiv \lim_n l_n$. Finally, passage of limit under integral sign $\bar{\bar{I}}$ is a direct consequence of definition of $\bar{\bar{I}}$ itself. As a matter of fact

$$\bar{\bar{I}}(l) = \bar{I}(F) - \bar{I}(G) = \lim_n \big[\bar{I}(F_n) - \bar{I}(G_n)\big].$$

Since $l_n \equiv F_n - G_n$, by monotonicity of $\bar{\bar{I}}$, $\bar{\bar{I}}(l_n) = \bar{\bar{I}}(F_n - G_n) = \bar{I}(F_n) - \bar{I}(G_n)$, and this ends the proof. □

2.6.2 Dominated Convergence Theorem

Repeatedly applying Monotone Convergence Theorem 2.6.3 (Beppo Levi's Theorem), we obtain Lebesgue's Dominated Convergence Theorem:

Theorem 2.33. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. Let (l_n) be a sequence in $\mathcal{R}^I - \mathcal{R}^I$ such that*

a) $|l_n| \ll h$ where $h \in \mathcal{R}^I - \mathcal{R}^I$ and $n \in \mathbb{N}$;
b) (l_n) converges a.e.,

then, there exists $l \in \mathcal{R}^I - \mathcal{R}^I$ such that $l \equiv \lim_n l_n$ and $\bar{\bar{I}}(l) = \lim_n \bar{\bar{I}}(l_n)$.

PROOF OF THEOREM 2.33. For all n and k in $\mathbb{N}$ we have that

$$\mathcal{R}^I - \mathcal{R}^I \ni g_{n,k} := l_{n+k} \wedge \cdots \wedge l_{n+1} \wedge l_n \leq l_n \leq l_n \vee l_{n+1} \vee \cdots \vee l_{n+k} =: f_{n,k} \in \mathcal{R}^I - \mathcal{R}^I.$$

Fix $n \in \mathbb{N}$. The sequences $(-g_{n,k})$ and $(f_{n,k})$ are Riesz sequences for $(\bar{\bar{I}}, \mathcal{R}^I - \mathcal{R}^I)$. By the Monotone Convergence Theorem 2.6.3 there exist g_n and f_n in $\mathcal{R}^I - \mathcal{R}^I$ such that

$$g_n \equiv \lim_k g_{n,k} \qquad \text{and} \qquad f_n \equiv \lim_k f_{n,k} \ .$$

Also,

$$g_n \ll g_{n+1} \ll l_n \ll f_{n+1} \ll f_n \qquad \forall n \in \mathbb{N}.$$

By hypothesis a), the sequences (g_n) and $(-f_n)$ satisfy the hypothesis of Theorem 2.6.3, so there exists $\tilde{l}$ and $\hat{l}$ in $\mathcal{R}^I - \mathcal{R}^I$ such that

$$\tilde{l} \equiv \lim_n g_n \qquad \text{and} \qquad \hat{l} \equiv \lim_n f_n.$$

From hypothesis b) it follows that $\tilde{l} \equiv \lim_n l_n \equiv \hat{l}$, so $\bar{\bar{I}}(\tilde{l}) = \bar{\bar{I}}(\hat{l})$ (notice we are not saying $\lim_n l_n \in \mathcal{R}^I - \mathcal{R}^I$). Now recall that

$$\bar{\bar{I}}(\tilde{l}) = \lim_n \bar{\bar{I}}(g_n) = \lim_n \lim_k \bar{\bar{I}}(g_{n,k}) \leq \bar{\bar{I}}(l_i) \leq \lim_n \lim_k \bar{\bar{I}}(f_{n,k}) = \lim_n \bar{\bar{I}}(f_n) = \lim_n \bar{\bar{I}}(\hat{l}).$$

This implies $\lim_n \bar{\bar{I}}(l_n) = \bar{\bar{I}}(\tilde{l}) = \bar{\bar{I}}(\hat{l})$. To end the proof it suffices to choose $l = \tilde{l}$ or $l = \hat{l}$. □

