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# MARCZEWSKI FIELDS AND IDEALS†

**Abstract**

For an $X \neq \emptyset$ and a given family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$, we consider the Marczewski field $S(\mathcal{F})$ which consists of sets $A \subset X$ such that each set $U \in \mathcal{F}$ contains a set $V \in \mathcal{F}$ with $V \subset A$ or $V \cap A = \emptyset$. We also study the respective ideal $S^0(\mathcal{F})$. We show general properties of $S(\mathcal{F})$ and certain representation theorems. For instance we prove that the interval algebra in $[0, 1)$ is a Marczewski field. We are also interested in situations where $S(\mathcal{F}) = S(\tau \setminus \{\emptyset\})$ for a topology $\tau$ on $X$. We propose a general method which establishes $S(\mathcal{F})$ and $S^0(\mathcal{F})$ provided that $\mathcal{F}$ is the family of perfect sets with respect to $\tau$, and $\tau$ is a certain ideal topology on $\mathbb{R}$ connected with measure or category.

## 1 General properties

The notions of $(s)$-sets and $(s^0)$-sets are due to Marczewski [Sz]. They have been investigated by many authors. (See [Mi1], [Mi2], [BrCo] and also [Br2], [Co], [Wa].) The scheme defining $(s)$-sets and $(s^0)$-sets was used for more general settings in several publications (see e.g. [Mo], [Bre], [Pa], [R], [BR]). We observe that this scheme turns out interesting without any essential restrictions on a generating family of sets. Namely, let $\mathcal{F}$ be a family of nonempty subsets of a given set $X$. We put

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$S(\mathcal{F}) = \{A \subset X : (\forall\, U \in \mathcal{F})\ (\exists V \in \mathcal{F})\ (V \subset U \cap A\ \vee\ V \subset U \setminus A)\}$,
$S^0(\mathcal{F}) = \{A \subset X : (\forall\, U \in \mathcal{F})\ (\exists V \in \mathcal{F})\ V \subset U \setminus A\}$,
$H(\mathcal{F}) = \{A \subset X : (\forall\, B \subset A)\ B \in \mathcal{F}\}$.
Note that $H(\mathcal{F})$ is the maximal hereditary family contained in $\mathcal{F}$. In the case when $\mathcal{F}$ consists of all perfect subsets of a given Polish space, $S(\mathcal{F})$ and $S^0(\mathcal{F})$ are exactly the families of classical Marczewski $(s)$-sets and $(s^0)$-sets.

Our notation is standard. By $\mathcal{P}(X)$ we denote the power set of $X$. Throughout the paper $X \neq \emptyset$.

**Proposition 1.1.** *Let $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$. Then we have*

**(1)** *$S(\mathcal{F})$ is a field of sets,*

**(2)** *$S^0(\mathcal{F}) \subset H(S(\mathcal{F}))$, and $S^0(\mathcal{F})$ is an ideal of sets,*

**(3)** *$\mathcal{F} \cap S^0(\mathcal{F}) = \emptyset$,*

**(4)** *$(\forall\, U \in S(\mathcal{F}) \setminus S^0(\mathcal{F}))(\exists V \in \mathcal{F}) V \subset U$,*

**(5)** *$\mathcal{F} \subset S(\mathcal{F}) \Leftrightarrow (\forall U, V \in \mathcal{F})(\exists W \in \mathcal{F})(W \subset U \cap V\ \vee\ W \subset U \setminus V)$,*

**(6)** *if $\{x\} \in \mathcal{F}$ for all $x \in X$ then $S(\mathcal{F}) = \mathcal{P}(X)$ and $S^0(\mathcal{F}) = \{\emptyset\}$.*

Proof. (1) From the definition of $S(\mathcal{F})$ it immediately follows that, if $A \in S(\mathcal{F})$ then $X \setminus A \in S(\mathcal{F})$. Now, assume that $A, B \in S(\mathcal{F})$. Let $C \in \mathcal{F}$. If there is a $D \in \mathcal{F}$ such that either $D \subset C \cap A$ or $D \subset C \cap B$ then $D \subset C \cap (A \cup B)$. If such $D$ does not exist, there is a $D_1 \in \mathcal{F}$ such that $D_1 \subset C \setminus A$ and there is a $D_2 \in \mathcal{F}$ such that $D_2 \subset D_1 \setminus B$. Thus $D_2 \subset C \setminus (A \cup B)$. Hence $A \cup B \in S(\mathcal{F})$.

Similarly, we show that $S^0(\mathcal{F})$ is an ideal (condition (2)). The remaining statements, except for (4), can be checked directly without troubles.

(4) Suppose that there is a $U_0 \in S(\mathcal{F}) \setminus S^0(\mathcal{F})$ such that $V \setminus U_0 \neq \emptyset$ for each $V \in \mathcal{F}$. Since $U_0 \notin S^0(\mathcal{F})$, there is a $V_0 \in \mathcal{F}$ such that $W \cap U_0 \neq \emptyset$ for each $W \in \mathcal{F}$, $W \subset V_0$. Since $U_0 \in S(\mathcal{F})$, there is a $W_0 \in \mathcal{F}$ such that $W_0 \subset V_0 \cap U_0$. But then $W_0 \setminus U_0 = \emptyset$, a contradiction. □

**Corollary 1.1.** *If $\mathcal{F} \subset \mathcal{P}(X)$ is a field of sets, then*

$$\mathcal{F} \setminus \{\emptyset\} \subset S(\mathcal{F} \setminus \{\emptyset\}) \setminus S^0(\mathcal{F} \setminus \{\emptyset\}).$$

Proof. Use Proposition 1.1 (3) and (5). □

Note that the classical $(s)$-sets and $(s^0)$-sets form a $\sigma$-field and a $\sigma$-ideal, respectively (the proof of $\sigma$-additivity for $(s^0)$-sets is based on the fusion lemma; see [Sz]). Additionally, in that case $H(S(\mathcal{F})) = S^0(\mathcal{F})$ [Sz, 3.1]. On the other hand, there are families $\mathcal{F}$ for which $H((S(\mathcal{F})) \neq S^0(\mathcal{F})$ [R, Cor. 1.10]. Observe that there are cases when $S(\mathcal{F})$ forms a $\sigma$-field but $S^0(\mathcal{F})$ is not a $\sigma$-ideal. That happens if $X$ is an infinite set and $\mathcal{F}$ stands for the family of all infinite subsets of $X$; then $S(\mathcal{F}) = \mathcal{P}(X)$ and $S^0(\mathcal{F})$ consists of all finite subsets of $X$. In [Pa, Lemma 2] it was proved that if $\mathcal{F} \subset S(\mathcal{F})$ and $S^0(\mathcal{F})$ is $\sigma$-additive then $S(\mathcal{F})$ is a $\sigma$-field.

