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THE HAUSDORFF MEASURE AND THE PACKING MEASURE ON A PERTURBED CANTOR SET

Abstract

Baek in [2] and [3] defines the covering measures h^s and Q^s for a perturbed Cantor set F . He shows that when s is the dimension of the covering measures h^s and Q^s on F , then s is the Hausdorff dimension and the packing measure dimension on F . In this paper, it is shown that for the perturbed Cantor set F , the Hausdorff measure is equal to the covering measure h^s on F . Under more restrictions on the set F , the packing measure is equal to $2 \cdot Q^s$. Similar results are shown for the weakly convergent deranged Cantor set.

1 Preliminaries.

In [3], Baek defined a weakly convergent deranged Cantor set as follows: Let $I = [0, 1]$. The left subinterval $I_{\sigma,1}$ and the right subinterval $I_{\sigma,2}$ of I_σ are obtained by deleting the middle open interval of I_σ , labeled J_σ , inductively for each $\sigma \in \{1, 2\}^n$, where $n = 1, 2, 3, \dots$. Consider $E_n = \cup_{\sigma \in \{1, 2\}^n} I_\sigma$. Then $\{E_n\}$ is a decreasing sequence of closed sets. For each n , let $\frac{|I_{\sigma,1}|}{|I_\sigma|} = c_{\sigma,1}$ and $\frac{|I_{\sigma,2}|}{|I_\sigma|} = c_{\sigma,2}$ for all $\sigma \in \{1, 2\}^n$, where $|I|$ denotes the diameter of I . Then $F = \cap_{n=1}^{\infty} E_n$ is a weakly convergent deranged Cantor set. Note that if $c_{\sigma,1} = a_{n+1}$ and $c_{\sigma,2} = b_{n+1}$ for all $\sigma \in \{1, 2\}^n$, then $F = \cap_{n=1}^{\infty} E_n$ is a perturbed Cantor set as in [2].

The purpose of this paper is to give explicit formulas for the exact value of the Hausdorff measure and for the exact value of the packing measure of F at

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the critical dimension. In fact, we prove that if s equals the common value of the Hausdorff dimension and the packing dimension of the perturbed Cantor set F and $\inf_k \{a_k, b_k, 1 - (a_k + b_k)\} > 0$, then the s -dimensional Hausdorff measure, $H^s(F)$, of F is given by $H^s(F) = \liminf_n \prod_{k=1}^n (a_k^s + b_k^s)$, and if, in addition, $1 - (a_k + b_k) \geq \frac{1}{2} \max \{a_k, b_k\}$ for all k , then the s -dimensional packing measure, $p^s(F)$, of F is given by $p^s(F) = 2 \cdot \limsup_n \prod_{k=1}^n (a_k^s + b_k^s)$. We remark that only very few non-trivial examples of sets F for which the exact value of the Hausdorff measure and/or the packing measure of F at the critical dimension are known, cf. the almost forgotten papers by Marion [9], [10] and the more recent papers [1], [6], [8], [16], [17]. Also, Roy Davies has an unpublished proof that the Hausdorff 1-dimensional measure of $1/4$ Cantor set in the square is $\sqrt{2}$. The proof of this result of Davies can be found in [17] (The author wishes to thank the referee for the wording of the above paragraph.)

The s -dimensional Hausdorff measure of F is defined by

$$H^s(F) = \lim_{\delta \rightarrow 0} H_\delta^s(F)$$

where $H_\delta^s(F) = \inf \{ \sum_{n=1}^\infty |U_n|^s : \{U_n\}_{n=1}^\infty \text{ is a } \delta\text{-cover of } F \}$ and the Hausdorff dimension of F is defined by

$$\dim_H(F) = \sup \{ s > 0 : H^s(F) = \infty \} = \inf \{ s > 0 : H^s(F) = 0 \}.$$

The s -dimensional packing measure of F is defined by

$$p^s(F) = \inf \left\{ \sum_{n=1}^\infty P^s(F_n) : \cup_{n=1}^\infty F_n = F \right\},$$

where $P^s(E) = \lim_{\delta \rightarrow 0} P_\delta^s(E)$ and

$$P_\delta^s(E) = \sup \left\{ \sum_{n=1}^\infty |U_n|^s : \{U_n\} \text{ is a } \delta\text{-packing of } E \right\}.$$

The packing dimension of F is

$$\dim_P(F) = \sup \{ s > 0 : p^s(F) = \infty \} = \inf \{ s > 0 : p^s(F) = 0 \}.$$

A *pseudo-packing* of a set E is any family \mathbf{B} of bounded subsets of the real line such that, if $B, B' \in \mathbf{B}$, then $B^c \cap E^c \neq \emptyset$ and $B \cap B' \cap E = \emptyset$ where B^c = the closure of B . A pseudo-packing \mathbf{B} of E by balls centered in E has the

following property. If $B_r(x), B_s(y) \in \mathbf{B}$, then $y \notin B_r(x)$. A pseudo-packing gives rise to a pre-measure defined as

$$R^s(E) = \lim_{\delta \rightarrow 0} \left[\sup \left\{ \sum_n |U_n|^s : \{U_n\} \text{ is a } \delta\text{-pseudo-packing of } E \text{ by closed symmetric intervals centred in } E \text{ with radii less than } \delta \right\} \right]$$

and the *pseudo-packing measure* is $r^s(E) = \inf \{ \sum_k R^s(E_k) : E \subset \cup_k E_k \}$. Raymond and Tricot [12] defined the above measure and showed that for any set F , $r^s(F) = p^s(F)$. [Proposition 10.1]