2.7 Density Results

Proposition 2.34. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$; if $l \in \mathcal{R}^I - \mathcal{R}^I$, then there exists a Cauchy sequence (γ_n) in $(\mathcal{R}, I(|\cdot|))$ such that $l(x) = \lim_n \gamma_n(x) \quad \forall x \in \Omega_l$.*

PROOF. We know that $l \in \mathcal{R}^I - \mathcal{R}^I$ if and only if there exist two Riesz sequences (ϕ_n) and (ψ_n) in $\mathcal{R}$ such that

$$l(x) = \lim_n \phi_n(x) - \lim_n \psi_n(x) \qquad \forall x \in \Omega_l\ .$$

From Proposition 2.10 we know that every Riesz sequence for $(I, \mathcal{R})$ is a Cauchy sequence in $\big(\mathcal{R}, I(|\cdot|)\big)$. Putting $\gamma_n := \phi_n - \psi_n$, we have that (γ_n), being a sum of two Cauchy sequences in $\big(\mathcal{R}, I(|\cdot|)\big)$, is itself a Cauchy sequence in $\big(\mathcal{R}, I(|\cdot|)\big)$. □

Proposition 2.35. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. If (γ_n) is a Cauchy sequence in $(\mathcal{R}, I(|\cdot|))$, then there exists a subsequence (γ_{n_k}) and an element l of $\mathcal{R}^I - \mathcal{R}^I$ such that*

$$i)\ \lim_k \gamma_{n_k} \equiv l \quad \textit{and} \quad ii)\ \lim_n \bar{\bar{I}}(|l - \gamma_n|) = 0.$$

PROOF OF i). Since (γ_n) is a Cauchy sequence in $\big(\mathcal{R}, I(|\cdot|)\big)$, we can extract from it a subsequence (γ_{n_k}) such that the number series

$$\sum_{k=1}^{\infty} I\big(|\gamma_{n_{k+1}} - \gamma_{n_k}|\big)$$

converges. This means that the following sequences

$$\sigma_N := |\gamma_{n_1}| + \sum_{k=1}^{N-1} |\gamma_{n_{k+1}} - \gamma_{n_k}|$$
$$\phi_N = \big(\gamma_{n_1}\big)^+ + \sum_{k=1}^{N-1} \big(\gamma_{n_{k+1}} - \gamma_{n_k}\big)^+$$
$$\psi_N = \big(\gamma_{n_1}\big)^- + \sum_{k=1}^{N-1} \big(\gamma_{n_{k+1}} - \gamma_{n_k}\big)^-$$

are all Riesz sequences for $(I, \mathcal{R})$ and $\phi_N - \psi_N = \gamma_N$. We know they converge respectively to some $s, f, g \in \mathcal{R}^I$. Now it suffice to note that $\forall x \in \Omega_f \cap \Omega_g$

$$l(x) := f(x) - g(x) = \lim_N (\phi_N(x) - \psi_N(x)) =$$
$$= \lim_N \left\{ \gamma^+_{n_1}(x) - \gamma^-_{n_1}(x) + \sum_{k=1}^{N-1} \Big[\big(\gamma_{n_{k+1}}(x) - \gamma_{n_k}(x)\big)^+ - \big(\gamma_{n_{k+1}}(x) - \gamma_{n_k}(x)\big)^- \Big] \right\} =$$
$$= \lim_N \left[\gamma_{n_1}(x) + \sum_{k=1}^{N-1} \big(\gamma_{n_{k+1}}(x) - \gamma_{n_k}(x)\big) \right] = \lim_N \gamma_{n_N}(x). \qquad \square$$

PROOF OF *ii*). We start from the foregoing result:

$$\lim_N \gamma_{n_N} \equiv l = f - g, \quad \text{where } f, g \in \mathcal{R}^I$$

$$\begin{aligned} |l - \gamma_n| &\ll |l - \gamma_{n_N}| + |\gamma_{n_N} - \gamma_n| = |f - g - \gamma_{n_N}| + |\gamma_{n_N} - \gamma_n| \\ &\ll |f - \phi_N| + |g - \psi_N| + |\gamma_{n_N} - \gamma_n|. \end{aligned}$$

