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THE TRANSLATION $\frac{1}{2}$ IN THE THEORY OF DIRICHLET SERIES*

Abstract

In three different problems concerning Dirichlet series, we study in detail the role and optimality of a translation by 1/2 on the abscissas. In particular, we show the non-existence of Rudin-Shapiro like Dirichlet polynomials.

1 Introduction and Basic Notation

In the theory of Dirichlet series $A(s) = \sum_1^\infty a_n n^{-s}$ it frequently occurs that a hypothesis on the behavior of A at some point $\alpha \in \mathbb{R}$ is followed by a conclusion on the behavior of the same series, or of a related one, at α shifted by $\frac{1}{2}$ to the right; that is, $s = \alpha$ (or $\mathcal{R}s > \alpha$) becomes $s = \alpha + 1/2$ (or $\mathcal{R}s > \alpha + 1/2$), and while the role of this translation is not clear, it sometimes turns out to be "optimal" and it may be a non-trivial matter to prove this optimality. The aim of this paper is the detailed study of three specific examples with either new proofs or new results. In each case, we shall try to explain why 1/2 comes into the matter and why it is optimal. We shall also study some related phenomena. Before stating our three examples, it is convenient to recall some definitions and facts. There are three abscissas connected to the convergence of $A(s) = \sum_1^\infty a_n n^{-s}$ in $[-\infty, \infty]$. First is the abscissa of convergence σ_c. A converges

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for $\mathcal{R}s > \sigma_c$ and diverges for $\mathcal{R}s < \sigma_c$. Second is the abscissa of uniform convergence σ_u. A converges uniformly in each half-plane $\mathcal{R}s \geq \sigma$ $(\sigma > \sigma_u)$, and does not converge uniformly in any half-plane $\mathcal{R}s \geq \sigma \quad (\sigma < \sigma_u)$. Third is the abscissa of absolute convergence σ_a. A converges absolutely for $\mathcal{R}s > \sigma_a$ and does not converge absolutely for $\mathcal{R}s < \sigma_a$.

If A diverges at zero, these three abscissas are given by Hadamard-like formulas (see [Q1] for example).

$$\sigma_c = \overline{\lim} \frac{\log |A_N|}{\log N},\ \sigma_u = \overline{\lim} \frac{\log U_N}{\log N} \text{ and } \sigma_a = \overline{\lim} \frac{\log A_N^*}{\log N}$$

where

$$A_N = a_1 + \cdots + a_N,\ U_N = \sup_{t\in\mathbb{R}} \Big|\sum_1^N a_n n^{it}\Big| \text{ and } A_N^* = |a_1| + \cdots + |a_N|.$$

If $f_N(t) = \sum_1^N a_n n^{it}$, then clearly $\sum_1^N |a_n|^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f_N(t)|^2 dt \leq U_N^2$, whence $A_N^* \leq N^{1/2} U_N$ by the Cauchy-Schwarz inequality (note the appearance of 1/2); so that

$$\sigma_a \leq \sigma_c + 1 \text{ and } \sigma_a \leq \sigma_u + 1/2. \tag{1}$$

The Dirichlet product $C = AB$ of two Dirichlet series $A(s) = \sum_1^\infty a_n n^{-s}$ and $B(s) = \sum_1^\infty b_n n^{-s}$ is formally defined by $C(s) = \sum_1^\infty c_n n^{-s}$, where $c_n = \sum_{ij=n} a_i b_j$. (More generally, if $A(s) = \sum_1^\infty a_n e^{-\lambda_n s}$, $B(s) = \sum_1^\infty b_n e^{-\mu_n s}$ and ν_n is the set of $\lambda_i + \mu_j$ rearranged in increasing order, then $C(s) = A(s)B(s) = \sum_1^\infty c_n e^{-\nu_n s}$, where $c_n = \sum_{\lambda_i+\mu_j=\nu_n} a_i b_j$.) It should be noted that, formally, $\sum_1^\infty a_n n^{it} = \sum_1^\infty a_n e^{it\log n}$. Thus, a Dirichlet series looks like a trigonometric series, but the frequences run over the set $\{\log n\}$ in contrast to the set of integers for a trigonometric series.
We now come to our three examples.

Example 1. If $\sum_1^\infty |a_n|^2 < \infty$, then $\sum_1^\infty a_n n^{-1/2+it}$ converges for almost all t (with respect to Lebesgue measure on $\mathbb{R}$) and this is optimal; that is $\sum_1^\infty a_n n^{-\alpha+it}$ may diverge for all t and all $\alpha < 1/2$.

This example is a recent result of Hedenmalm and Saksman [HS] reminiscent of the celebrated Carleson's theorem. If $\sum_1^\infty |a_n|^2 < \infty$, then $\sum_1^\infty a_n e^{int}$ converges for almost all t. We shall give a very simple proof of Example 1 based on Carleson's theorem for integrals.

Example 2. If $A(s) = \sum_1^\infty a_n n^{-s}$ and $B(s) = \sum_1^\infty b_n n^{-s}$ converge at $s = 0$, then $C = AB$ converges at $s = 1/2$; moreover, this is optimal in general; i.e., $\sum_1^\infty n^{-\delta} c_n$ may diverge for all $\delta < 1/2$.

This example is an old result of Stieltjes as concerns the convergence at 1/2 and has a funny story. It was first conjectured that $\sum_1^\infty n^{-\delta} c_n$ converges for any $\delta > 0$. This was refuted by Landau (see [T] p. 127) using the so-called Lindelöf (or order) function of a Dirichlet series, the functional equation of the zêta function and Stirling's formula for the gamma function: one must take $\delta \geq 1/8$. Later Bohr ([B]) refined Landau's argument by constructing an ad hoc series A with the largest possible order function. One has in general to take $\delta \geq \frac{1}{2}$. A simplified version of Bohr's construction, using the Baire category theorem, was given in [Q2]. The optimality of various extensions (some due to Landau) of Stieltjes's result, using or not the order function, was studied in [KQ]. We will give here a proof of Example 2 (and of some extensions) which does not use the order function but only the Banach-Steinhaus theorem for bilinear forms (Baire's theorem after all!), and which we hope explains clearly the appearance of the translation 1/2.

Example 3. (1) The inequality $\sigma_a \leq \sigma_u + 1/2$ of (1) is optimal in general. This fact is an old and non-trivial result of Bohnenblust and Hille (see [BoH]). In view of the Hadamard formulas for σ_u, σ_a, it amounts to the following.

(1′) For each integer $N \geq 1$, find scalars $a_1, \ldots, a_N$ such that the quotient A_N^*/U_N is as close as possible to $N^{1/2}$. In other words, we have to find a Dirichlet polynomial $\sum_1^N a_n n^{it} = P_N(t)$ for which $\| P_N \|_\infty = \sup_{t\in\mathbb{R}} |P_N(t)|$ is as small as possible compared to $\sum_1^N |a_n|$. Let us recall that (cf. [R]) if $P_N(t) = \sum_0^N \delta_n e^{int}$, where (δ_n) is the ± 1-valued Rudin-Shapiro sequence, inductively defined by $\delta_0 = 1$, $\delta_{2n} = \delta_n$, $\delta_{2n+1} = (-1)^n \delta_n$, then $\| P_N \|_\infty \leq (2+\sqrt{2})\sqrt{N+1}$, while $|\delta_0| + \cdots + |\delta_N| = N+1$. Here, we cannot use ± 1-valued sequences because of Bohr's inequality (see [Q1]): $\| P_N \|_\infty \geq \sum_{\substack{p \leq N \\ p \text{ prime}}} |a_p| \sim \frac{N}{\log N}$. But, we can use $\pm 1, 0$-valued sequences (see [Q1]) to obtain a Dirichlet polynomial $P_N(t) = \sum_1^N a_n n^{it}$ for which the following holds.