The s -dimensional packing measure is also equal to the following definition of a *symmetric derivation basis measure* where the positive real function $\delta(x)$ replaces the constant $\delta > 0$. A $\delta(\cdot)$ -packing of the set F is any disjoint collection of symmetric intervals $B_{r_i}(x_i)$ where $x_i \in F$, $r_i < \delta(x_i)$, and $P_{\delta(\cdot)}^s(F) = \sup \{ \sum_i (2r_i)^s : \{B_{r_i}(x_i)\} \text{ is a } \delta(\cdot)\text{-packing of } F \}$. The symmetric derivation basis measure is $p^s(F) = \inf \{ P_{\delta(\cdot)}^s(F) : \delta(\cdot) \text{ is any positive real function} \}$. The same notation is being used as for the packing measure because it was shown in [11] that the two measures are equal for all sets F .

In [3] for the weakly convergent deranged Cantor set, Baek introduces the covering functions

$$h^s(F) = \liminf_{n \rightarrow \infty} \min_{\sigma \in \{1,2\}^n} (c_1^s + c_2^s) \left(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s \right) \cdots \left(c_{\sigma|n,1}^s + c_{\sigma|n,2}^s \right)$$

and

$$Q^s(F) = \limsup_{n \rightarrow \infty} \max_{\sigma \in \{1,2\}^n} (c_1^s + c_2^s) \left(c_{\sigma|1,1}^s + c_{\sigma|1,2}^s \right) \cdots \left(c_{\sigma|n,1}^s + c_{\sigma|n,2}^s \right).$$

For the perturbed Cantor set in [2],

$$h^s(F) = \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) \text{ and } Q^s(F) = \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s).$$

In [2] and [3], Baek uses $h^s(F)$ and $Q^s(F)$ to show that if $c_{\sigma,1}^{s\sigma} + c_{\sigma,2}^{s\sigma} = 1$ (or $a_k^{s_k} + b_k^{s_k} = 1$) and $\lim_{n \rightarrow \infty} s_\sigma = s$, for all $\sigma \in \{1,2\}^n$ (or $\lim_{k \rightarrow \infty} s_k = s$), then the dimension $h^s(F)$ (likewise $Q^s(F)$) equals s which is the Hausdorff dimension and the packing measure dimension.

2 Main Results.

Let F denote a perturbed Cantor set determined by the contracting ratios $a_{n+1} = \frac{|I_{\sigma,1}|}{|I_\sigma|}$, $b_{n+1} = \frac{|I_{\sigma,2}|}{|I_\sigma|}$ for all $\sigma \in \{1,2\}^n$ and gap ratio $d_{n+1} = 1 -$

$(a_{n+1} + b_{n+1})$. Assume that the contracting ratios are uniformly bounded away from 0 in the sense that there exists an $a > 0$ such that a_n , b_n and d_n are greater than or equal to a for all $n = 1, 2, \dots$. The covering measure h^s is defined as

$$h^s(F) = \liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s = \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s).$$

The following theorem shows that the covering measure h^s is equal to the Hausdorff measure on a perturbed Cantor set.

Theorem 1. *Let F be a perturbed Cantor set with contracting ratios a_n and b_n such that $a_n^{s_n} + b_n^{s_n} = 1$. Assume that $s_n \rightarrow s$ as $n \rightarrow \infty$. Then the Hausdorff measure $H^s(F) = h^s(F) = \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s)$.*

PROOF. The dimension of the Hausdorff measure $H^s(F)$ and the dimension of the covering measure $h^s(F)$ are both equal to s by [2]. Note that $\prod_{k=1}^n (a_k^s + b_k^s) = \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$. Also note that for each number n , $\{I_\sigma\}_{\sigma \in \{1,2\}^n}$ is a covering of F . Therefore, $H_\delta^s(F) \leq \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$ where $|I_\sigma| < \delta$ for all $\sigma \in \{1,2\}^n$. Since $\lim_{n \rightarrow \infty} |I_\sigma| = 0$ for each $\sigma \in \{1,2\}^n$, there exists an N such that if $n \geq N$, then $|I_\sigma| < \delta$ for all $\sigma \in \{1,2\}^n$. Therefore, $H_\delta^s(F) \leq \inf_{n \geq N} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$ and $H_\delta^s(F) \leq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$. Hence $H^s(F) \leq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$. Let $r > 0$ such that $H^s(F) < r$. Then $H_\delta^s(F) < r$ for all $\delta > 0$ which implies that there exists a δ -covering $\{U_n^\delta\}_n$ of F such that $\sum_n |U_n^\delta|^s < r$ for each $\delta > 0$. Suppose that there exists a natural number N such that if $n \geq N$, then $\sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s > r$. Consider $\sum_{\sigma \in \{1,2\}^N} |I_\sigma|^s$. Let $\delta_0 = \max \{ |I_\sigma| : \sigma \in \{1,2\}^N \}$. Then, $\sum_n |U_n^{\delta_0}|^s < r < \sum_{\sigma \in \{1,2\}^N} |I_\sigma|^s$. This implies that there exists an M such that $U_M^{\delta_0} \neq I_\sigma$ for all $\sigma \in \{1,2\}^N$ and for all n . (For, if not, and each $U_n^{\delta_0} = I_\sigma$ where $\sigma \in \{1,2\}^m$ for some m , then by a finite process described in [7], p. 16, Remark 2.12 which is a method used by A. F. Beardon, *On the Hausdorff dimension of the general Cantor set*, Proc. Cambridge Phil. Soc., **61** (1965), 679-94 and I. J. Good, *The fractal dimension theory of continued fractions*, Proc. Cambridge Phil. Soc., **37**(1941), 199-288 there would exist an n such that $\sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s < r$.) (The remainder of this paragraph is referred to as Argument (\ddagger), later in the paper.) Therefore, there exists a $J_\sigma \subset I_\sigma$ where $\sigma \in \{1,2\}^N$ such that J_σ is not contained in $U_M^{\delta_0}$; however, $U_M^{\delta_0} \supset J_\gamma$ where $\gamma \in \{1,2\}^K$, $K < N$, and $|J_\sigma| > |J_\gamma|$. Now, $I_\sigma \subset I_\gamma$ which implies that $|I_\sigma| < (\frac{1}{2})^{N-K} |I_\gamma|$. Therefore,