So we have

$$\bar{\bar{I}}(|l - \gamma_n|) \le \bar{\bar{I}}(|f - \phi_N|) + \bar{\bar{I}}(|g - \psi_N|) + I(|\gamma_{n_N} - \gamma_n|)$$
$$= \bar{I}(f) - I(\phi_N) + \bar{I}(g) - I(\psi_N) + I(|\gamma_{n_N} - \gamma_n|). \qquad \square$$

Proposition 2.36. *Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$, then $\mathcal{R}$ is dense in $\Big(\mathcal{R}^I - \mathcal{R}^I, \bar{\bar{I}}(|\cdot|)\Big)$.*

PROOF. By Proposition 2.34, we can associate to each element l in $\mathcal{R}^I - \mathcal{R}^I$ a Cauchy sequence (γ_n) in $(\mathcal{R}, I(|\cdot|))$ such that $l(x) = \lim_n \gamma_n(x)$ $\forall x \in \Omega_f$ Finally, by Proposition 2.35 we have that $\lim_n \bar{\bar{I}}(|l - \gamma_n|) = 0$. $\square$

2.8 Completeness

Theorem 2.37.

$$\left(\mathcal{R}^I - \mathcal{R}^I, \bar{\bar{I}}(|\cdot|)\right) \text{ is complete.}$$

PROOF. Let (l_n) be a Cauchy sequence in $\left(\mathcal{R}^I - \mathcal{R}^I, \bar{\bar{I}}(|\cdot|)\right)$. Retracing the same reasoning exhibited in Proposition 2.35, one can deduce the existence of a subsequence (l_{n_k}) such that the real series $\sum_{k=1}^{\infty} \bar{I}(|l_{n_{k+1}} - l_{n_k}|)$ is convergent. The three sequences

$$\begin{aligned} s_N &:= \sum_{k=1}^{N-1} |l_{n_{k+1}} - l_{n_k}| \\ f_N &:= l_{n_1}^+ + \sum_{k=1}^{N-1} \left(l_{n_{k+1}} - l_{n_k}\right)^+ \\ g_N &:= l_{n_1}^- + \sum_{k=1}^{N-1} \left(l_{n_{k+1}} - l_{n_k}\right)^- \end{aligned}$$

are all increasing sequences in $(\mathcal{R}^I - \mathcal{R}^I, \bar{\bar{I}})$ whose integrals are uniformly bounded. The Beppo Levi Theorem implies they converge almost everywhere. Thus, the subsequence (l_{n_k}) converges almost everywhere. Indeed, $l_{n_N} \equiv f_N - g_N$. Moreover, $\left|l_{n_N}\right| \leq \sum_{k=1}^{\infty} |l_{n_{k+1}} - l_{n_k}| + |l_{n_1}| \in \mathcal{R}^I - \mathcal{R}^I$. Lebesgue's Dominated Convergence Theorem implies that there exists a function $l \in \mathcal{R}^I - \mathcal{R}^I$ such that $\lim_k l_{n_k} \equiv l$; so $\lim_k \bar{\bar{I}}(|l - l_{n_k}|) = 0$. Finally, we simply observe that $\bar{\bar{I}}(|l - l_n|) \leq \bar{\bar{I}}(|l - l_{n_k}|) + \bar{\bar{I}}(|l_{n_k} - l_n|)$. □

2.9 Functions Equal Almost Everywhere

At Section 2.4 in Notation 2.24 we introduced relation $\equiv$. Then we noticed its role was marginal in extending I if compared with relation $\ll$. Now it should be evident how to apply Corollary 2.27 to extend $\bar{\bar{I}}$ to functions that are equal almost everywhere to elements in $\mathcal{R}^I - \mathcal{R}^I$.

Definition 2.38. Let I be an integral on a Riesz space $\mathcal{R}(\Omega)$. We define $\mathcal{F} = \mathcal{F}(I, \mathcal{R})$ as the class of real functions a.e. defined on Ω.