The operation $S$ can be iterated. For a family $\mathcal{F} \subset \mathcal{P}(X)$ we define $S_0(\mathcal{F}) = \mathcal{F}$ and $S_\alpha(\mathcal{F}) = S(\cup_{\gamma<\alpha} S_\gamma(\mathcal{F}) \setminus \{\emptyset\})$ for any ordinal $\alpha > 0$. Of course we may consider only $\alpha \leq 2^{2^\kappa}$ where $\kappa$ is the cardinality of $X$. The families $S_\alpha(\mathcal{F})$, $\alpha > 0$, are fields and from Corollary 1.1 it follows that $S_\gamma(\mathcal{F}) \subset S_\alpha(\mathcal{F})$ for any ordinals $\gamma, \alpha$ with $0 < \gamma < \alpha$. In our future studies, we plan to establish the maximal number of different fields that can be obtained in a sequence of type $\langle S_\alpha(\mathcal{F})\colon \alpha > 0\rangle$. In the former version of the paper we claimed incorrectly that this number is 2. The referee has observed that it is at least 3. (See Remark 2.2.)[1]

We say that two families $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{P}(X) \setminus \{\emptyset\}$ are *mutually coinitial* if

$$(\forall\, U \in \mathcal{F}_1)\ (\exists\, V \in \mathcal{F}_2)\ V \subset U$$

$$\text{and } (\forall\, U \in \mathcal{F}_2)\ (\exists\, V \in \mathcal{F}_1)\ V \subset U.$$

**Proposition 1.2.** *Let $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{P}(X) \setminus \{\emptyset\}$.*

**(i)** *If $\mathcal{F}_1, \mathcal{F}_2$ are mutually coinitial then $S(\mathcal{F}_1) = S(\mathcal{F}_2)$ and $S^0(\mathcal{F}_1) = S^0(\mathcal{F}_2)$.*

**(ii)** *Assume that $\mathcal{F}_1 \subset S(\mathcal{F}_1)$ and $\mathcal{F}_2 \subset S(\mathcal{F}_2)$. If $S(\mathcal{F}_1) = S(\mathcal{F}_2)$ and $S^0(\mathcal{F}_1) = S^0(\mathcal{F}_2)$ then $\mathcal{F}_1, \mathcal{F}_2$ are mutually coinitial.*

Proof. (i) is evident.
(ii) Let $U \in \mathcal{F}_1$. Then $U \notin S^0(\mathcal{F}_1)$ by Proposition 1.1(3). Hence $U \in S(\mathcal{F}_1) \setminus S^0(\mathcal{F}_1) = S(\mathcal{F}_2) \setminus S^0(\mathcal{F}_2)$ and by Proposition 1.1(4) there is a $V \in \mathcal{F}_2$ such that $V \subset U$. Analogously, for each $U \in \mathcal{F}_2$ there is a $V \in \mathcal{F}_1$ such that $V \subset U$. □

Note that an idea similar to that contained in Proposition 1.2 was used in [Mo, Th. 1, p. 23]. The referee has asked whether the converse of (i) is true. The answer is “no”which follows from Remark 2.1 in the next section.

Now, consider a field $\Sigma$ (respectively, an ideal $\mathcal{I}$) of subsets of $X$. We say that:

[1] S. Wroński has recently proved that this number is exactly 3.

- $\Sigma$ (respectively, $\mathcal{I}$) is a *topological field* (respectively, a *topological ideal*) if there is a topology $\tau$ on $X$ such that $\Sigma$ consists of all sets with $\tau$-nowhere dense boundary (respectively, $\mathcal{I}$ consists of $\tau$-nowhere dense sets). (Cf. [Ku, §8.V].) We thus write $\Sigma = \Sigma(\tau)$ and $\mathcal{I} = NWD(\tau)$.
- $\Sigma$ (respectively, $\mathcal{I}$) is a *Marczewski field* (respectively, a *Marczewski ideal*) if there is a family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ such that $\Sigma = S(\mathcal{F})$ (respectively, $\mathcal{I} = S^0(\mathcal{F})$).

Note that papers [BET] and [BBC] use different terminology for Marczewski fields: the authors of [BET] say that "$\mathcal{F}$ is a basis for a Marczewski-Burstin-like characterization of $\Sigma$", and in [BBC], $\Sigma$ is called "Marczewski-Burstin representable".

An easy connection between the above notions is contained in the following

**Proposition 1.3.** *(Cf. [BR].) If $\tau$ is a topology on $X$ then $S(\tau \setminus \{\emptyset\}) = \Sigma(\tau)$ and $S^0(\tau \setminus \{\emptyset\}) = NWD(\tau)$. Consequently, every topological field (ideal) is a Marczewski field (ideal).*

From Proposition 1.2(i) we derive

**Proposition 1.4.** *If a family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ is mutually coinitial with a topological base on $X$ then the field $S(\mathcal{F})$ is topological.*

In Section 2 we shall show that the interval algebra is a Marczewski field. This will imply that there are Marczewski fields that are not topological fields. A Boolean-theoretical characterization of topological fields was given in [Wr1]. Article [CJ] was devoted to extensive studies of topological ideals; the authors considered also an additional requirement stating that an ideal consists of meager sets in some topology. In Section 2 we discuss some connections between Marczewski fields, topological fields and category bases (introduced by John Morgan II, see [Mo]).