$\frac{1}{|I_\sigma|} > \frac{2^{N-K}}{|I_\gamma|}$ which implies that

$$d_{N+1} = \frac{|J_\sigma|}{|I_\sigma|} > \frac{2^{N-K}|J_\sigma|}{|I_\gamma|} > \frac{2^{N-K}|J_\gamma|}{|I_\gamma|} = 2^{N-K}d_{K+1} > 2^{N-K}a$$

where $a > 0$ is the uniform lower bound for a_k , b_k , and d_k for all k . Consider $\sum_{\sigma \in \{1,2\}^{N+1}} |I_\sigma|^s > r$. Again let $\delta_1 = \max \left\{ |I_\sigma| : \sigma \in \{1,2\}^{N+1} \right\}$. Then, there exists a $U_{M_1}^{\delta_1} \neq I_\sigma$ for all $\sigma \in \{1,2\}^n$ and for all n which implies that there exists a $\sigma_1 \in \{1,2\}^{N+1}$ such that $J_{\sigma_1} \subset I_{\sigma_1}$ and J_{σ_1} is not contained in $U_{M_1}^{\delta_1}$; however, $U_{M_1}^{\delta_1} \supset J_{\gamma_1}$ for some $\gamma_1 \in \{1,2\}^{K_1}$ and $K_1 < N+1$ with $|J_{\sigma_1}| > |J_{\gamma_1}|$. (1) Suppose $K_1 = N$. Then $|J_{\sigma_1}| > |J_{\gamma_1}|$ implies that $d_{N+2} > 2d_{N+1} > 2(2^{N-K}a) \geq 2^2a$. (2) Suppose $K_1 < N$. Then $d_{N+2} > 2^{N+1-K_1}d_{K_1+1} > 2^{N+1-K_1}a = 2(2^{N-K_1}a) \geq 2^2a$. Again, consider $\sum_{\sigma \in \{1,2\}^{N+2}} |I_\sigma|^s > r$ and $\delta_2 = \max \left\{ |I_\sigma| : \sigma \in \{1,2\}^{N+2} \right\}$. Then there exists a $U_{M_2}^{\delta_2} \neq I_\sigma$ for all $\sigma \in \{1,2\}^n$ and all n which implies there exists a $\sigma_2 \in \{1,2\}^{N+2}$ such that $J_{\sigma_2} \subset I_{\sigma_2}$, $J_{\sigma_2} \not\subset U_{M_2}^{\delta_2}$ but $J_{\gamma_2} \subset U_{M_2}^{\delta_2}$ where $\gamma_2 \in \{1,2\}^{K_2}$ and $K_2 < N+2$ with $|J_{\sigma_2}| > |J_{\gamma_2}|$. (1) Suppose $K_2 = N+1$. Then $d_{N+3} > 2d_{N+2} > 2(2^2a) = 2^3a$. (2) Suppose $K_2 = N$. Then $d_{N+3} > 2^2d_{N+1} > 2^2(2^{N-K}a) > 2^3a$. (3) Suppose $K_2 < N$. Then $d_{N+3} > 2^{N+3-K_2}d_{K_2+1} > 2^3a$. Continuing this process for each natural number k produces a sequence $d_{N+K} > 2^Ka \rightarrow \infty$ as $K \rightarrow \infty$. However, since $a_k > a$, and $b_k > a$ for all k , $d_k = 1 - (a_k + b_k) < 1 - 2a$ which is a contradiction.

Thus, there exists infinitely many n such that $\sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s < r$ which implies that $\liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s < r$. Since $r > 0$ with $r > H^s(F)$ was arbitrary, $\liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s \leq H^s(F)$ and, therefore,

$$H^s(F) = \liminf_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s = \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s). \quad \square$$

The following corollary gives the result that the Hausdorff measure is equal to one when the left contracting ratio is equal to a and the right contracting ratio is equal to b for all k .

Corollary 2. *Let F be a perturbed Cantor set with contracting ratios $a_k = a$ and $b_k = b$ for all k . Let $a^s + b^s = 1$. Then $H^s(F) = 1$.*

PROOF. The Hausdorff dimension of F is equal to s by [2]. Also $H^s(F) = \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s)$ by Theorem 1. Since $a_k = a$, $b_k = b$ and $a^s + b^s = 1$, $H^s(F) = 1$. \square