Definition 2.39. Let I be an integral on a Riesz space $\mathcal{R}(\Omega)$. Let $\mathcal{A} \subset \mathcal{F}(I, \mathcal{R})$, we define

$$\mathcal{A}^- := \left\{g \in \mathcal{F} \ : \ g \equiv f \text{ for some } f \in \mathcal{A}\right\}.$$

Extension 2.40. Let I be an integral on a Riesz space $\mathcal{R} = \mathcal{R}(\Omega)$. Let f be in $\mathcal{R}^I - \mathcal{R}^I$, if $g \equiv f$, then we define the integral of g as $\bar{\bar{I}}(f)$.

Contrary to the usual treatments of the Riesz approach, we delayed this extension as far as possible to emphasize that we can get complete function spaces and Limit Theorems without adding functions that lack a clear representation.

3 Concrete Settings

3.1 Basic Function Spaces

3.1.1 Riemann Integrable Functions

Let $\mathcal{I}$ denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are zero outside some bounded interval, and are Riemann integrable on this interval, that is

$$f \in \mathcal{I} \iff \exists\, [a,b] \quad : \left\{ \begin{array}{rl} i) & -\infty < a < b < +\infty \\ ii) & \forall x \notin [a,b] \quad f(x) = 0 \\ iii) & f \text{ is Riemann integrable over } [a,b]. \end{array} \right.$$

In this case we will write

$$\int f := \int_a^b f(x)\, dx \ .$$

3.1.2 Step Functions

By a step function we mean any function $\phi : \mathbb{R} \to \mathbb{R}$ that takes a finite number of non-Zero real values $c_1, \ldots, c_m$ over bounded intervals (possibly degenerate [13])$\Sigma_1 \ , \ldots , \Sigma_m$ respectively:

$$\phi(x) = \sum_{k=1}^{m} c_k \cdot 1_{\Sigma_k}(x) \quad \text{where} \quad 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

We will let $\mathcal{S}$ denote the set of step functions. It is simple to verify that $\mathcal{S}$ is a Riesz space and that the Riemann integral is a well defined pointwise increasing linear functional on $\mathcal{S}$. Less simple is to verify the Daniell continuity of the Riemann integral: [14]

[13] By a degenerate interval we mean a singleton or the empty set.
[14] See for instance [10].

Lemma 3.1. Let (δ_n) be a Riesz sequence for $\left(\int, \mathcal{S}\right)$, such that

$$\lim_n \delta_n(x) = 0 \qquad \forall x \in \mathbb{R}. \qquad \text{Then,} \qquad \lim_n \int \delta_n = 0.$$

3.1.3 Continuous Function with Compact Support

Another Riesz space, often used to construct the Lebesgue integral via the Riesz method, is the space $\mathcal{K}(\Omega) = C_c(\Omega)$ of continuous functions with compact support[15] in an open set $\Omega \subset \mathbb{R}^N$.

We simply note that (pointwise) Daniell continuity of the Riemann integral on $\mathcal{K}(\Omega)$ follows directly from Dini's Theorem:

Dini's Theorem. *Let f_n be a monotone sequence of continuous functions defined on a compact set K. If the sequence converges pointwise to a continuous function f, then it converges uniformly on K.*

3.2 Complete Function Spaces

Consider the space $\mathcal{S}$. It is well known that the (semi)normed space $(\mathcal{S}, \int |\cdot|)$ is not complete. We know equally that any (semi)normed space has an abstract completion. However, such abstract completion, being a set of equivalence classes of Cauchy sequences, is not a space of functions. Since many crucial theorems used to prove existence of functions (in differential equations, calculus of variations, etc.) rest on completeness, we need to represent as functions the elements of such completions.

Functional representation of complete spaces and extension of the Riemann integral are closely connected problems. Indeed, we have to find a space of functions $\hat{\mathcal{S}}$ and an integral $\hat{\int} : \hat{\mathcal{S}} \to \mathbb{R}$ such that

$$\begin{array}{ll} 1. & \mathcal{S} \subset \hat{\mathcal{S}} \\ 2. & \hat{\int} \text{ extends } \int \\ 3. & \mathcal{S} \text{ is dense in } \left(\hat{\mathcal{S}}, \hat{\int}(|\cdot|)\right) \\ 4. & \left(\hat{\mathcal{S}}, \hat{\int}(|\cdot|)\right) \text{ is complete.} \end{array} \tag{9}$$