The class of Marczewski fields seems to be rich. From [Bu] it follows that the Lebesgue measurable sets in $\mathbb{R}$ form a Marczewski field. (Note that paper [Bu] is much earlier than [Sz].) Also the sets with the Baire property in $\mathbb{R}$ constitute a Marczewski field [Br1], [BET]. When we started to prepare our paper, it was not even known whether there exists a non-Marczewski field of subsets of $\mathbb{R}$. Now, our knowledge is wider. Namely, the forthcoming paper [BBC] contains a construction of a non-Marczewski field on $\mathbb{R}$ provided $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Another result of [BBC] states that $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ imply that the Borel subsets of $\mathbb{R}$ form a Marczewski field.

For any filter $\mathcal{F}$ of the algebra $\mathcal{A} = \mathcal{P}(X)$ we denote $-\mathcal{F} = \{X \setminus E : E \in \mathcal{F}\}$ and $\mathcal{A}_{\mathcal{F}} = \mathcal{F} \cup -\mathcal{F}$.

**Proposition 1.5.** *For any filter $\mathcal{F}$ of $\mathcal{A}$ we have $S(\mathcal{F}) = \mathcal{A}_{\mathcal{F}}$ and $S^0(\mathcal{F}) = -\mathcal{F}$.*

Proof. We easily check that $\mathcal{F} \subset S(\mathcal{F})$ and $-\mathcal{F} \subset S^0(\mathcal{F})$. Since $\mathcal{A}_{\mathcal{F}}$ is the smallest field containing $\mathcal{F}$, we have $\mathcal{A}_{\mathcal{F}} \subset S(\mathcal{F})$. To show the reverse inclusion consider a $U \in S(\mathcal{F})$. Since $X \in \mathcal{F}$, we can find a $V \in \mathcal{F}$ such that either $V \subset U$ or $V \subset X \setminus U$. Hence either $U \in \mathcal{F}$ or $X \setminus U \in \mathcal{F}$ which means that $U \in \mathcal{A}_{\mathcal{F}}$. Thus $S(\mathcal{F}) \subset \mathcal{A}_{\mathcal{F}}$. It can be similarly shown that $S^0(\mathcal{F}) \subset -\mathcal{F}$. $\square$

**Proposition 1.6.** *For a set $X$ of cardinality $|X| = \kappa$ there are $2^{2^\kappa}$ nonisomorphic Marczewski fields on $X$ containing all singletons.*

Proof. We follow the argument given in [F]. Let $\Phi$ consist of all filters in $\mathcal{A} = \mathcal{P}(X)$ which are intersections of two free ultrafilters. Then $|\Phi| = 2^{2^\kappa}$ and $\mathcal{A}_{\mathcal{F}_1} \neq \mathcal{A}_{\mathcal{F}_2}$ for any distinct $\mathcal{F}_1, \mathcal{F}_2 \in \Phi$. Additionally, $\{x\} \in \mathcal{A}_{\mathcal{F}}$ for any $x \in X$ and $\mathcal{F} \in \Phi$. Thus, by Proposition 1.5, there are $2^{2^\kappa}$ Marczewski fields on $X$ containing all singletons. Any isomorphism between subalgebras of $\mathcal{P}(X)$ containing all singletons is induced by a bijection from $X$ to $X$. Hence each isomorphism class of such subalgebras has at most $2^\kappa$ elements. Finally, observe that if $h$ is a bijection of $X$ onto $X$ and $\mathcal{F} \in \Phi$ then

$$\{h[U] :\ U \in S(\mathcal{F})\} = S(\{h[V] :\ V \in \mathcal{F}\}) \text{ and } \{h[V] :\ V \in \mathcal{F}\} \in \Phi.$$

Thus there are $2^{2^\kappa}$ different classes of isomorphic Marczewski fields on $X$ containing all singletons. $\square$

## 2 Marczewski fields, topological fields and category bases

Let $X = [0, 1)$. The family of all finite unions of half-open intervals $[a, b)$ (where $0 \le a < b \le 1$) form a field of subsets of $X$. It is called the *interval algebra* of $X$ [K, 1.11].

**Theorem 2.1.** *The interval algebra $\mathcal{A}$ of $X = [0, 1)$ is a Marczewski field.*

Proof. Let $\mathbb{Q}$ stand for the set of all rationals and let $\mathfrak{c}$ denote the cardinality of $\mathbb{R}$. Consider the equivalence relation $x \sim y \iff x - y \in \mathbb{Q}$. Let $F\colon [0, 1] \to \mathbb{R}/\sim$ be a one-to-one function such that $x \notin F(x)$ for $x \in [0, 1]$. (Note that $F$ can be easily constructed by transfinite induction. Indeed, arrange all points of $[0, 1]$ into a one-to-one sequence $x_\gamma$, $\gamma < \mathfrak{c}$, and consider an $\alpha < \mathfrak{c}$. If the values $F(x_\gamma)$ for $\gamma < \alpha$ have been defined, we pick $x \in [0, 1] \setminus \bigcup_{\gamma<\alpha}[F(x_\gamma)] \setminus [x_\alpha]$ and put $F(x_\alpha) = [x]$ where $[x]$ denotes the respective equivalence class.)

For $x \in [0, 1]$ let
$\mathcal{F}_r(x) = \{([x, x + \varepsilon) \setminus F(x)) \cap X : \varepsilon > 0\}$,

$\mathcal{F}_l(x) = \{((x-\varepsilon, x) \setminus F(x)) \cap X : \varepsilon > 0\}$,
$\mathcal{F}(x) = \mathcal{F}_l(x) \cup \mathcal{F}_r(x)$.
Note that $\mathcal{F}_r(1) = \mathcal{F}_l(0) = \emptyset$, otherwise $\mathcal{F}_r(x)$ and $\mathcal{F}_l(x)$ are nonempty. Finally, let $\mathcal{F} = \bigcup_{x\in[0,1]} \mathcal{F}(x)$.