(1″) $A_N^*/U_N \geq \alpha\sqrt{N}\exp(-\beta\sqrt{\log N \log_2 N})$. Here, α and β are absolute positive constants, $\log N$ the natural logarithm of N, $\log_2 N = \log\log N$ the iterated logarithm. In view of (1″), a natural question is: can one go further and obtain, in the case of Dirichlet polynomials, something analogous to the example of Rudin and Shapiro, which could give that $A_N^*/U_N \geq \delta\sqrt{N}$? We will see that this is <u>never the case</u> and that, up the exact value of the constants α and β, (1″) is optimal: there exist numerical constants α and $\beta > 0$ such that, for any integer $N \geq 3$ and any complex numbers $a_1, \ldots, a_N$, one has the following inequality.

$$(1''') \quad \frac{A_N^*}{U_N} \leq \alpha\sqrt{N}\exp(-\beta\sqrt{\log N \log_2 N}).$$

The paper is organized as follows. The truth (and optimality) of Example 1 is proved in Section 2, the optimality of Example 2 is proved in Section 3, and various extensions are given ; the non-existence of Rudin-Shapiro like sequences for Dirichlet polynomials, under the form of $(1''')$, is proved in Section 4. Finally, Section 5 is devoted to some concluding remarks and questions.

2 Almost Everywhere Convergence of Dirichlet Series

We are going to prove the following theorem.

Theorem 2.1. *a) Suppose that $\sum_1^\infty |a_n|^2 < \infty$. Then, the series $\sum_1^\infty a_n n^{-1/2+it}$ converges for almost all $t \in \mathbb{R}$ (with respect to the Lebesgue measure).*

b) There exists a square-summable sequence (a_n) such that $\sum_1^\infty a_n n^{-\alpha+it}$ diverges for each $\alpha < 1/2$ and each $t \in \mathbb{R}$.

c) "Carleson's condition" $\sum_1^\infty |a_n|^2 < \infty$ is optimal in the following sense : if (A_n) is a sequence of nonnegative numbers, (a_n) a sequence of complex numbers such that $\sum_1^\infty A_n^2 = \infty$, the sequence $(A_n n^{1/2})$ is nonincreasing, $|a_n| \geq A_n$ for $n \geq n_0$, then there exists a sequence (ε_n), $\varepsilon_n = \pm 1$ for all n, such that the series $\sum_1^\infty \varepsilon_n a_n n^{-1/2+it}$ diverges for each $t \in \mathbb{R}$.

Proof of a) We shall give a very simple proof. Let $f : [1, \infty[\to \mathbb{C}$ be defined by $f(x) = a_n$ if $n \leq x < n+1$ and $n = 1, 2, \ldots$. Clearly, $\int_1^\infty |f(x)|^2\, dx = \sum_1^\infty |a_n|^2$. Now, set $g(y) = f(e^y)e^{y/2}$ for $y \geq 0$ (Observe how 1/2 comes into play.) and make the change of variable $x = e^y$ to get $\int_0^\infty |g(y)|^2\, dy = \sum_1^\infty |a_n|^2$; i.e., $g \in L^2(\mathbb{R}^+)$. Using Carleson's theorem for integrals (see for example [BPW]), we get that $\int_0^\infty g(y)e^{ity}\, dy = \lim_{A\to\infty} \int_0^A g(y)e^{ity}\, dy$ exists a.e.in t.

Equivalently $\int_1^\infty \frac{f(x)}{\sqrt{x}} e^{it\log x}\,dx$ converges a.e. Now

$$\begin{aligned}\int_1^{N+1} \frac{f(x)}{\sqrt{x}} e^{it\log x}\,dx &= \sum_{n=1}^{N} \int_n^{n+1} a_n \frac{e^{it\log x}}{\sqrt{x}}\,dx \\ &= \sum_1^N \int_n^{n+1} a_n \Big(\frac{e^{it\log x}}{\sqrt{x}} - \frac{e^{it\log n}}{\sqrt{n}}\Big)\,dx + \sum_{n=1}^{N} a_n n^{-1/2+it} \\ &=: \sum_1^N b_n(t) + \sum_1^N a_n n^{-1/2+it}\end{aligned}$$

with $|b_n(t)| = O(n^{-3/2})$, since $a_n \to 0$ and $\frac{d}{dx}(\frac{e^{it\log x}}{\sqrt{x}}) = O(x^{-3/2})$. The integral $\int_1^\infty \frac{f(x)}{\sqrt{x}} e^{it\log x}\,dx$ and the series $\sum_1^\infty a_n n^{-1/2+it}$ are thus equiconvergent, and the result follows.

PROOF OF b) Take $a_n = n^{-1/2}\big(\log(n+1)\big)^{-1}$, so that $\sum_1^\infty |a_n|^2 < \infty$ and that the abscissa of convergence is $\sigma_c = 1/2$. Therefore, $\sum_1^\infty a_n n^{-\alpha+it}$ diverges for each $\alpha < \frac{1}{2}$ and each $t \in \mathbb{R}$. Observe that $\sum_1^\infty a_n n^{-1/2+it}$ converges for all $t \neq 0$.

PROOF OF c) We follow the papers [N], [S], [S2], [DE], [G], where similar results were proved for the divergence of power and trigonometric series. We set $A_n = n^{-1/2}\delta_n$, where δ_n decreases, and we assume, without loss of generality, that $n_0 = 1$ and that $A_n n^{-1/2} \to 0$. The strategy is to build closed intervals $\Delta_k = [t_k - \ell_k, t_k + \ell_k]$, signs $\varepsilon_n = \pm 1$, and blocks $D_k(t) = \sum_{p_k \le n < p_{k+1}} \varepsilon_n a_n n^{-1/2+it}$ of the corresponding Dirichlet series such that:

$$\text{Each } t \in \mathbb{R} \text{ belongs to infinitely many intervals } \Delta_k \tag{2}$$

$$|D_k(t)| \ge 1/4 \text{ for each } k \ge 1 \text{ and each } t \in \Delta_k. \tag{3}$$

The first condition will hold if we have

$$\sum_1^\infty \ell_k = \infty. \tag{4}$$

In fact, it is then possible to construct consequently intervals $\Delta_1, \dots, \Delta_{k_1}$ covering $[-1, 1]$, intervals $\Delta_{k_1+1}, \dots, \Delta_{k_2}$ covering $[-2, 2]$, and so on. To obtain (3), we first choose inductively integers $p_1 < \dots < p_k < \dots$ so that $A_n n^{-1/2} \le 1$ for $n \ge p_1$ and that

$$1 < \sum_{p_k \le n < p_{k+1}} A_n n^{-1/2} \le 2. \tag{5}$$

This is possible since $\sum_1^\infty A_n n^{-1/2} \geq \sum_1^\infty A_1^{-1} A_n^2 = \infty$ and since $A_n n^{-1/2} \leq 1$ for $n \geq p_1$. It will be useful for the sequel to observe that

$$\sum_1^\infty \frac{1}{\log \frac{p_{k+1}}{p_k}} = \infty. \tag{6}$$

Indeed,

$$\sum_{p_k \leq n < p_{k+1}} A_n^2 = \sum_{p_k \leq n < p_{k+1}} \frac{\delta_n^2}{n} \leq \delta_{p_k} \sum_{p_k \leq n < p_{k+1}} \frac{\delta_n}{n} \leq 2\delta_{p_k}$$

in view of (5), so that $\Sigma \delta_{p_k} = \infty$. Moreover, (5) implies that

$$2 \geq \sum_{p_{k-1} \leq n < p_k} \frac{\delta_n}{n} \geq \delta_{p_k} \sum_{p_{k-1} \leq n < p_k} \frac{1}{n} \geq \delta_{p_k} \log \frac{p_k}{p_{k-1}}.$$