The space $L^1(\mathbb{R})$ of all Lebesgue integrable functions has such properties. While elements in $\mathcal{S}$ have a quite explicit representation, functions in $L^1(\mathbb{R})$ are no more so apparent. In order to describe functions in $L^1(\mathbb{R})$ one usually

[15]See [4], [6] and [13].

writes:

$$L^1(\mathbb{R}) = \big\{u \overset{\text{a.e.}}{=} \lim_n \gamma_n \ ; \quad \text{where } (\gamma_n) \text{ is a Cauchy sequence in } \big(\mathcal{S}, \int |\cdot|\big)\big\}.$$

Thus, contrary to what we might have expected, not every Cauchy sequence is apt to represent elements in $L^1(\mathbb{R})$ as an almost everywhere limit. As a matter of fact there exist Cauchy sequences in $\big(\mathcal{S}, \int |\cdot|\big)$ that converge nowhere.

The classical Riesz approach allows a more explicit representation of functions in $L^1(\mathbb{R})$:

$$L^1(\mathbb{R}) = \big\{u \overset{\text{a.e.}}{=} \lim_n \phi_n - \lim_n \psi_n \ ; \text{where} (\phi_n), (\psi_n) \text{ are Riesz sequences for} \big(\int, \mathcal{S}\big)\big\}.$$

Theorem 2.37 shows that the space $\mathcal{S}^{\int} - \mathcal{S}^{\int} =$

$$\big\{u = \lim_n \phi_n - \lim_n \psi_n \ ; \quad \text{where } (\phi_n), (\psi_n) \text{ are Riesz sequences for } \big(\int, \mathcal{S}\big)\big\}$$

is a completion of $\big(\mathcal{S}, \int |\cdot|\big)$ as well. This last result shows how simple it is to extend the Riemann integral to obtain a complete space. As a matter of fact the extended Riemann integral (i.e., the Lebesgue integral) of a function $u \in \mathcal{S}^{\int} - \mathcal{S}^{\int}$ is simply $\int u := \lim_n \int \phi_n - \lim_n \int \psi_n$.

3.3 Null Sets with Respect to the Riemann Integral

The following theorem tells us that if we consider as Riesz space $\mathcal{S}$, and as integral the Riemann integral, null sets for $\big(\int, \mathcal{S}\big)$ are reasonably small sets.

Theorem 3.2. *Let N be a subset of $\mathbb{R}$. The following properties are equivalent:*

1. *N is a null set for $\big(\int, \mathcal{S}\big)$ (see Definition 2.19)*
2. *Some Riesz sequence for $\big(\int, \mathcal{S}\big)$ diverges on N*
3. *Some Riesz sequence for $\big(\int, \mathcal{S}^{\int}\big)$ diverges on N*
4. *N is a Lebesgue null set.*

Equivalence [1. $\Leftrightarrow$ 2.] holds by Definition 2.19. Equivalence [2. $\Leftrightarrow$ 3.] holds by Monotone Convergence Theorem 2.6.1. Proof of [1. $\Longleftarrow$ 4.] can be found in [12] on in [9].

4 Examples

In this section I will represent the Riemann integral.

Example 4.1. Let $C \subset [0,1]$ be the Cantor set [16]. Consider the function $f(x) = 1_C(x)$. Then, $f \in \mathcal{I}$ (see Section 3.1), $f \in \mathcal{S}^I - \mathcal{S}^I$, and $f \notin \mathcal{S}^I$.

Indeed the following more general result holds:

Proposition 4.2. *If $A \subset \mathbb{R}$ is any uncountable null set [17], then it cannot exist in $\mathcal{S}^I$ any function g such that:*

$$g(x) \begin{cases} \leq 0 & \text{if } x \notin C \\ > 0 & \text{otherwise.} \end{cases} \tag{10}$$

Let A° denote the interior of A. We just recall an apparent lemma:

Lemma 4.3. Let $\phi \in \mathcal{S}$. If $c \geq 0$, then

$$\big\{\phi > c\big\}^\circ = \emptyset \iff \big\{\phi > c\big\} \text{ is finite or empty.}$$