From the definitions of $\mathcal{F}$ and $S(\mathcal{F})$ it easily follows that $[a, b) \in S(\mathcal{F})$ for any $a, b$ with $0 \le a < b \le 1$. Since $S(\mathcal{F})$ is a field of sets, we have $\mathcal{A} \subset S(\mathcal{F})$.

**Claim 1.** Let $x \in [0,1]$ and $k \in \{r, l\}$. If $U \in \mathcal{F}_k(x)$ then for each $y \in [0,1]$ and for each $V \in \mathcal{F}(y)$ such that $V \subset U$, we have $y = x$, and moreover $V \in \mathcal{F}_k(x)$.

Indeed, suppose that $y \neq x$. Let $U = I \setminus F(x)$ and $V = J \setminus F(y)$ where $I$ and $J$ are the respective intervals. From $V \subset U$ and the density of $F(x)$ it follows that $\emptyset \neq J \cap F(x) \subset J \cap F(y)$ which contradicts the disjointness of $F(x)$ and $F(y)$. Thus $y = x$ and so, $V \in \mathcal{F}_m(x)$ for some $m \in \{r, l\}$. However, $m = k$ since otherwise $U \cap V = \emptyset$.

We have already observed that $\mathcal{A} \subset S(\mathcal{F})$. To prove that $S(\mathcal{F}) \subset \mathcal{A}$ fix an $A \in S(\mathcal{F}) \setminus \{\emptyset\}$.

**Claim 2.** For each $x \in A$ there exists an $\varepsilon > 0$ such that $[x, x+\varepsilon) \subset A$. For each $x \in X \setminus A$ there exists an $\varepsilon > 0$ such that $[x, x+\varepsilon) \cap A = \emptyset$.

The latter assertion follows from the former applied to $X \setminus A$. To show the former assertion, suppose $x \in A$ and $[x, x+\varepsilon) \setminus A \neq \emptyset$ for each $\varepsilon > 0$. Consider a $U \in \mathcal{F}_r(x)$. Since $A \in S(\mathcal{F})$, there is a $V \in \mathcal{F}$ such that either $V \subset U \cap A$ or $V \subset U \setminus A$. By Claim 1 we have $V \in \mathcal{F}_r(x)$. Since $x \in A \cap V$, we infer that $V \subset U \cap A$. Let $V = ([x, x+\varepsilon) \setminus F(x)) \cap X$ where $\varepsilon > 0$. We may assume that $x + \varepsilon \le 1$. By our supposition, pick a $y \in (x, x+\varepsilon) \setminus A$. Let $\tilde{V} = [y, x+\varepsilon) \setminus F(y)$. Then $\tilde{V} \in \mathcal{F}_r(y)$ and since $A \in S(\mathcal{F})$, there is a $W \in \mathcal{F}$ such that either $W \subset \tilde{V} \cap A$ or $W \subset \tilde{V} \setminus A$. Again, by Claim 1, we have $W \in \mathcal{F}_r(y)$, so we may assume that $W = [y, y+\varepsilon_1) \setminus F(y)$ where $y + \varepsilon_1 \le x + \varepsilon$. Since $y \notin A$, we have $W \cap A = \emptyset$. The set $[y, y+\varepsilon_1) \setminus (F(x) \cup F(y))$ is nonempty (uncountable) contained in $[x, x+\varepsilon) \setminus F(x) = V \subset A$ and simultaneously in $[y, y+\varepsilon_1) \setminus F(y) = W \subset X \setminus A$. Contradiction.

**Claim 3.** For each $x \in (0,1]$ there exists an $\varepsilon > 0$ such that either $(x-\varepsilon, x) \subset A$ or $(x-\varepsilon, x) \cap A = \emptyset$.

To show the claim, suppose that there exists an $x \in (0,1]$ such that $(x - \varepsilon, x) \setminus A \neq \emptyset$ and $(x-\varepsilon, x) \cap A \neq \emptyset$ for each $\varepsilon > 0$. Let $U \in \mathcal{F}_l(x)$. Since $A \in S(\mathcal{F})$, there is a $V \in \mathcal{F}$ such that either $V \subset U \cap A$ or $V \subset U \setminus A$. By Claim 1 we have $V \in \mathcal{F}_l(x)$ and we may assume that $V = (x-\varepsilon, x) \setminus F(x)$

where $x-\varepsilon \geq 0$. If $V \subset A$, by our supposition we can pick $y \in (x-\varepsilon, x)\setminus A$. By Claim 2 there is an $\varepsilon_1 > 0$ such that $[y, y+\varepsilon_1)\cap A = \emptyset$ and we may assume that $y+\varepsilon_1 \leq x$. On the other hand, $\emptyset \neq [y, y+\varepsilon_1)\setminus F(x) \subset (x-\varepsilon, x)\setminus F(x) \subset V \subset A$, a contradiction. If $V \cap A = \emptyset$, by our supposition we can pick $y \in (x-\varepsilon, x)\cap A$. By Claim 2 there is an $\varepsilon_1 > 0$ such that $[y, y+\varepsilon_1) \subset A$ and we may assume that $y+\varepsilon_1 \leq x$. On the other hand, $\emptyset \neq [y, y+\varepsilon_1)\setminus F(x) \subset (x-\varepsilon, x)\setminus F(x) = V \subset X \setminus A$, a contradiction.

From Claim 2 it follows that each connected component $I$ of $A$ is a nondegenerate interval with $b = \sup I \notin I$. Denote $a = \inf I$ and observe that $a \in I$. Indeed, suppose that $a \notin I$. We know that $[a, b) \in S(\mathcal{F})$. Thus $A \cap [a, b) = (a, b) \in S(\mathcal{F})$ and consequently $[a, b) \setminus (a, b) = \{a\} \in S(\mathcal{F})$ which contradicts Claim 2.