Now, let F denote a weakly convergent deranged Cantor set determined by the contracting ratios c_σ and the gap ratios $d_\sigma = 1 - (c_{\sigma,1} + c_{\sigma,2})$ with $\sigma \in \{1, 2\}^n$ where $n = 1, 2, \dots$. Assume that the c_σ and d_σ are locally uniformly bounded away from 0 in the sense that for each $\sigma \in \{1, 2\}^\infty$, $\{c_{\sigma|k}\}_{k=1}^\infty$ and $\{d_{\sigma|k}\}_{k=1}^\infty$ are uniformly bounded away from 0. Note that if $x \in F$, then there is an $\sigma \in \{1, 2\}^\infty$ such that $\bigcap_{k=1}^\infty I_{\sigma|n} = \{x\}$. (Here $\sigma|k = (i_1, i_2, \dots, i_k)$ where $\sigma = (i_1, i_2, \dots, i_k, i_{k+1}, \dots)$) Also, assume for the following corollary that for a fixed N , the left ratios of each subinterval of $I_{\sigma|N}$ are equal to $C_{\sigma|N+k,1}$ meaning that for any $\sigma^* \in \{1, 2\}^\infty$ such that $\sigma^*|N = \sigma|N$, $c_{\sigma^*|N+k,1} = C_{\sigma|N+k,1}$. Assume similarly for $C_{\sigma|N+k,2}$ in each right ratio of each subinterval of $I_{\sigma|N}$.

Corollary 3. *Let F be a weakly convergent deranged Cantor set with contracting ratios $c_{\sigma,1}$ and $c_{\sigma,2}$ such that $c_{\sigma,1}^s + c_{\sigma,2}^s = 1$. Assume $\lim_{n \rightarrow \infty} s_\sigma = s$ for each $\sigma \in \{1, 2\}^n$. For any n , let $\{I_\sigma\}_{\sigma \in \{1,2\}^n}$ be the closed intervals such that $E_n = \cup_{\sigma \in \{1,2\}^n} I_\sigma$ with $F = \bigcap_{n=1}^\infty E_n$. Assume there exists an N such that for each interval $I_{\sigma|N}$, $\sigma \in \{1, 2\}^\infty$, the left ratios for $n \geq N$ equal $C_{\sigma|n,1}$ and the right ratios equal $C_{\sigma|n,2}$. Then*

$$H^s(F) = \sum_{\sigma \in \{1,2\}^N} H^s(F \cap I_\sigma) = \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \liminf_{m \rightarrow \infty} \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right)$$

where $|I_\sigma|^s = \prod_{k=1}^N \left(c_{\sigma|k}^s \right)$ for each $\sigma \in \{1, 2\}^N$.

PROOF. Since the contracting ratios $\{c_\sigma\}_{\sigma \in \{1,2\}^n}$ and gap ratios $\{d_\sigma\}_{\sigma \in \{1,2\}^n}$ are locally bounded away from 0, for each $I_{\sigma|N}$ with $\sigma \in \{1, 2\}^\infty$, the contracting ratios $\{c_{\sigma|k}\}_{k=1}^\infty$ and gap ratios $\{d_{\sigma|k}\}_{k=1}^\infty$ are uniformly bounded away from 0. Therefore, $\{C_{\sigma|k,1}\}_{k=N}^\infty$ and $\{C_{\sigma|k,2}\}_{k=N}^\infty$ are uniformly bounded away from 0. By Theorem 1, $H^s(F \cap I_\sigma) = \liminf_{m \rightarrow \infty} \sum_{\gamma \in \{1,2\}^m} |I_{\sigma*\gamma}|^s$ where $\sigma * \gamma$ is the sequence $\sigma \in \{1, 2\}^N$ followed by the sequence $\gamma \in \{1, 2\}^m$ such that $\sigma * \gamma \in \{1, 2\}^{N+m}$. Note that $|I_\sigma|^s = \prod_{k=1}^N c_{\sigma|k}^s$, $C_{\sigma|N,1} = \frac{|I_{\sigma|N,1}|}{|I_{\sigma|N}|}$, and $C_{\sigma|N,2} = \frac{|I_{\sigma|N,2}|}{|I_{\sigma|N}|}$ where $|I_{\sigma|N}| = |I_\sigma|$. Therefore,

$$H^s(F \cap I_\sigma) = \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \liminf_{m \rightarrow \infty} \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right).$$

Hence,

$$\begin{aligned} H^s(F) &= \sum_{\sigma \in \{1,2\}^N} H^s(F \cap I_\sigma) = \sum_{\sigma \in \{1,2\}^N} \liminf_{m \rightarrow \infty} \sum_{\gamma \in \{1,2\}^m} |I_{\sigma * \gamma}|^s \\ &= \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \liminf_{m \rightarrow \infty} \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right). \quad \square \end{aligned}$$

The analogous results for the packing measure is unknown to the author. However, if the gap ratios in the perturbed Cantor set are restricted to being greater than or equal to $\frac{1}{2} \max\{a_k, b_k\}$, the following result holds.

Theorem 4. *Let F be a perturbed Cantor set with contracting ratios a_k and b_k such that $a_k^{s_k} + b_k^{s_k} = 1$. Assume that the gap ratios $d_k = 1 - (a_k + b_k)$ satisfies $d_k \geq \frac{1}{2} \max\{a_k, b_k\}$ and that $\lim_{k \rightarrow \infty} s_k = s$. Then*

$$p^s(F) = 2 \cdot Q^s(F) = 2 \cdot \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s).$$