PROOF OF PROPOSITION 4.2. Assume that the thesis is false, then there exists in $\mathcal{R}^I$ a function g that satisfies (10). By construction there exists a Riesz sequence (ϕ_n) in $\mathcal{S}$ such that $g(x) = \lim_n \phi_n(x)$. Put

$$A_n := \big\{x \ : \ \phi_n(x) > 0\big\}.$$

Since (ϕ_n) is a Riesz sequence, $A_n \subset A_{n+1}$, and also

$$C = \big\{x \ : \ \lim_n \phi_n(x) > 0\big\} = \bigcup_{n=1}^{\infty} A_n.$$

As every null set has empty interior, each $A_n \subset C$ has empty interior. By Lemma 4.3 it follows that every set A_n is finite or empty, but this contradicts uncountability of C. $\square$

5 Order and Algebraic Extension of $\mathbb{R}$

In this paragraph we will show why we cannot extend real vector space operations on $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ in a way that would be compatible with its extended order structure.

The order $\leq$ relation on $\mathbb{R}$ is extended to $\mathbb{R} \cup \{-\infty, +\infty\}$ in the following way

$$\forall x \in \mathbb{R} \qquad -\infty \leq x \leq +\infty. \tag{11}$$

[16] For a classical definition see [6].

[17] As, for instance, the Cantor set.

We begin trying to extend the sum. We want it still to be commutative and associative. Besides we want 0 to be the neutral element. That is,

$$\forall x \in \mathbb{R} \cup \{-\infty, +\infty\} \qquad x + 0 = x.$$

We begin by attempting to assign a value to $x + (+\infty)$ when $x \in \mathbb{R}$. One can intuitively guess: $x + (+\infty) = +\infty$; i.e.,

$$\forall x \in \mathbb{R} \qquad x + (+\infty) = +\infty. \tag{12}$$

We will show that this is the only choice compatible with the property:

$$\forall x, y, z \in \mathbb{R} \cup \{-\infty, +\infty\} \qquad x \leq y \implies x + z \leq y + z. \tag{13}$$

Indeed, suppose (12) is not true, that is $\exists \bar{x} \in \mathbb{R}$ such that $\bar{x} + (+\infty) \neq +\infty$. Two choices are possible:

1. $\bar{x} = -\infty$, or
2. $\exists \hat{x} \in \mathbb{R}$ such that $\bar{x} + (+\infty) = \hat{x}$.

Suppose $\bar{x} + (+\infty) = -\infty$. By (11) we know that $-\infty \leq \bar{x}$. Applying (13) we have $-\infty + (-\bar{x}) = \bar{x} + (+\infty) + (-\bar{x}) = +\infty \leq 0$. Now consider the case where $\exists \hat{x} \in \mathbb{R}$ such that $\bar{x} + (+\infty) = \hat{x}$. By (11) we know that$\forall x \in \mathbb{R}$ $x \leq +\infty$. Applying (13) with $z = \bar{x}$, we have that $\forall x \in \mathbb{R}$ $x + \bar{x} \leq +\infty + \bar{x} = \hat{x}$. Thus, we arrive at the contradiction: $\forall x \in \mathbb{R}$ $x \leq \hat{x} - \bar{x} \in \mathbb{R}$.

So, for every x in $\mathbb{R}$ we are forced to choose $x + (+\infty) = +\infty$ and analogously, $x + (-\infty) = -\infty$. Since $(\mathbb{R}, +)$ is an abelian group, now we have to find an element c in $\mathbb{R} \cup \{-\infty, +\infty\}$ such that $c + (+\infty) = 0$. We have just seen that such a $c \notin \mathbb{R}$, so we have only two possibilities, either $c = +\infty$ or $c = -\infty$.

Suppose $+\infty + (+\infty) = +\infty$. We have showed that $\forall x \in \mathbb{R}$ $x + (+\infty) = +\infty$; that implies: $\forall x \in \mathbb{R} \quad x + (+\infty) + (+\infty) = +\infty + (+\infty)$; that is: $\forall x \in \mathbb{R} \quad x = 0$. Finally, one obtains the same contradiction assuming $+\infty + (-\infty) = 0$.

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