From the above we infer that $A$ is a union of at most countable family of pairwise disjoint intervals of type $[a, b)$. This family however cannot be infinite. Indeed, suppose that $A = \bigcup_{n=1}^{\infty}[a_n, b_n)$ with $[a_n, b_n) \subset X$, $n \geq 1$, pairwise disjoint. Pick a strictly monotonic subsequence $(a_{k_n})$ of $(a_n)$. If $a_{k_n} \searrow x$, we apply Claim 2 to $x$ and we obtain a contradiction. If $a_{k_n} \nearrow x$, we apply Claim 3 to $x$ and we obtain a contradiction.

Thus we have proved that $A \in \mathcal{A}$. Consequently, $S(\mathcal{F}) \subset \mathcal{A}$. $\square$

**Remark 2.1.** *Observe that in the above construction, we can choose, for $i = 1, 2$, one-to-one functions $F^{(i)} \colon [0, 1] \to \mathbb{R}/\sim$ with disjoint ranges, and such that $x \notin F^{(i)}(x)$ for each $x \in [0, 1]$. Then $\mathcal{A} = S(\mathcal{F}^{(1)}) = S(\mathcal{F}^{(2)})$ where $\mathcal{F}^{(i)}$ $(i = 1, 2)$ is associated with $F^{(i)}$ as in the proof of Theorem 2.1. Since $H(\mathcal{A}) = \{\emptyset\}$, we have $S^0(\mathcal{F}^{(1)}) = S^0(\mathcal{F}^{(2)}) = \{\emptyset\}$ by Proposition 1.1(2). However, the argument for Claim 1 shows that $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are not mutually coinitial. Thus the converse of (i) in Proposition 1.2 is false.*

**Remark 2.2.** *Since $\mathcal{A}\setminus\{\emptyset\}$ and the family of nonempty open sets in $[0, 1)$ are mutually coinitial, the field $\Sigma = S(\mathcal{A} \setminus \{\emptyset\})$ consists of all sets in $[0, 1)$ with nowhere dense boundary, and $S(\Sigma\setminus\{\emptyset\}) = \mathcal{P}([0, 1))$ by Proposition 1.1(6). So we have 3 different fields obtained by the iteration of $S(\cdot)$.*

**Corollary 2.1.** *There exists an Marczewski field which is not a topological field.*

PROOF. This follows from Theorem 2.1 since every topological field has an atom [Wr1] and the algebra $\mathcal{A}$ has no atoms. $\square$

Although the class of topological subfields of $\mathcal{P}(X)$ is smaller than the class of Marczewski fields, the former can be used to get the following representation result:

**Theorem 2.2.** *Every field $\Sigma$ of subsets of $X$ is equal to the intersection of all topological fields containing $\Sigma$.*

Proof. If $\Sigma = \mathcal{P}(X)$, the assertion is obvious. Assume that $\Sigma \neq \mathcal{P}(X)$. It suffices to show that for each $A \notin \Sigma$ there is a topological field $\Sigma_A \supset \Sigma$ with $A \notin \Sigma_A$. So, let $A \notin \Sigma$. By [Wr2, Lemma 2] we find an ultrafilter $\mathcal{F}_A$ of the field $\Sigma$ such that no subset of $A$ is in $\mathcal{F}_A$ and no subset of $X \setminus A$ is in $\mathcal{F}_A$. Thus $A \notin S(\mathcal{F}_A)$. Put $\Sigma_A = S(\mathcal{F}_A)$. Observe that $\Sigma_A$ is a topological field since $\mathcal{F}_A$ forms a topological base, and thus Proposition 1.4 can be used. Because $\mathcal{F}_A$ is an ultrafilter of $\Sigma$, we have $\Sigma \subset S(\mathcal{F}_A)$. □

Recall [Mo] that a pair $(X, \mathcal{C})$ is said to be a *category base* if $\mathcal{C}$ stands for a family of subsets of a nonempty set $X$, and nonempty sets in $\mathcal{C}$, called *regions*, satisfy the following axioms:

$1^0$ $\bigcup \mathcal{C} = X$,

$2^0$ Let $A$ be a region and $\mathcal{D}$ – a nonempty family of disjoint regions with $|\mathcal{D}| < |\mathcal{C}|$. Then

- if $A \cap (\bigcup \mathcal{D})$ contains a region then there is a region $D \in \mathcal{D}$ such that $A \cap D$ contains a region;
- if $A \cap (\bigcup \mathcal{D})$ contains no region then there is a region $B \subset A \setminus \bigcup \mathcal{D}$.

A set $E \subset X$ is called *singular* if $E \in S^0(\mathcal{C} \setminus \{\emptyset\})$.

**Theorem 2.3.** *Assume that for a family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$ we have $X \in \mathcal{F} \subset S(\mathcal{F})$ and $\bigcup (S^0(\mathcal{F}) \cap \{U \cap V : V \in \mathcal{F}\}) \in S^0(\mathcal{F})$ for each $U \in \mathcal{F}$. Then $(X, \mathcal{F})$ is a category base whose ideal of singular sets equals $S^0(\mathcal{F})$.*