PROOF. From [2] $p^s(F) = Q^s(F) = s$. Since the pseudo-packing measure $r^s(F)$ is equal to the packing measure by [12]-Proposition 10.1], it suffices to show that $r^s(F) = 2 \cdot Q^s(F)$. Also, since F is a closed, bounded set, $P^s(F) = p^s(F) = r^s(F) = R^s(F)$. For each I_σ , with $\sigma \in \{1,2\}^n$, let $S^l(I_\sigma)$ be the symmetric open interval whose center is the left endpoint of I_σ and whose radius is $\frac{1}{2}|I_\sigma|$ and let $S^r(I_\sigma)$ be the symmetric open interval whose center is the right endpoint of I_σ and whose radius is $\frac{1}{2}|I_\sigma|$. Note that $|S^l(I_\sigma)|^s + |S^r(I_\sigma)|^s = 2 \cdot |I_\sigma|^s$ and $\{S^l(I_\sigma), S^r(I_\sigma)\}_{\sigma \in \{1,2\}^n}$ is a pseudo-packing of F since $d_k \geq \frac{1}{2} \max\{a_k, b_k\}$ which means that each interval satisfies $|J_\sigma| \geq \frac{1}{2}|I_{\sigma,1}|$ and $|J_\sigma| \geq \frac{1}{2}|I_{\sigma,2}|$. Therefore, $\sum_{\sigma \in \{1,2\}^n} [|S^l(I_\sigma)|^s + |S^r(I_\sigma)|^s] \leq R_\delta^s(F)$ when $\frac{1}{2}|I_\sigma| < \delta$ for every $\sigma \in \{1,2\}^n$. Hence,

$$2 \cdot Q^s(F) = \limsup_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} 2 \cdot |I_\sigma|^s \leq R_\delta^s(F)$$

for any $\delta > 0$ which implies that $2 \cdot Q^s(F) \leq R^s(F)$. Let $r > 0$ be a positive real number such that $0 < r < R^s(F)$ which implies that $r < R_\delta^s(F)$ for any $\delta > 0$. Therefore, there exists a pseudo-packing $\{U_n^\delta\}_n$ such that $\sum_n |U_n^\delta|^s > r$ for each $\delta > 0$. Suppose that there exists a natural number N such that if $n \geq N$. Then $2 \cdot \prod_{k=1}^n (a_k^s + b_k^s) = \sum_{\sigma \in \{1,2\}^n} 2 \cdot |I_\sigma|^s < r$ for each $n \geq N$. Fix $\delta > 0$. Since $\sum_n |U_n^\delta|^s > r$, there exists an M such that $U_M^\delta = (x_M - r_M, x_M + r_M)$

where $x_M \in F$, $r_M < \delta$ but x_M is not an endpoint of any I_σ where $\sigma \in \{1, 2\}^n$ and $n \geq N$. (For, if not, and each midpoint of a U_n^δ is an endpoint of some I_σ , then by a finite process described in [7, p. 16, Remark 2.12] there would exist an n such that $\sum_{\sigma \in \{1, 2\}^n} 2 \cdot |I_\sigma|^s > r$.) Therefore, x_M is a limit point of F which implies that there exists a $\sigma \in \{1, 2\}^\infty$ such that $x_M \in I_{\sigma|k}$, where $\sigma|k \in \{1, 2\}^k$, for $k = 1, 2, \dots$ and $x_M = \bigcap_{k=1}^\infty I_{\sigma|k}$. Let N_0 be the first natural number larger than N such that $\frac{1}{2}|I_\sigma| < \delta$ for every $\sigma \in \{1, 2\}^{N_0}$. Then

$$r < \sum_n |U_n^\delta|^s \leq |U_M^\delta|^s + \sum_{\sigma \neq \gamma \in \{1, 2\}^{N_0}} (|S^l(I_\gamma)|^s + |S^r(I_\gamma)|^s).$$

However,

$$(|S^l(I_{\sigma|N_0})|^s + |S^r(I_{\sigma|N_0})|^s) + \sum_{\sigma|N_0 \neq \gamma \in \{1, 2\}^{N_0}} [|S^l(I_\gamma)|^s + |S^r(I_\gamma)|^s] < r$$

where $\sigma|N_0 \in \{1, 2\}^{N_0}$. This implies that

$$2 \cdot |I_{\sigma|N_0}|^s = |S^l(I_{\sigma|N_0})|^s + |S^r(I_{\sigma|N_0})|^s < |U_M^\delta|^s = (2r_M)^s \quad (1)$$

which means that $r_M > |I_{\sigma|N_0}|$. Without loss of generality, assume that $I_{\sigma|N_0} = I_{\sigma|(N_0-1),1}$ and that $\frac{|I_{\sigma|N_0}|}{|I_{\sigma|(N_0-1)}|} = a_{N_0}$. In order for the inequality (1) to hold with $s < 1$, the gap interval $J_{\sigma|N_0-1}$ in the interval $I_{\sigma|N_0-1}$ (where $d_{N_0} = \frac{|J_{\sigma|N_0-1}|}{|I_{\sigma|N_0-1}|}$) must be very large and $|I_{\sigma|N_0}|$ must be very small.

However, by Argument (‡) in Theorem 1, $J_{\sigma|N_0-1}$ cannot be disproportionately large. Therefore, $J_{\sigma|N_0-1}$ must be very off center and $I_{\sigma|N_0}$ is very small which implies that $a_{N_0} = r_{N_0} b_{N_0}$ where $r_{N_0} < 1$. Again, let $\delta_1 = \max \left\{ \frac{1}{2}|I_\sigma| : \sigma \in \{1, 2\}^{N_0+1} \right\}$. Then, there exists a packing $\{U_n^{\delta_1}\}$ such that $r < \sum_n |U_n^{\delta_1}|^s$ and $\sum_{\sigma \in \{1, 2\}^{N_0+1}} [|S^l(I_\sigma)|^s + |S^r(I_\sigma)|^s] < r$. Therefore, again, there exists an M_1 such that $U_{M_1}^{\delta_1} = (x_{M_1} - r_{M_1}, x_{M_1} + r_{M_1})$ where $x_{M_1} \in F$, $r_{M_1} < \delta_1$ and x_{M_1} is not an endpoint of any I_σ , $\sigma \in \{1, 2\}^n$ and $n \geq N$. Likewise, there exists a $\sigma_1 \in \{1, 2\}^\infty$ such that $\{x_{M_1}\} = \bigcap_{k=1}^\infty I_{\sigma_1|k}$ and $|S^l(I_{\sigma_1|N_0+1})|^s + |S^r(I_{\sigma_1|N_0+1})|^s < |U_{M_1}^{\delta_1}|^s$ which implies that $r_{M_1} > |I_{\sigma_1|N_0+1}|$ and $a_{N_0+1} = r_{N_0+1} b_{N_0+1}$ where $r_{N_0+1} < 1$. Continuing this process for each natural number k produces the equation for the interval $I_{\sigma|N_0}$