Therefore $\dfrac{1}{\log \frac{p_k}{p_{k-1}}} \geq \frac{\delta_{p_k}}{2}$, and this proves (6), allowing in (4) the choice, to which we will stick:

$$\ell_k = \frac{1}{4 \log \frac{p_{k+1}}{p_k}}. \tag{7}$$

Finally, we choose the numbers ε_n blockwise so that $\varepsilon_n = 1$ for $n < p_1$, $\epsilon_n = \pm 1$ for all n and that

$$\left| \sum_{p_k \leq n < p_{k+1}} \varepsilon_n a_n n^{-1/2 + it_k} \right| = \left| D_k(t_k) \right| \geq \frac{1}{2} S_k \geq \frac{1}{2}. \tag{8}$$

(Here, we have set $S_k = \sum_{p_k \leq n < p_{k+1}} |a_n| n^{-1/2}$, and used $|a_n| \geq A_n$ and (5).) This is possible, because if $z_1, \ldots, z_N \in \mathbb{C}$, one can find signs $\varepsilon_1, \ldots, \varepsilon_N$ such that $\varepsilon_n = \pm 1$ for all n and $\left| \sum_1^N \pm z_n \right| \geq \frac{1}{2} \sum_1^N |z_n|$ ([KQ] p. 491). (3) is now a consequence of (7) and (8), because D_k oscillates little in the small interval Δ_k around t_k. More precisely, setting $C_k(t) = p_k^{-it} \sum_{p_k \leq n < p_{k+1}} \varepsilon_n a_n n^{-\frac{1}{2} + it} = p_k^{-it} D_k(t)$, we have that

$$\| C_k' \|_\infty \leq \sum_{p_k \leq n < p_{k+1}} |a_n| n^{-1/2} \log \frac{n}{p_k} \leq S_k \log \frac{p_{k+1}}{p_k}$$

and that, for $t \in \Delta_k$

$$\begin{aligned} |D_k(t)| = &|C_k(t)| \geq |C_k(t_k)| - |t - t_k| \, \| C_k' \|_\infty \\ \geq &\frac{1}{2} S_k - \ell_k S_k \log \frac{p_{k+1}}{p_k} = \frac{1}{4} S_k \geq \frac{1}{4}, \end{aligned}$$

where we used (7) and (8). Therefore, (2) and (3) hold, and this clearly ends the proof of c). $\square$

Remark 2.1. A condition of regularity in c) is essential. Take $a_n = 1$ if $n = 2^k$ for some integer k and $a_n = 0$ otherwise. Then $\sum_1^\infty a_n^2 = \infty$, but the Dirichlet series $\sum_1^\infty a_n n^{-\sigma+it}$ is absolutely convergent for each $\sigma > 0$.

Remark 2.2. Take B_n decreasing to zero such that $\Sigma A_n^2 B_n^2 = \infty$ and now choose the $p_k' s$ in (5) so as to have $1 < \sum_{p_k \le n < p_{k+1}} A_n B_n n^{-1/2} \le 2$. Leave the rest unchanged. For $t \in \Delta_k$ we now have

$$|D_k(t)| \ge \frac{1}{4} S_k \ge \frac{1}{4} \sum_{p_k \le n < p_{k+1}} A_n n^{-1/2} \ge \frac{1}{4B_{p_k}},$$

which shows that the Dirichlet series $\sum_1^\infty \varepsilon_n a_n n^{-1/2+it}$ diverges unboundedly for each $t \in \mathbb{R}$.

Remark 2.3. Note that part b) of Theorem 2.1 does not follow from c).

3 Convergence of Products of Dirichlet Series

We first recall some general facts. Let $(\lambda_n)_{n\ge1}$ and $(\mu_n)_{n\ge1}$ be two sequences of positive numbers strictly increasing to $+\infty$, and (ν_n) the sequence of $\lambda_i + \mu_j$ rearranged in increasing order. If $A(s) = \sum_1^\infty a_n e^{-\lambda_n s}$ and $B(s) = \sum_1^\infty b_n e^{-\mu_n s}$, we set formally $C(s) = A(s)B(s) = \sum_1^\infty c_n e^{-\nu_n s}$, with $c_n = \sum_{\lambda_i+\mu_j=\nu_n} a_i b_j$. We also set $A_n = a_1 + \cdots + a_n$, $\alpha(x) = \sum_{\lambda_n \le x} a_n$, and define similarly $B_n, C_n, \beta(x), \gamma(x)$. We then have the following useful identity ([HaR]), easy to check

$$\int_0^x \gamma(t)dt = \int_0^x \alpha(x-t)\beta(t)dt \text{ for all } x \ge 0. \tag{9}$$

Let us note in passing that (9) provides a particularly clear proof of the generalized Mertens theorem, "If the three series $(a_n), (b_n), (c_n)$ converge with respective sums a, b, c, then $c = ab$." Just divide the two members of (9) by x and let x tend to infinity to get the result. A corollary of (9) is that

$$\text{If } \sum_1^\infty a_n \text{ and } \sum_1^\infty b_n \text{converge, then } \sum_1^N \frac{C_n}{n} = O(\log N). \tag{10}$$

In fact, putting $\lambda_n = \log n$, $\mu_n = \log n$, $x = \log(N+1)$ in (9), we get

$$\begin{aligned} O(\log N) = \sum_{n=1}^{N} C_n\big(\log(n+1) - \log n\big) &= \sum_{1}^{N} C_n \log(1+\frac{1}{n}) \\ &= \sum_{1}^{N} C_n\big(\frac{1}{n} + O(\frac{1}{n^2})\big) \\ &= \sum_{1}^{N} \frac{C_n}{n} + O(\log N), \end{aligned}$$

since clearly $C_n = O(n)$. Let us also recall the Kronecker Lemma ([K]).

$$\text{If } \sum_{1}^{\infty} \frac{c_n}{\varphi_n} \text{ converges and } \varphi_n \text{ increases to } \infty, \text{ then } \frac{C_N}{\varphi_N} \to 0. \tag{11}$$

Finally, we recall the following result of Landau ([K]), which extends that of Stieltjes quoted in the introduction.