Proof. It is enough to check condition $2^0$ defining a category base. Let $F, F_t \in \mathcal{F}$ for $t \in T$ (where $T$ is an arbitrary set of indices). Assume that $F \cap \bigcup_{t\in T} F_t$ contains a set from $\mathcal{F}$. Then there exists a $t_0 \in T$ such that $F \cap F_{t_0}$ contains a set from $\mathcal{F}$. Indeed, suppose that it is not the case. Since $\mathcal{F} \subset S(\mathcal{F})$, we have $F \cap F_t \in S(\mathcal{F})$ for each $t \in T$. Hence by Proposition 1.1(4) we get $F \cap F_t \in S^0(\mathcal{F})$. Thus, by assumption, we have $\bigcup_{t\in T} (F \cap F_t) \in S^0(\mathcal{F})$ which yields a contradiction (cf. Proposition 1.1(3)). Assume now that $F \cap \bigcup_{t\in T} F_t$ contains no set from $\mathcal{F}$. Then every set $F \cap F_t$, $t \in T$, contains no set from $\mathcal{F}$. As before we infer that $F \cap F_t \in S(\mathcal{F})$ for $t \in T$ and moreover $F \cap F_t \in S^0(\mathcal{F})$. By assumption we have $\bigcup_{t\in T} (F \cap F_t) \in S^0(\mathcal{F})$ and consequently,

$$F \setminus \bigcup_{t\in T} (F \cap F_t) \in S(\mathcal{F}) \setminus S^0(\mathcal{F}),$$

since $F \in S(\mathcal{F}) \setminus S^0(\mathcal{F})$. Hence, by Proposition 1.1(4), there is a set from $\mathcal{F}$ contained in $F \setminus \bigcup_{t \in T} F_t$. $\square$

## 3 Marczewski fields and perfect sets

Let $(X, \tau)$ be a topological space. By a *$\tau$-perfect set* we mean a nonempty $\tau$-closed set without isolated points. Let $\mathrm{Perf}(\tau)$ stand for the family of all $\tau$-perfect subsets of $X$. As it was mentioned in Section 1, if $X$ is a Polish space with topology $\tau$ then $S(\mathrm{Perf}(\tau))$ and $S^0(\mathrm{Perf}(\tau))$ are exactly the classical families of Marczewski $(s)$-sets and $(s^0)$-sets. In [R], studies of $S(\mathrm{Perf}(\tau))$ and $S^0(\mathrm{Perf}(\tau))$ for other topological spaces were initiated. (See also [BR].) In this section we propose a general method which enables us to reprove some results of [R] and [BR], and to show new applications.

If $\tau$ is a given topology on $X$ and we want to characterize $S(\mathrm{Perf}(\tau))$, we shall use the following scheme:

$1^0$ we conjecture that $S(\mathrm{Perf}(\tau)) = \Sigma$ where $\Sigma$ is a known field of sets,

$2^0$ we know that $\Sigma = S(\mathcal{F})$ for some family $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\emptyset\}$;

$3^0$ in aim to confirm our conjecture $1^0$, it is enough (by Proposition 1.2(i)) to check that $\mathrm{Perf}(\tau)$ and $\mathcal{F}$ are mutually coinitial.

A similar method works for $S^0(\mathrm{Perf}(\tau))$. The above scheme will be illustrated by examples dealing with some "ideal topologies" on $\mathbb{R}$.

Let $\tau$ be a topology on $X$, and let $\mathcal{I} \subset \mathcal{P}(X)$ be a $\sigma$-ideal containing all singletons. The family

$$\mathcal{B}_{\mathcal{I}}^{\star} = \{U \setminus A : \ U \in \tau \ \& \ A \in \mathcal{I}\}$$

forms a base for a topology $\tau_{\mathcal{I}}^{\star}$, on $X$, stronger than $\tau$, which will be called the *Hashimoto topology* associated with $\tau$ and $\mathcal{I}$. If $\tau$ is second countable (or even hereditary Lindelöf) then $\tau_{\mathcal{I}}^{\star} = \mathcal{B}_{\mathcal{I}}^{\star}$. (See [H], [JH], [LMZ].) The following property is well known.

**Lemma 3.1.** *(Cf. [BR]). Let $\mathcal{I}$ be a $\sigma$-ideal of subsets of a separable metric space $X$ and let $\mathcal{I}$ contain all singletons. A set $F \subset X$ is $\tau_{\mathcal{I}}^{\star}$-perfect if and only if $F$ is $\tau$-perfect and $U \cap F \notin \mathcal{I}$ for each $U \in \tau$ with $U \cap F \neq \emptyset$.*

By $\mathcal{M}$ and $\mathcal{N}$ we denote, respectively, the $\sigma$-ideals all meager (i.e. of the first category) sets and of all Lebesgue null sets in $\mathbb{R}$. We shall consider the Hashimoto topologies $T_{\mathcal{M}}^{\star}$ and $T_{\mathcal{N}}^{\star}$ where $T$ stands for the natural topology on $\mathbb{R}$. Let $\mathcal{D}_{\mathcal{N}}$ denote the density topology on $\mathbb{R}$. (See e.g. [O] or [CLO].) Wilczyński in [W1] introduced the category analogue of the density topology which will be denoted by the $\mathcal{D}_{\mathcal{M}}$. Since topology $\mathcal{D}_M$ is less known, let us give necessary definitions. A number $x \in \mathbb{R}$ is called a *category density point* of a set $A \subset \mathbb{R}$ with the Baire property if each increasing sequence $\{n_k\}$ of positive integers has a subsequence $\{n_{m_k}\}$ such that the sequence of characteristic functions

$$\chi_{[-1,1]\cap n_{m_k}(A-x)}(t)$$

(where $n_{m_k}(A-x) = \{n_{m_k}(a-x) : a \in A\}$) tends to $\chi_{[-1,1]}(t)$ for all points $t \in \mathbb{R}$ except for those belonging to a meager set. If $[-1,1]$ is replaced by $[-1,0]$ or $[0,1]$, we get the respective notions of one-sided category density points. Topology $\mathcal{D}_{\mathcal{M}}$ consists of all sets $A \subset \mathbb{R}$ with the Baire property, such that each point of $A$ is a category density point of $A$. There are many analogies between $\mathcal{D}_{\mathcal{M}}$ and $\mathcal{D}_{\mathcal{N}}$; for details, see [PWW], [W2] and [CLO].