$$|I_{\sigma|N_0+k,1}| = \prod_{j=0}^{k+1} (r_{N_0+j} b_{N_0+j}) |I_{\sigma|N_0-1}|$$

which implies that $a_{N_0+k} = \left(\prod_{j=0}^k r_{N_0+j}\right) b_{N_0+k} < \left(\prod_{j=0}^k r_{N_0+j}\right) \frac{1}{2}$. Since each $J_{\sigma|N_0+k}$ must be significantly off center, there exists an $\varepsilon > 0$ such that $r_{N_0+k} < 1 - \varepsilon$ for all k . Therefore,

$$0 < a < a_{N_0+k} < \left(\prod_{j=0}^k r_{N_0+j}\right) \frac{1}{2} < (1 - \varepsilon)^k \frac{1}{2} \rightarrow 0 \text{ as } k \rightarrow \infty$$

which is a contradiction. Hence, there exists infinitely many n such that $\sum_{\sigma \in \{1,2\}^n} (|S^l(I_\sigma)|^s + |S^r(I_\sigma)|^s) > r$ which implies that

$$\limsup_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} 2 \cdot |I_\sigma|^s \geq r \text{ and } 2 \cdot \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) \geq r.$$

Since $r > 0$ was arbitrary, $\limsup_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} 2|I_\sigma|^s \geq R^s(F)$ which implies that

$$2 \cdot \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) = R^s(F) \quad \square$$

The following corollary shows that under the restriction that the gap ratio $d \geq \frac{1}{2} \max\{a, b\}$ when a is the left contracting ratio for the simple perturbed Cantor set and b is the right contracting ratio, then the packing measure is 2.

Corollary 5. *Let F be the perturbed Cantor set with contracting ratios $a_k = a$ and $b_k = b$ for all k . Assume that the gap ratio $d = 1 - (a + b)$ satisfies $d \geq \frac{1}{2} \max\{a, b\}$ and $a^s + b^s = 1$. Then $P^s(F) = 2$.*

PROOF. By [2], s is the packing measure dimension of F . By Theorem 4, $p^s(F) = 2 \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) = 2 \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a^s + b^s) = 2$. □

Now for the result analogous to Corollary 3 for the more general weakly convergent deranged Cantor set F .

Corollary 6. *Let F be a weakly convergent deranged Cantor set with contracting ratios $c_{\sigma,1}$ and $c_{\sigma,2}$ such that $c_{\sigma,1}^{s_\sigma} + c_{\sigma,2}^{s_\sigma} = 1$. Assume that the gap ratios $d_\sigma = 1 - (c_{\sigma,1} + c_{\sigma,2})$ satisfy $d_\sigma \geq \frac{1}{2} \max\{c_{\sigma,1}, c_{\sigma,2}\}$, and there exists an N such that for each interval $I_{\sigma|N}$, $\sigma \in \{1, 2\}^\infty$, the left ratios of each left subinterval of $I_{\sigma|N}$ equal $C_{\sigma|n,1}$ and the right ratios equal $C_{\sigma|n,2}$ for $n \geq N$.*

Assume also that $\lim_{n \rightarrow \infty} s_\sigma = s$ for each $\sigma \in \{1, 2\}^n$. Then

$$\begin{aligned} p^s(F) &= \sum_{\sigma \in \{1,2\}^N} p^s(F \cap I_\sigma) \\ &= \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) 2 \cdot \limsup_{m \rightarrow \infty} \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right). \end{aligned}$$

PROOF. The packing measure dimension of F is s by [3]. Let $\{I_\sigma\}_{\sigma \in \{1,2\}^N}$ be the set of closed intervals such that $E_N = \cup_{\sigma \in \{1,2\}^N} I_\sigma$ where the left ratios for each left subinterval of I_σ equal $C_{\sigma|n,1}$ and the right ratios equal $C_{\sigma|n,2}$ for $n \geq N$. Then, the $C_{\sigma|n,1}, C_{\sigma|n,2}$ and $D_{\sigma|n}$ are uniformly bounded. By Theorem 4, $p^s(F \cap I_\sigma) = \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \limsup_{m \rightarrow \infty} 2 \cdot \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right)$. Therefore, since $\{I_\sigma\}_{\sigma \in \{1,2\}^N}$ are pairwise disjoint closed intervals,

$$\begin{aligned} p^s(F) &= \sum_{\sigma \in \{1,2\}^N} p^s(F \cap I_\sigma) \\ &= \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) 2 \cdot \limsup_{m \rightarrow \infty} \prod_{k=N}^{N+m} \left(C_{\sigma|k,1}^s + C_{\sigma|k,2}^s \right). \quad \square \end{aligned}$$

Returning now to the perturbed Cantor set F where the gap ratios have no restrictions other than the uniform lower bound $a > 0$ which is the same as the lower bound for the left contracting ratios a_k and the right contracting ratios b_k , the following lower bounds and upper bounds hold for the s -dimensional packing measure.