Theorem 3.1. *If $A(s) = \sum_1^\infty a_n n^{-s}$ and $B(s) = \sum_1^\infty b_n n^{-s}$ converge respectively at $s = \rho_1$ and $s = \rho_2$, where $\rho_1, \rho_2 \in \mathbb{R}$ and $|\rho_1 - \rho_2| < 1$, then $C(s) = A(s)B(s) = \sum_1^\infty c_n n^{-s}$ converges at $s = \frac{1}{2}(\rho_1 + \rho_2 + 1)$.*

PROOF. We just give a proof for $\rho_1 = \rho_2 = 0$; not the simplest one, but one which, we believe, clearly shows the role of the translation 1/2 and the key role of the term $N^{-1/2}C_N$. The main point is

$$C_N = o(N^{1/2}). \tag{12}$$

The exponent 1/2 will appear through the hyperbola method of Dirichlet, when we write

$$\begin{aligned} C_N = \sum_{ij \le N} a_i b_j &= \Big(\sum_{i \le \sqrt{N}} a_i\Big)\Big(\sum_{j \le \sqrt{N}} b_j\Big) + \sum_{i \le \sqrt{N}} a_i \Big(\sum_{\sqrt{N} < j \le \frac{N}{i}} b_j\Big) \\ &+ \sum_{j \le \sqrt{N}} b_j \Big(\sum_{\sqrt{N} < i \le \frac{N}{j}} a_i\Big) = O(1) + \sum_{i \le \sqrt{N}} o(1) + \sum_{j \le \sqrt{N}} o(1) = o(\sqrt{N}), \end{aligned}$$

proving (12). Now, two Abel's summations by parts, where we set $S_n =$

$\sum_{k=1}^{n} \frac{C_k}{k}$ and use (10) and (12), give

$$\sum_1^N \frac{c_n}{\sqrt{n}} = \frac{C_N}{\sqrt{N}} + \sum_1^{N-1} C_n\Big(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\Big) = \frac{C_N}{\sqrt{N}} + \sum_1^{N-1} \frac{C_n}{2n^{3/2}} + \varepsilon_N$$
$$= \frac{C_N}{\sqrt{N}} + \frac{S_{N-1}}{2(N-1)^{1/2}} + \sum_1^{N-2} \frac{S_n}{4n^{3/2}} + \varepsilon'_N,$$

where $\varepsilon_N, \varepsilon'_N$ are convergent sequences and $\sum_1^\infty \frac{S_n}{n^{3/2}}$ an absolutely convergent series from (10). By (12), this ends the proof and shows that the converse of Kronecker's lemma is essentially true: $\frac{C_N}{\sqrt{N}} \to 0$, "therefore" $\sum_1^\infty \frac{c_n}{\sqrt{n}}$ converges. The fact that $\sum_1^\infty c_n n^{-1/2} = \big(\sum_1^\infty a_n n^{-1/2}\big)\big(\sum_1^\infty b_n n^{-1/2}\big)$ is now a consequence of Mertens theorem. □

What we will now show is the optimality of Theorem 9 in a way which avoids the use of the order function of Lindelöf although being non explicit. (As far as we know, no explicit example giving the optimality, even for $\rho_1 = \rho_2 = 0$, has ever been given.)

Theorem 3.2. *Let $\rho_1, \rho_2 \in \mathbb{R}$ with $|\rho_1 - \rho_2| < 1$, and (φ_n) a sequence of positive numbers such that $\frac{\varphi_n}{n^{\rho_2}}$ increases to infinity and such that, for every pair of Dirichlet series A and B converging at ρ_1 and ρ_2 respectively, one has the convergence of $\sum_1^\infty \frac{c_n}{\varphi_n}$, where $\sum_1^\infty c_n n^{-s} = A(s)B(s)$. Then*

$$\varphi_n \geq \delta n^{\frac{1}{2}(\rho_1+\rho_2+1)} \ (\delta \text{ positive constant}). \tag{13}$$

In particular, one can find A and B converging at ρ_1 and ρ_2 such that $C = AB$ diverges at every $\sigma < \frac{1}{2}(\rho_1 + \rho_2 + 1)$, and Theorem 3.1 is optimal.

Proof. By shifting, we may and shall assume that $\rho_2 = 0$. Let E be the Banach space of sequences $a = (a_n)$ such that the series $\sum_1^\infty a_n n^{-\rho_1}$ converges, equipped with the norm $\| a \| = \sup_n |\sum_{k=1}^n a_k k^{-\rho_1}|$. Define F similarly, replacing ρ_1 by 0. The hypothesis and Kronecker's Lemma (11) imply that $\dfrac{c_1 + \ldots + c_n}{\varphi_n} \to 0$, in particular is bounded. In other words, if (L_n) is the sequence of bilinear forms on $E \times F$ defined by

$$L_n(a, b) = \frac{c_1 + \ldots + c_n}{\varphi_n},$$

one has $\sup_n |L_n(a, b)| < \infty$ for all $a \in E, b \in F$. It now follows from the Banach-Steinhaus Theorem for the space $B(E\times F)$ of continuous bilinear forms on $E \times F$ (the space $B(E \times F)$ can be identified with the space $\mathcal{L}(E, F^*)$ of continuous linear maps from E to the dual F^* of F) that $M = \sup_n \|L_n\| < \infty$; i.e., that

$$|L_n(a,b)| \leq M \parallel a \parallel\parallel b \parallel \text{ for each } (a,b) \in E \times F. \tag{14}$$

First fix a perfect square n, set $B_m = b_1 + \cdots + b_m$ and $\lambda_i = [\frac{n}{i}]$ where [ ] denotes the integral part. Then observe that $c_1 + \cdots + c_n = \sum_{ij \leq n} a_i b_j = \sum_{i \leq n} a_i \big(\sum_{j \leq \frac{n}{i}} b_j\big) = \sum_1^n a_i B_{\lambda_i}$, so that (14) can be rewritten as

$$\big|\sum_1^n a_i B_{\lambda_i}\big| \leq M\varphi_n \parallel a \parallel\parallel b\| \text{ for each } (a,b) \in E \times F. \tag{15}$$

Now, the following simple observation is the key to understand the appearance of the translation 1/2.

$$\text{If } 1 \leq i < j \leq \sqrt{n} < k \leq n, \text{ then } \lambda_i > \lambda_j > \lambda_k. \tag{16}$$

In fact, $\frac{n}{i} - \frac{n}{j} \geq \frac{n}{j-1} - \frac{n}{j} = \frac{n}{j(j-1)} \geq \frac{n}{j^2} \geq 1$; so that $\lambda_i - \lambda_j \geq 1$. Moreover, $\lambda_k < \sqrt{n} \leq \lambda_j$. (Note that $\sqrt{n}$ is an integer.) The relation (16) shows that the $\lambda_i' s$ $(i \leq \sqrt{n})$ are distinct, and distinct from the $\lambda_k' s$ $(k > \sqrt{n})$. One can therefore choose $b \in F$ so that $a_i B_{\lambda_i} = |a_i|,\ i \leq \sqrt{n},\ B_{\lambda_k} = 0,\ k > \sqrt{n}$ and $\|b\| = 1$. Testing (15) on this special b gives

$$\sum_{i \leq \sqrt{n}} |a_i| \leq M\varphi_n \|a\|, \text{ for any } a \in E. \tag{17}$$

Now take $a_i = (-1)^i i^{\rho_1},\ i \leq \sqrt{n}$ and $a_i = 0,\ i > \sqrt{n}$, so that $\|a\| = 1$. For this special a, (17) gives $\sum_{i \leq \sqrt{n}} i^{\rho_1} \leq M\varphi_n$. Since $|\rho_1| < 1$, there exists $\alpha = \alpha(\rho_1) > 0$ such that $\sum_{i \leq \sqrt{n}} i^{\rho_1} \geq \alpha n^{\frac{\rho_1+1}{2}}$ and we obtain $\varphi_n \geq \frac{\alpha}{M} n^{\frac{\rho_1+1}{2}}$ for n a perfect square. For a general n, interpolate between two squares (recall that φ_n increases) to get (13) with $\delta = \alpha M^{-1} 2^{-\frac{\rho_1+1}{2}}$. □

If we allow $|\rho_1 - \rho_2| = 1$, we have the following result, where a logarithmic factor enters.