It is known that the only possible inclusions between the above-mentioned topologies are the following $T \subsetneq T^{\star}_{\mathcal{M}} \subsetneq \mathcal{D}_{\mathcal{M}}$ and $T \subsetneq T^{\star}_{\mathcal{N}} \subsetneq \mathcal{D}_{\mathcal{N}}$. (See [LJW].)

The following proposition is due to W. Wilczyński (oral communication).

**Proposition 3.1.** *Every $\mathcal{D}_{\mathcal{M}}$-perfect set has nonempty $T$-interior.*

Proof. Let $F$ be a $\mathcal{D}_{\mathcal{M}}$-perfect set. Then $F$ is nonmeager. Indeed, if $F \in \mathcal{M}$ and $x \in F$ then $U = (\mathbb{R} \setminus F) \cup \{x\} \in \mathcal{D}_{\mathcal{M}}$ and $F \cap U = \{x\}$ which contradicts the fact that $F$ is a $\mathcal{D}_{\mathcal{M}}$-perfect set. Since $F$ is a nonmeager set with the Baire property, there is an open interval $V$ such that $F \cap V$ is comeager in $V$. We claim that $V \subset F$. Let $x \in V$ and let $W$ be a $\mathcal{D}_{\mathcal{M}}$-neighborhood of $x$. Thus $x$ is a category density point of the both sets $V$ and $W$. Consequently, $V \cap W \notin \mathcal{M}$. Since $F \cap V$ is comeager in $V$, we thus have $F \cap W \neq \emptyset$. Hence $x$ belongs to the the $\mathcal{D}_{\mathcal{M}}$-closure of $F$ (equal to $F$). □

Let $\Sigma_{\mathcal{M}}$ and $\Sigma_{\mathcal{N}}$ denote, respectively, the $\sigma$-fields of all sets with the Baire property and of all Lebesgue measurable sets in $\mathbb{R}$. As in Section 1, $\Sigma(T)$ stands for the field of all subsets of $\mathbb{R}$ with nowhere dense boundary, and $NWD(T)$ - the ideal of nowhere dense subsets of $\mathbb{R}$.

**Theorem 3.1.** **(a)** *$S(Perf(\mathcal{D}_{\mathcal{N}})) = \Sigma_{\mathcal{N}}$, $S^0(Perf(\mathcal{D}_{\mathcal{N}})) = \mathcal{N}$. (See [R].)*

**(b)** *$S(Perf(T^{\star}_{\mathcal{N}})) = \Sigma_{\mathcal{N}}$, $S^0(Perf(T^{\star}_{\mathcal{N}})) = \mathcal{N}$. (See [BR].)*

**(c)** *$S(Perf(T^{\star}_{\mathcal{M}})) = \Sigma(T)$, $S^0(Perf(T^{\star}_{\mathcal{M}})) = NWD(T)$. (See [BR].)*

**(d)** *$S(Perf(\mathcal{D}_{\mathcal{M}})) = \Sigma(T)$, $S^0(Perf(\mathcal{D}_{\mathcal{M}})) = NWD(T)$.*

Proof. (a) Let $\mathcal{F}$ denote the family of all $T$-perfect sets of positive measure. Burstin [Bu] proved that $\Sigma_{\mathcal{N}} = S(\mathcal{F})$. It is not hard to prove that $\mathcal{F}$ and $\mathrm{Perf}(\mathcal{D}_{\mathcal{N}})$ are mutually coinitial (cf. [R, Lemma 3.2]). Thus by our scheme we have $S(\mathrm{Perf}(\mathcal{D}_{\mathcal{N}})) = \Sigma_{\mathcal{N}}$. Moreover $S^0(\mathcal{F}) \subset H(S(\mathcal{F})) = H(\Sigma_{\mathcal{N}}) = \mathcal{N}$ and $\mathcal{N} \subset \Sigma_{\mathcal{N}} = S(\mathcal{F})$. But $\mathcal{N} \subset S(\mathcal{F})$ easily implies that $\mathcal{N} \subset S^0(\mathcal{F})$. Hence $\mathcal{N} = S^0(\mathcal{F}) = S^0(\mathrm{Perf}(\mathcal{D}_{\mathcal{N}}))$.

(b) If $\mathcal{F}$ is as in (a), Lemma 3.1 easily implies that $\mathcal{F}$ and $\mathrm{Perf}(T^\star_{\mathcal{N}})$ are mutually coinitial (cf. [BR]). The rest is the same as in (a).

(c) From Lemma 3.1 it follows that $T^\star_{\mathcal{M}}$-perfect sets have nonempty $T$-interior. Also, a nonempty $T$-open set contains a $T^\star_{\mathcal{M}}$-perfect set (a closed nondegenerate interval). Hence $\mathrm{Perf}(T^\star_{\mathcal{M}})$ and $T \setminus \{\emptyset\}$ are mutually coinitial. Thus the assertion follows from Proposition 1.3.

(d) As in (c) it suffices to prove that $\mathrm{Perf}(\mathcal{D}_{\mathcal{M}})$ and $T \setminus \{\emptyset\}$ are mutually coinitial. Firstly, by Proposition 3.1, every $\mathcal{D}_{\mathcal{M}}$-perfect set has nonempty $T$-interior. Secondly, let us show that a nondegenerate interval $[a, b]$ is a $\mathcal{D}_{\mathcal{M}}$-perfect set. Indeed, $[a, b]$ is obviously $\mathcal{D}_{\mathcal{M}}$-closed. Let $x \in [a, b]$ and $x \in U \in \mathcal{D}_{\mathcal{M}}$. Then $x$ is a category density point of $U$ and $x$ is at least a one-sided category density point of $[a, b]$. Hence $U \cap [a, b] \notin \mathcal{M}$ and so, $x$ is a $\mathcal{D}_{\mathcal{M}}$-accumulation point of $[a, b]$. □

Note that statements (a) and (b),(c) of Theorem 3.1 can be extended to cases dealing with spaces more general than $\mathbb{R}$, as it was mentioned in [R] and [BR]. Topology $\mathcal{D}_{\mathcal{M}}$ can be considered in certain linear topological spaces and statement (d) of Theorem 3.1 then holds.