Theorem 7. *Let F be a perturbed Cantor set with contracting ratios a_k and b_k such that $a_k^{s_k} + b_k^{s_k} = 1$. Then*

$$2^{s \cdot} \cdot \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) \leq p^s(F) \leq \left[\frac{2(1-2a) + 1}{a} \right]^s \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s).$$

PROOF. Let $B_r(x)$ be any symmetric interval where x is an endpoint of a I_σ for any $\sigma \in \{1, 2\}^n$, $n = 1, 2, \dots$ and $r > 0$. Let \mathbf{B} be the collection of all $B_r(x)$ such that $\frac{a}{2(1-2a) + 1} \leq \frac{\text{diam}(B_r(x) \cap F)}{\text{diam } B_r(x)} \leq \frac{1}{2}$ where $d_k \leq 1-2a$ for all k and $0 < a < a_k, b_k$, and d_k for all k . Let $y \in F$ be any point that is not an endpoint of any $I_\sigma, \sigma \in \{1, 2\}^n$ and $n = 1, 2, \dots$. Then, there exists a sequence

of positive numbers $\{t_i(y)\}_i$, depending of y , such that $\lim_{i \rightarrow \infty} t_i(y) = 0$

$$0 < \frac{a}{[2(1-2a)+1]} - \varepsilon \leq \frac{\text{diam}(B_{t_i(y)}(y) \cap F)}{\text{diam} B_{t_i(y)}(y)} \leq \frac{1}{2} + \varepsilon < 1$$

where $0 < \varepsilon < \min\{\frac{1}{2}, \frac{a}{[2(1-2a)+1]}\}$. Therefore,

$$\mathbf{B}^0 = \mathbf{B} \cup \{B_{t_i(y)}(y) : y \in F \text{ and } y \text{ is not an endpoint of any } I_\sigma \text{ for } \sigma \in \{1, 2\}^n, n = 1, 2, \dots\}$$

is a Vitali covering of F which implies that there exists a disjoint collection of $\{B_{r_i}(x_i)\}_i \subset \mathbf{B}^0$ such that $p^s(F \setminus \cup_i B_{r_i}(x_i)) = 0$ and $H^s(F \setminus \cup_i B_{r_i}(x_i)) = 0$. The inequality $\text{diam}(B_{r_i}(x_i) \cap F) \leq (\frac{1}{2} + \varepsilon) \text{diam} B_{r_i}(x_i)$ implies that

$$\left(\frac{1}{2} + \varepsilon\right)^{-s} (\text{diam}(B_{r_i}(x_i) \cap F))^s \leq (2r_i)^s$$

and consequently that

$$\left(\frac{1}{2} + \varepsilon\right)^{-s} H^s(B_{r_i}(x_i) \cap F) \leq \left(\frac{1}{2} + \varepsilon\right)^{-s} (\text{diam}(B_{r_i}(x_i) \cap F))^s \leq (2r_i)^s$$

for all $i = 1, 2, \dots$. Hence,

$$\begin{aligned} \left(\frac{1}{2} + \varepsilon\right)^{-s} H^s(F) &= \left(\frac{1}{2} + \varepsilon\right)^{-s} \sum_i H^s(B_{r_i}(x_i) \cap F) \\ &\leq \left(\frac{1}{2} + \varepsilon\right)^{-s} \sum_i [\text{diam}(B_{r_i}(x_i) \cap F)]^s \quad (2) \\ &\leq \sum_i (2r_i)^s \leq P_\delta^s(F) \end{aligned}$$

where $r_i \leq \delta$ for all $i = 1, 2, \dots$. Now, let $\delta(x) > 0$ be any positive real function and restrict the radii in \mathbf{B}^0 to those where $r_i < \delta(x_i)$. Then, again, a Vitali covering occurs and the inequality (2) is true for any positive function $\delta(x)$. Hence, $(\frac{1}{2} + \varepsilon)^{-s} H^s(F) \leq p^s(F)$.

Let $\varepsilon_0 > 0$ be any positive number, and let $\delta(x)$ be any positive real function such that $P_{\delta(\cdot)}^s(F) < p^s(F) + \varepsilon_0$. Then there exists a $\delta(\cdot)$ -packing $\{U_n\}_n$ such that $p^s(F) \leq \sum_n |U_n|^s \leq p^s(F) + \varepsilon_0$. Suppose that there exists a U_M such that

$$\frac{a}{2(1-2a)+1} - \varepsilon > \frac{\text{diam}(U_M \cap F)}{\text{diam} U_M} \quad (3)$$

where $U_M = (x_M - r_M, x_M + r_M)$. Since $\frac{a}{2(1-2a)+1} - \varepsilon < \frac{1}{2}$, we have $\text{diam}(U_M \cap F) < \frac{1}{2} \text{diam} U_M$. Also, since (3) holds, $\text{diam}(U_M \cap F)$ is very small and x_M must be an endpoint of an I_σ for some n and some $\sigma \in \{1, 2\}^n$. Likewise, the gap intervals on both sides of I_σ must be very large, one of which must be $J_{\sigma|n-1}$. Because of argument (‡) in Theorem 1, assume $J_{\sigma|n-1}$ is not the gap interval with endpoint x_M and that $r_M \leq |J_{\sigma|n-1}| + |I_\sigma|$. Then, $\frac{r_M}{|I_{\sigma|n-1}|} \leq \frac{|J_{\sigma|n-1}|}{|I_{\sigma|n-1}|} + \frac{|I_\sigma|}{|I_{\sigma|n-1}|} \leq d_k + a_k \leq (1-2a) + \frac{1}{2}$ (similarly, for $d_k + b_k$). Hence,