Theorem 3.3. *Let A, B be two Dirichlet series and $C = AB$ their product. We have the following :*

a) If A converges at -1 and B converges at 0, then $\sum_2^\infty \frac{c_n}{\log n}$ converges.

b) Let $(\varphi_n)_{n \geq 1}$ be a sequence of positive numbers increasing to ∞ such that $\sum_1^\infty \frac{c_n}{\varphi_n}$ converges as soon as A converges at -1, B converges at 0, and $C = AB$. Then $\varphi_n \geq \delta \log n$ for some positive constant δ. In particular, the conclusion of a) is best possible.

PROOF OF a). We can assume that a_1 (and therefore c_1) vanishes. We shall argue as in Theorem 3.2, where now

$$E = \{a = (a_n); \sum_1^\infty n a_n \text{ converges and } a_1 = 0\},$$

with norm $\|a\| = \sup_n \bigl|\sum_1^n k a_k\bigr|$ and

$$F = \{b = (b_n); \sum_1^\infty b_n \text{ converges}\}$$

with norm $\|b\| = \sup_n \bigl|\sum_1^n b_k\bigr|$. Recall that we have (cf. the proof of (10))

$$\bigl|\sum_{n=1}^N C_n\bigl(\log(n+1) - \log n\bigr)\bigr| =: |S_N| \leq 2\log(N+1)\|a\|\|b\|. \tag{18}$$

(Here, $C_n = c_1 + \cdots + c_n$.) Also note that

$$|C_n| \leq 4\log(N+1)\|a\|\|b\|. \tag{19}$$

In fact (see the proof of Theorem 3.2) we have $C_N = \sum_1^N a_i B_{\lambda_i}$, whence $|C_n| \leq \|b\| \sum_1^N |a_i| \leq 2\|a\|\|b\| \sum_1^N \frac{1}{i} \leq 4\log(N+1)\|a\|\|b\|$. The two relations (18) and (19) will replace (11) and (12). Now, we want to prove that if $L_N(a,b) = \sum_2^N \frac{c_n}{\log n}$, the bilinear forms L_N are pointwise convergent on $E \times F$. Since the finitely supported sequences are dense in E and F, it suffices to find a constant M such that

$$|L_N(a,b)| \leq M\|a\|\|b\| \quad \text{for each } (a,b) \in E \times F. \tag{20}$$

We shall use the notation $U \ll V$ to indicate that $|U| \leq \alpha|V|$, where α is a constant ; recall that $S_n = \sum_{k=1}^n C_k\bigl(\log(k+1) - \log k\bigr)$. Now two summation by parts and the use of (18), (19) give

$$
\begin{aligned}
L_N(a,b) =& \sum_2^N \frac{C_n - C_{n-1}}{\log n} = \frac{C_N}{\log N} + \sum_2^{N-1} C_n \big(\frac{1}{\log n} - \frac{1}{\log(n+1)} \big) \\
=& \frac{C_N}{\log N} + \sum_2^{N-1} \frac{S_n - S_{n-1}}{\log n \log(n+1)} = \frac{C_N}{\log N} + \frac{S_{N-1}}{\log(N-1)\log N} \\
&+ \sum_2^{N-2} S_n \Big(\frac{1}{\log n \log(n+1)} - \frac{1}{\log(n+1)\log(n+2)} \Big) \\
\ll& \|a\|\|b\| + \|a\|\|b\| + \|a\|\|b\| \sum_2^{N-2} \Big(\frac{1}{\log n} - \frac{1}{\log(n+2)} \Big) \\
\ll& \|a\|\|b\| \sum_2^{\infty} \frac{1}{n \log^2 n}, \text{ which proves (20) and therefore a).}
\end{aligned}
$$

Proof of b). Now set $L_n(a,b) = \frac{c_1 + \ldots + c_n}{\varphi_n} = \frac{C_n}{\varphi_n}$. As in Theorem 3.2, the Banach-Steinhaus Theorem provides a constant $M(= \sup \|L_n\|) < \infty$ such that

$$
\big| \sum_1^n a_i B_{\lambda_i} \big| = \big| \varphi_n L_n(a,b) \big| \leq M \varphi_n \|a\|\|b\|.
$$

If n is a perfect square, the same choice of b gives $\sum_1^{\sqrt{n}} |a_i| \leq M\varphi_n \|a\|$, for any $a \in E$. The choice $a_i = \frac{(-1)^i}{i}$ $(i \leq \sqrt{n})$ and $a_i = 0$ $(i > \sqrt{n})$ gives (since $\|a\| = 1$) $\frac{1}{2} \log n \leq M\varphi_n$. Interpolating between two perfect squares ends the proof. □

Remark 3.1. With the logarithmic scale appearing here, it would not be possible to prove the optimality of a) using only the Lindelöf function associated with A and B. Part a) appears to be new, as well as the proof of b). Another proof of b), very close to the use of Baire's theorem, was given in ([KQ]). In the proofs of Theorem 3.2 and b) of Theorem 3.3, we seem to use only a weakened form (through Kronecker's lemma) of the hypotheses, but as we noticed in the proof of Theorem 3.1, we are in situations where the convergence of the series $\sum \frac{c_n}{\varphi_n}$ and the convergence to zero of the sequence $\frac{C_n}{\varphi_n}$ are equivalent, and this explains why the proof works.

4 Non-Existence of Rudin-Shapiro Like Dirichlet Polynomials

Some notation: $\mathbb{N} = \{0, 1, \cdots\}$ will denote the set of natural integers, $\mathbb{N}^* = \{1, 2, \dots\}$ will denote the set $\mathbb{N} \setminus \{0\}$.

Here we will make here strong use of the number theoretic function $\Omega : \mathbb{N}^* \to \mathbb{N}$ defined by

$$\Omega(n) = \alpha_1(n) + \cdots + \alpha_k(n) \text{ if } n = p_1^{\alpha_1(n)} \dots p_k^{\alpha_k(n)}$$

is decomposed into its prime factors. (We will allow the $\alpha_j's$ to be zero.) Thus, $\Omega(n)$ counts the prime factors of n with their multiplicity and, although completely additive, it has in some respects a more complicated behavior than the companion function $\omega(n) = \sum_{\alpha_j(n)>0} 1$, which counts the prime factors of n without their multiplicity. (The function ω is only additive.) But the complicated behavior of Ω will not be pertinent here, since we will only need (see Lemmas 4.1 and 4.2 below) rather crude information on this function. We will fix a real number $x \geq 3$ (which should be thought of as an integer) and consider a Dirichlet polynomial

$$A(t) = \sum_{n \leq x} a_n n^{it}.$$

Denote by $p_1 < \cdots < p_k$ the prime numbers less than or equal to x, so that $k = \pi(x)$, the number of prime numbers less than or equal to x. Let Γ be the unit circle of the complex plane, and let

$$P(z) = P(z_1, \dots, z_k) = \sum_{n \leq x} a_n z_1^{\alpha_1(n)} \dots z_k^{\alpha_k(n)},$$

where $z = (z_1, \dots, z_k) \in \Gamma^k$. It is Bohr's inspired observation that, thanks to the rational independence of $\log p_1, \dots, \log p_k$ and to Kronecker's approximation theorem ([KS]), one has the following equality (see [Q1]).

$$\|A\|_\infty = \|P\|_\infty := \sup\{|P(z_1, \dots, z_k)|; |z_1| = \dots = |z_k| = 1\}. \tag{21}$$

We will make strong use of (21) to obtain non-trivial upper bounds for $\|A\|_\infty$. We will also make use of the following result of ([Q1]); theorem 4.1 p. 49) which uses Blei's theory of p-Sidon sets.