Assertions (a),(d) and (b),(c) of Theorem 3.1 show a kind of asymmetry between measure and category. Knowing (a),(b) we rather expected to obtain $\Sigma_{\mathcal{M}}$ and $\mathcal{M}$ as the respective Marczewski families in (c),(d). So the following problem appears:

**Problem 3.1.** Find a topology $\tau$ on $\mathbb{R}$ such that $\Sigma_{\mathcal{M}}$ (the $\sigma$-field of all subsets of $\mathbb{R}$ with the Baire property) is of the form $S(\mathrm{Perf}(\tau))$.

Note that Brown [Br1] (see also [BET]) showed the equalities $\Sigma_{\mathcal{M}} = S(\mathcal{G})$ and $\mathcal{M} = S^0(\mathcal{G})$ provided that $\mathcal{G}$ consists of sets of the form $U \setminus F$ where $U$ is open and $F$ is an $F_\sigma$ meager set. This easily implies that $\Sigma_{\mathcal{M}} = S(T^\star_{\mathcal{M}})$ and $\mathcal{M} = S^0(T^\star_{\mathcal{M}})$ since $\mathcal{G}$ and $T^\star_{\mathcal{M}}$ are mutually coinitial. Thus $\Sigma_{\mathcal{M}}$ is a topological field.

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## References

[BBC] M. Balcerzak, A. Bartoszewicz, K. Ciesielski, *On Marczewski-Burstin representations of certain algebras of sets*, Real Anal. Exchange, **26** (2000-2001).

[BR] M. Balcerzak, J. Rzepecka, *Marczewski sets in the Hashimoto topologies for measure and category*, Acta Univ. Carolin. Math. Phys., **39** (1998), 93–97.

[Bre] J. Brendle, *Strolling through paradise*, Fund. Math. **148** (1995), 1–25.

[Br1] J. B. Brown, *The Ramsey sets and related sigma algebras and ideals*, the lecture during Special Session on Measure Theory and Descriptive Set Theory, Atlanta Meeting of the AMS, January 1988.

[Br2] J. B. Brown, *The Ramsey sets and related sigma algebras and ideals*, Fund. Math. **136** (1990), 179–185.

[BrCo] J. B. Brown, G. V. Cox, *Classical theory of totally imperfect spaces*, Real Anal. Exchange **7** (1982), 1–39.

[BET] J. B. Brown, H. Elalaoui-Talibi, *Marczewski-Burstin-like characterizations of $\sigma$-algebras, ideals, and measurable functions*, Colloq. Math., **82** (1999), 277–286.

[Bu] C. Burstin, *Eigenschaften messbaren und nichtmessbaren Mengen*, Wien Ber. **123** (1914), 1525–1551.

[CJ] K. Ciesielski, J. Jasinski, *Topologies making a given ideal nowhere dense or meager*, Topology Appl. **63** (1995), 277–298.

[CLO] K. Ciesielski, L. Larson, K. Ostaszewski, *$\mathcal{I}$-density continuous functions*, Mem. Amer. Math. Soc. **107** (515) (1994).

[Co] P. Corazza, *Ramsey sets, the Ramsey ideal, and other classes over* $\mathbf{R}$, J. Symb. Logic **57** (1992), 1441–1468.

[F] F. J. Freniche, *The number of nonisomorphic Boolean subalgebras of a power set*, Proc. Amer. Math. Soc. **91** (1984), 199–201.

[H] H. Hashimoto, *On the $^\star$topology and its application*, Fund. Math. **91** (1976), 5–10.

[JH] D. Jankovič and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97**, No 4, April 1990, 295–310.

[K] S. Kopelberg, *Handbook of Boolean Algebras*, vol.1, North Holland, 1989.

[Ku] K. Kuratowski, *Topology*, vol.1, Academic Press, New York, 1966.

[LMZ] J. Lukeš, J. Malý and L. Zajiček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. **1189**, Springer, New York, 1986.

[LJW] E. Łazarow, R. A. Johnson, W. Wilczyński, *Topologies related to sets having the Baire property*, Demonstr. Math. **22** (1989), 179–191.

[Mi1] A. W. Miller, *Special subsets of the real line*, in Handbook of Set-Theoretic Topology, North-Holland, Amsterdam-New York, 1984, 201–233.

[Mi2] A. W. Miller, *Special sets of reals*, Israel Math. Conf. Proc., **6** (1993), 415–431.

[Mo] J. C. Morgan II, *Point Set Theory*, Marcel Dekker, New York, 1990.

[O] J. C. Oxtoby, *Measure and Category*, Springer Verlag, New York 1971.

[Pa] J. Pawlikowski, *Parametrized Ellentuck theorem*, Topology Appl. **37** (1990), 65–73.

[PWW] W. Poreda, E. Wagner-Bojakowska, W. Wilczyński, *A category analogue of the density topology*, Fund. Math. **125** (1985), 167–173.

[R] P. Reardon, *Ramsey, Lebesgue and Marczewski sets and the Baire property*, Fund. Math. **149** (1996), 191–203.

[Sz] E. Szpilrajn (Marczewski), *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.

[Wa] J. T. Walsh, *Marczewski sets, measure and the Baire property*, Fund. Math. **129** (1988), 83–89.

[W1] W. Wilczyński, *A generalization of the density topology*, Real Anal. Exchange **8** (1982-83), 16–20.

[W2] W. Wilczyński, *A category analogue of the density topology, approximate continuity and the approximate derivative*, Real Anal. Exchange **10** (1984-85), 241–265.

[Wr1] S. Wroński, *On fields of sets with a nowhere dense boundary*, Demonstr. Math. **12** (1979), 373–377.

[Wr2] S. Wroński, *On proper subuniverses of a Boolean algebra*, Acta Univ. Lodziensis, Folia Mathematica **9** (1997), 69–76.