$$\begin{aligned} \frac{\text{diam}(U_M \cap F)}{\text{diam} U_M} &= \frac{\text{diam}(U_M \cap F)}{2r_M} \geq \frac{|I_\sigma|}{2(|J_{\sigma|n-1}| + |I_\sigma|)} \\ &\geq \frac{|I_\sigma|}{|I_{\sigma|n-1}|} \cdot \frac{|I_{\sigma|n-1}|}{[2(|J_{\sigma|n-1}| + |I_\sigma|)]} \geq a_k \cdot \frac{1}{2(d_k + a_k)} \\ &\geq \frac{a}{2(1-2a) + 2\frac{1}{2}} = \frac{a}{2(1-2a) + 1} \end{aligned}$$

(where the same inequalities are true for b_k) which is a contradiction to (3). This implies that the $\delta(\cdot)$ -packing $\{U_n\}_n$ is contained in \mathbf{B}^0 . Therefore, $\left[\frac{a}{2(1-2a)+1} - \varepsilon\right] < \frac{\text{diam}(U_n \cap F)}{\text{diam} U_n}$ or $\left[\frac{a}{2(1-2a)+1} - \varepsilon\right] \text{diam} U_n < \text{diam}(U_n \cap F)$ which implies $\left[\frac{a}{2(1-2a)+1} - \varepsilon\right]^s (2r_n)^s < \text{diam}(U_n \cap F)^s$. Therefore,

$$\begin{aligned} \left[\frac{a}{2(1-2a)+1} - \varepsilon\right]^s p^s(F) &\leq \left[\frac{a}{2(1-2a)+1} - \varepsilon\right]^s \sum_n (2r_n)^s \\ &< \sum_n \text{diam}(U_n \cap F)^s < \sum_n |I_{\sigma_n}|^s \end{aligned}$$

where $\{I_{\sigma_n}\}_n$ are disjoint and $U_n \cap F \subset I_{\sigma_n}$ for each n . However, $\sum_n |I_{\sigma_n}|^s$ can be replaced by $\sum_{\sigma \in \{1,2\}^N} |I_\sigma|^s$ for some N by the finite process described in [7]. Hence, $\left(\frac{a}{2(1-2a)+1} - \varepsilon\right)^s p^s(F) \leq \sup_{n \geq N} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$. Since ε_0 was arbitrary, $\left(\frac{a}{2(1-2a)+1} - \varepsilon\right)^s p^s(F) \leq \limsup_{n \rightarrow \infty} \sum_{\sigma \in \{1,2\}^n} |I_\sigma|^s$ which implies that

$$p^s(F) \leq \left[\frac{a}{2(1-2a)+1} - \varepsilon\right]^{-s} \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s).$$

Since ε was arbitrary,

$$2^s \liminf_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s) \leq p^s(F) \leq \left[\frac{2(1-2a)+1}{a} \right]^s \limsup_{n \rightarrow \infty} \prod_{k=1}^n (a_k^s + b_k^s)$$

for any perturbed Cantor set F . □

For the simple perturbed Cantor set with left contracting ratio equal to a and right contracting ratio equal to b , the packing measure has the following upper and lower bounds.

Corollary 8. *Let F be the perturbed Cantor set with contracting ratios $a_k = a$ and $b_k = b$ for all k . Assume that $a^s + b^s = 1$. Then $2^s \leq p^s(F) \leq \left(\frac{2(1-2c)+1}{c} \right)^s$ where $c = \min\{a, b\}$.*

For the weakly convergent deranged Cantor set, the packing measure has the following upper and lower bounds when $a_{\sigma|N}$ is defined to be the uniform lower bound for all the contracting ratios of all the left subinterval, right subinterval and gap subintervals of I_σ , where $\sigma \in \{1, 2\}^\infty$.

Corollary 9. *Let F be a weakly convergent deranged Cantor set with contracting ratios $c_{\sigma,1}$ and $c_{\sigma,2}$ such that $c_{\sigma,1}^{s_\sigma} + c_{\sigma,2}^{s_\sigma} = 1$. Assume $\lim_{n \rightarrow \infty} s_\sigma = s$ for each $\sigma \in \{1, 2\}^n$ and that there exists an N such that for each interval $I_{\sigma|N}$, $\sigma \in \{1, 2\}^\infty$, the left ratio of each left subinterval of $I_{\sigma|N}$ is equal to $C_{\sigma|n,1}$ and the right ratios equal $C_{\sigma|n,2}$ for $n \geq N$. Let $a_{\sigma|N}$ be the uniform lower bound for $I_{\sigma|N}$, $\sigma \in \{1, 2\}^\infty$. Then*

$$\begin{aligned} & 2^s \cdot \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \liminf_{m \rightarrow \infty} \prod_{k=N}^{N+m} (C_{\sigma|k,1}^s + C_{\sigma|k,2}^s) \leq p^s(F) \\ & \leq \sum_{\sigma \in \{1,2\}^N} \left(\prod_{k=1}^N c_{\sigma|k}^s \right) \left[\frac{2(1-2a_{\sigma|N})+1}{a_{\sigma|N}} \right]^s \limsup_{m \rightarrow \infty} \prod_{k=N}^{N+m} (C_{\sigma|k,1}^s + C_{\sigma|k,2}^s). \end{aligned}$$

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