Theorem 4.1. *Let $m \in \mathbb{N}^*$. Set $E_m = \{n \leq x; \Omega(n) = m\}$. Then*

$$\Big(\sum_{n \in E_m} |a_n|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \leq C_m \|P\|_\infty = C_m \|A\|_\infty \tag{22}$$

where $C_m = \left(\frac{2}{\sqrt{\pi}}\right)^{m-1} \frac{m^{m/2}(m+1)^{\frac{m+1}{2}}}{2^m} (m!)^{-\frac{m+1}{2m}}$.

The order of growth of C_m is important in what follows, so we begin by giving a more tractable upper bound for C_m, under the form of the following lemma.

Lemma 4.1. *With the previous notation, one has*

$$C_m \leq m^{m/2}. \tag{23}$$

Proof. One checks (23) by hand for $m = 1, 2, 3$. For $m \geq 4$, we will use Stirling's inequality $n! \geq n^n e^{-n}\sqrt{2\pi n}$.

We have to show that

$$\frac{\sqrt{\pi}}{2}\Big(\frac{1}{\sqrt{\pi}}\Big)^m (m+1)^{\frac{m+1}{2}} \leq (m!)^{\frac{m+1}{2m}}. \tag{24}$$

Now, the LHS of (24) is $\leq \frac{\sqrt{\pi}}{2}\big(\frac{1}{\sqrt{\pi}}\big)^m m^{\frac{m+1}{2}} e^{\frac{m+1}{2m}}$, while the RHS is $\geq \big(\frac{m}{e}\big)^{\frac{m+1}{2}}$ $(2\pi m)^{1/4}$ by Stirling's inequality. It remains to check that $\frac{\sqrt{\pi}}{2}\frac{1}{(\sqrt{\pi})^m} e^{\frac{m+1}{2m}} \leq$ $e^{-\frac{m+1}{2}}(2\pi m)^{1/4}$, or that $\frac{\sqrt{\pi}e}{2}\big(\frac{e}{\pi}\big)^{m/2} e^{1/2m} \leq (2\pi m)^{1/4}$, or by squaring both sides $\pi e^2 \big(\frac{e}{\pi}\big)^m \frac{e^{1/m}}{4} \leq (2\pi m)^{1/2}$. Now

$$LHS \leq 24 \times 0,57 \times 0,33 < 5 \text{ and } RHS \geq (25)^{1/2} = 5. \qquad \square$$

From now on, $\alpha, \beta, \alpha', \beta'$, etc. will denote positive constants which can vary from line to line. It was mentioned in the introduction that there exist Dirichlet polynomials $A(t) = \sum_{n\leq x} a_n n^{it}$ for which $\|A\|_\infty = 1$ and

$$\sum_{n\leq x} |a_n| \geq \alpha\sqrt{x}\exp(-\beta\sqrt{\log x \log_2 x}). \tag{24'}$$

By analogy with the Rudin-Shapiro trigonometric polynomials, it is natural to ask whether one can get $\sum_{n\leq x}|a_n| \geq \alpha\sqrt{x}$. This is not the case, and in fact the lower bound (24′) is best possible (if one ignores the precise value of α and β). But the proof of this latter fact is rather delicate. We therefore prefer to prove two different theorems; the second one contains the first, but its proof is just an elaboration of the first one, and things are probably clearer under this form.

Theorem 4.2. *Let* $A(t) = \sum_{n\leq x} a_n n^{it}$ *be a Dirichlet polynomial. Then one has*

$$\sum_{n\leq x} |a_n| \leq \alpha\sqrt{x}\exp\Big(-\beta\sqrt{\frac{\log x}{\log_2 x}}\Big)\|A\|_\infty.$$

In particular, there does not exist Rudin-Shapiro like Dirichlet polynomials.

Proof. We can assume that $\|A\|_\infty = 1$. Let $M \geq 1$ to be chosen later ; write (see Theorem 4.1)

$$\sum_{n\leq x} |a_n| = \sum_{m\leq M}\Big(\sum_{n\in E_m} |a_n|\Big) + \sum_{n\in E} |a_n| =: \Sigma_1 + \Sigma_2,$$

where $E = \cup_{m>M} E_m$. To bound $\sum_{n\in E_m} |a_n|$ for $m \leq M$, we use Hölder's inequality and (22), (23) to obtain

$$\begin{aligned}\sum_{n\in E_m} |a_n| &\leq \Big(\sum_{n\in E_m} |a_n|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} |E_m|^{\frac{m-1}{2m}} \\ &\leq m^{m/2}|E_m|^{\frac{m-1}{2m}} \leq M^{M/2}x^{\frac{m-1}{2m}} \leq M^{M/2}x^{\frac{M-1}{2M}} \\ &= x^{1/2}\exp\Big(\frac{M}{2}\log M - \frac{\log x}{2M}\Big).\end{aligned}$$

A good compromise for M is given by $M^2 \log M \approx \log x$, therefore we choose $M = \sqrt{\frac{\log x}{\log_2 x}}$. For this value, to which we will stick, one gets an inequality of the form

$$\sum_{n\in E_m} |a_n| \leq \alpha\sqrt{x}\exp(-\beta\sqrt{\log x \log_2 x}),\ m \leq M. \tag{25}$$

Adding the inequalities (25) for $m \leq M$ and changing α and β, one gets

$$\Sigma_1 \leq \alpha\sqrt{x}\exp(-\beta\sqrt{\log x \log_2 x}) \tag{26}$$

As we will see later, this is the right order of growth. But due to a bad upper bound for $|E|$, the cardinality of E, we will obtain something less precise than (26) for Σ_2; in fact the Cauchy-Schwarz inequality gives

$$\Sigma_2 \leq |E|^{1/2}\Big(\sum_{n\leq x} |a_n|^2\Big)^{1/2} = |E|^{1/2}\|P\|_2 \leq |E|^{1/2}\|P\|_\infty = |E|^{1/2}\|A\|_\infty.$$

To majorize $|E|$, we shall use the following simple lemma, kindly shown to us by Hugh. L. Montgomery.

Lemma 4.2. *Let c be fixed, $1 < c < 2$. The number $N(x, m)$ of $n \leq x$ such that $\Omega(n) \geq m$ is less than $x(\log x)^c c^{-m}$, uniformly for $m \geq 1$. In particular, this number is less than $\alpha x e^{-\beta m}$ for $m \geq \alpha' \log_2 x$.*

PROOF. Force a completely multiplicative function to come into play as follows.

$$N(x,m)=\sum_{\substack{n\le x\\ \Omega(n)\ge m}} 1 \le \frac{1}{c^m}\sum_{n\le x} c^{\Omega(n)} \le \frac{x}{c^m}\sum_{n\le x}\frac{c^{\Omega(n)}}{n}$$
$$\le \frac{x}{c^m}\sum_{\substack{n\\ p|n\Longrightarrow p\le x}}\frac{c^{\Omega(n)}}{n} = xc^{-m}\prod_{p\le x}\big(1-\frac{c}{p}\big)^{-1}$$

where p denotes a prime number and where we used Euler product formula. (Recall that $\sum_1^\infty \frac{g(n)}{n^s}=\Pi_p\big(1-\frac{g(p)}{p^s}\big)^{-1}$ for a completely multiplicative function g such that $|g(p)|<|p^s|$ for all primes p and the series in the formula is absolutely convergent.) Now, by Merten's theorem ([T]), $\Pi_{p\le x}\big(1-\frac{1}{p}\big)^{-1}\sim e^\gamma\log x$, where γ is the Euler constant, which clearly gives the result. (Observe that $\frac{c}{p}$ has to be <1, for each prime p. Therefore we must have $c<2$.) Now, Lemma 4.2 applied for $m>M$ gives

$$\Sigma_2\le |E|^{1/2}\le \alpha\sqrt{x}e^{-\beta M}\le \alpha\sqrt{x}\exp\Big(-\beta\sqrt{\frac{\log x}{\log_2 x}}\Big). \tag{27}$$

Adding (26) and (27) gives Theorem 4.2, for appropriate constants α and β. $\square$

As we already mentioned, Theorem 4.2 does not give the best possible estimate. But we can use a less brutal splitting of $\sum_{n\le x}|a_n|$, and this will allow us to prove our main theorem.

Theorem 4.3. *Let $A(t)=\sum_{n\le x}a_n n^{it}$ be a Dirichlet polynomial. Then*

a) for some numerical constant $\alpha,\beta>0$

$$\sum_{n\le x}|a_n|\le \alpha\sqrt{x}\exp\big(-\beta\sqrt{\log x\log_2 x}\big)\|A\|_\infty. \tag{28}$$

b) The estimation of a) is optimal in general: one can find a non-zero Dirichlet polynomial A such that

$$\sum_{n\le x}|a_n|\ge \alpha'\sqrt{x}\exp\big(-\beta'\sqrt{\log x\log_2 x}\big)\|A\|_\infty$$

for some numerical constants $\alpha',\beta'>0$.

PROOF. Throughout the proof assume that $\|A\|_\infty = 1$ and p will denote a prime number. Fix $0 < \delta < \frac{1}{2}$, set $y = (\log x)^\delta$ and

$$\begin{aligned} S_y =& \{n \le x; p|n \Rightarrow p \le y\}; \\ L_y =& \{n \le x; p|n \Rightarrow p > y\}. \end{aligned}$$

We call the elements of S_y y-small and those of L_y y-large. Of course, S_y and L_y also depend on x, but since we shall keep this x fixed, the notation S_y, L_y will cause no confusion. We still define k and s by $p_k \le x < p_{k+1}$ and $p_s \le y < p_{s+1}$. The proof now consists of three steps.

Step 1 :

Without loss of generality, we can assume that $a_n = 0$ if n is not y-large ($n \notin L_y$). In fact, we know from (21) that $\|A\|_\infty = \|P\|_\infty$, where $P(z) = \sum_{n\le x} a_n z_1^{\alpha_1(n)} \dots z_k^{\alpha_k(n)}$ and $k = \pi(x)$. Clearly, each integer $n \le x$ can be uniquely written as $n = m\ell$, where $m \in S_y$, $\ell \in L_y$, so that

$$\begin{aligned} P(z) =& \sum_m z_1^{\alpha_1(m)} \dots z_s^{\alpha_s(m)} \Big(\sum_\ell a_{m\ell} z_{s+1}^{\alpha_{s+1}(\ell)} \dots z_k^{\alpha_k(\ell)} \Big) \\ =:& \sum_m z_1^{\alpha_1(m)} \dots z_s^{\alpha_s(m)} P_m(z''), \end{aligned}$$

where $z = (z', z'')$, with $z' = (z_1, \dots, z_s)$ and $z'' = (z_{s+1}, \dots, z_k)$ for $z = (z_1, \dots, z_k) \in \Gamma^k$. Observe that

$$\|P_m\|_\infty \le \|P\|_\infty \text{ for } m \in S_y. \tag{29}$$

In fact, P_m appears as a Fourier coefficient of P.

$$P_m(z'') = \frac{1}{(2\pi)^s} \int_0^{2\pi} \dots \int_0^{2\pi} P(e^{i\theta_1}, \dots, e^{i\theta_s}, z'') e^{-i(\alpha_1(m)\theta 1 + \dots + \alpha_s(m)\theta_s)} \, d\theta_1 \dots d\theta_s.$$

Now, each P_m is associated to a Dirichlet polynomial $\sum_\ell a_{m\ell} \ell^{it}$, where all the $\ell' s$ are y-large. Therefore, if this case has been settled, we get, assuming that $\|P\|_\infty = 1$ and using (29) that

$$\sum_\ell |a_{m\ell}| \le \alpha \sqrt{x} \exp(-\beta \sqrt{\log x \log_2 x}). \tag{30}$$

Now observe that

$$|S_y| \le \alpha \exp\big(\beta (\log x)^\delta\big). \tag{31}$$

In fact, if $m \in S_y$, $m = p_1^{\alpha_1} \dots p_s^{\alpha_s} \le x$, so that $\alpha_j \le \frac{\log x}{\log 2}$ and that $|S_y| \le \big(1 + \frac{\log x}{\log 2}\big)^s \le \exp(2s \log_2 x)$. But $s \log s \sim y$ by the prime number theorem;

so $s \sim \frac{(\log x)^\delta}{\delta \log_2 x}$, and this gives (31). Now, use (30) and (31) to see that

$$\sum_{n \le x} |a_n| = \sum_{m \in S_y} \Big(\sum_{\substack{\ell \in L_y \\ m\ell \le x}} |a_{m\ell}| \Big) \le |S_y| \alpha \sqrt{x} \exp\big(-\beta \sqrt{\log x \log_2 x}\big)$$

$$\le \alpha' \sqrt{x} \exp(-\beta' \sqrt{\log x \log_2 x}),$$

since $\delta < 1/2$.

Step 2 :

We have already mentioned (see Lemma 4.2) that the bound $N(x,m) \le \alpha x e^{-\beta m}$ for large m is not sufficient to get the optimal form of (28) ; but a variant of Lemma 4.2 will allows us to obtain a good upper bound for $N(x,y,m)$, where

$$N(x,y,m) = \#\{n \le x; p|n \Rightarrow p > y \text{ and } \Omega(n) \ge m\} = \#\{n \in L_y \,; \Omega(n) \ge m\}.$$

Lemma 4.2′. *Fix a numerical constant c such that $c < 1$ and $c(\log 3)^\delta > 1$. Then*

$$N(x,y,m) \le \frac{x}{(cy)^m} \exp(\beta y \log_2 x).$$

PROOF. First observe that $cy = c(\log x)^\delta \ge c(\log 3)^\delta > 1$, and that there exists $\beta_0 > 0$ such that $1 - u \ge e^{-\beta_0 u}$ for $0 \le u \le c$. Then proceed as in Lemma 4.2 to get

$$N(x,y,m) = \sum_{\substack{n \in L_y \\ \Omega(n) \ge m}} 1 \le \frac{1}{(cy)^m} \sum_{n \in L_y} (cy)^{\Omega(n)}$$

$$\le \frac{x}{(cy)^m} \sum_{n \in L_y} \frac{(cy)^{\Omega(n)}}{n} \le \frac{x}{(cy)^m} \prod_{y < p \le x} \big(1 - \frac{cy}{p}\big)^{-1}$$

(by the Euler product formula). Now, for $y < p \le x$, we have $c\frac{y}{p} \le c$; so that $\big(1 - \frac{cy}{p}\big)^{-1} \le \exp\big(\beta_0 c \frac{y}{p}\big)$. This implies

$$N(x,y,m) \le \frac{x}{(cy)^m} \exp\Big(\sum_{y < p \le x} \beta_0 c \frac{y}{p} \Big)$$

$$\le \frac{x}{(cy)^m} \exp\Big(\sum_{p \le x} \beta_0 c \frac{y}{p} \Big) \le \frac{x}{(cy)^m} \exp\big(\beta y \log_2 x\big)$$

where we used the classical estimate $\sum_{p \le x} \frac{1}{p} = O(\log_2 x)$. $\square$

Step 3 :
The inequality (28) holds when $a_n = 0$ for $n \notin L_y$. This last step will be easy after Step 2. Argue exactly as in the proof of Theorem 4.2, the point being that now E_m is the set of those $n \leq x$ for which $\Omega(n) = m$ and moreover $n \in L_y$. So that, with the same value $M = \sqrt{\frac{\log x}{\log_2 x}}$, the set $E = \cup_{m>M} E_m$ is exactly the set of $n \in L_y$ such that $\Omega(n) > M$, and that $|E| \leq N(x, y, M)$. Lemma 4.2$'$ then implies

$$\begin{aligned} |E| \leq & x \exp\bigl(\beta(\log x)^\delta \log_2 x - M \log\bigl(c(\log x)^\delta\bigr)\bigr) \\ \leq & x \exp\bigl(\beta(\log x)^\delta \log_2 x - \beta' \sqrt{\log x \log_2 x}\bigr) \\ \leq & x \exp\bigl(-\beta'' \sqrt{\log x \log_2 x}\bigr), \end{aligned}$$

since $\delta < \frac{1}{2}$. This better upper bound on $|E|$ gives a) of Theorem 4.3, and the lower bound of b), with a precise value of β', has been proved in [Q1]. $\square$

5 Concluding Remarks and Questions

1. One way to express Theorem 4.3 is as follows. Recall that if G is a compact abelian group and Γ, its discrete dual, the p-Sidonicity constant ($1 \leq p < 2$) of a subset Λ of Γ is the smallest constant C such that, for any trigonometric polynomial $P = \sum_{\gamma \in \Lambda} a_\gamma \gamma$, one has $\bigl(\Sigma |a_\gamma|^p\bigr)^{1/p} \leq C \|P\|_\infty$. This smallest constant is denoted by $S_p(\Lambda)$. Now, take for G the Bohr compactification $\overline{\mathbb{R}}$ of $\mathbb{R}$, whose dual Γ is the discrete group $\mathbb{R}_d$ of real numbers, and take for Λ the set $\Lambda_N = \{\log 1, \dots, \log N\}$. For $x = N$, the assertions a) and b) Theorem 4.3 amount to saying that

$$\alpha_0 \sqrt{N} \exp\bigl(-\beta_0 \sqrt{\log N \log_2 N}\bigr) \leq S_1(\Lambda_N) \leq \alpha \sqrt{N} \exp\bigl(-\beta \sqrt{\log N \log_2 N}\bigr).$$

In fact, $A(t) = \sum_1^N a_n n^{it}$ can be viewed as the trigonometric polynomial $P(\bar{t}) = \sum_1^N a_n \exp(i < \log n, \bar{t} >)$ where $\bar{t}$ runs over $\bar{\mathbb{R}}$, and we have $\|A\|_\infty = \|P\|_\infty$ since $\mathbb{R}$ is dense in $\overline{\mathbb{R}}$. In this setting, Theorem 4.3 easily extends to the p-Sidon case.

Theorem 5.1. *The p-Sidonicity constant $S_p(\Lambda_N)$ of $\Lambda_N = \{\log 1, \dots, \log N\}$ satisfies*

$$\begin{aligned} \alpha'_p N^{\frac{1}{p} - \frac{1}{2}} \exp\Bigl(-\beta'_p \sqrt{\log N \log_2 N}\Bigr) \leq S_p(\Lambda_N) \\ \leq \alpha_p N^{\frac{1}{p} - \frac{1}{2}} \exp\Bigl(-\beta_p \sqrt{\log N \log_2 N}\Bigr). \end{aligned} \tag{32}$$

Proof. Let $A(t) = \sum_1^N a_n n^{it}$. If $\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}$ with $0 \leq \theta \leq 1$, Hölder's inequality gives via (28)

$$\begin{aligned}\Big(\sum_1^N |a_n|^p\Big)^{\frac{1}{p}} \leq & \Big(\sum_1^N |a_n|\Big)^{1-\theta}\Big(\sum_1^N |a_n|^2\Big)^{\theta/2} \\ \leq & \alpha^{1-\theta} N^{\frac{1-\theta}{2}} \exp\big(-\beta(1-\theta)\sqrt{\log N \log_2 N}\big)\|A\|_\infty^{1-\theta}\|A\|_\infty^{\theta} \\ = & \alpha_p N^{\frac{1}{p}-\frac{1}{2}} \exp\Big(-\beta_p\sqrt{\log N \log_2 N}\Big)\|A\|_\infty\end{aligned}$$

which is the upper bound in (32). For the lower bound, observe that the extremal example $A(t) = \sum_1^N a_n n^{it}$ of [Q1] has the following properties.

$\|A\|_\infty \leq \alpha\sqrt{N}\exp(\beta\sqrt{\log N \log_2 N})$;

$\sum_1^N |a_n| \geq \alpha' N \exp\big(-\beta'\sqrt{\log N \log_2 N}\big)$;

$a_n = +1, -1, 0.$

It follows that

$$\begin{aligned}S_p(\Lambda_N) \geq \frac{\Big(\sum_1^N |a_n|^p\Big)^{1/p}}{\|A\|_\infty} &= \frac{\Big(\sum_1^N |a_n|\Big)^{1/p}}{\|A\|_\infty} \\ &\geq \alpha'_p N^{\frac{1}{p}-\frac{1}{2}} \exp\big(-\beta'_p\sqrt{\log N \log_2 N}\big)\end{aligned}$$

with $\alpha'_p = \alpha'^{1/p}\alpha^{-1}$, $\beta'_p = \frac{\beta'}{p} + \beta$ which ends the proof of (32). □

2. The results of Theorems 3.2, 3.3, and 4.3 are not constructive, since their proofs use either topological (Banach-Steinhaus) or probabilistic arguments. It would be very interesting to obtain explicit examples, in particular to see what can replace the Rudin-Shapiro sequence in the case of Dirichlet polynomials.

3. We made no particular attempt to find the best constants $\alpha, \beta, \alpha', \beta'$ of Theorem 4.3. It follows from the precise estimate

$$S_1(\Lambda_N) \geq \delta\sqrt{N \log N} \exp\Big[-\sqrt{2 \log N(\log_2 N + \log_3 N}\Big]$$

(see theorem V.1 of [Q1]) that one should have $\beta \leq \sqrt{2}$, and it is likely that any constant $\beta < \sqrt{2}$ in (28) of Theorem 3.3 works.

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