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INFINITE PEANO DERIVATIVES

Abstract

Let $f_{(n)}$ and $\underline{f}_{(n)}$ denote the n^{th} Peano derivative and the n^{th} lower Peano derivative of the function $f : [a, b] \to \mathbb{R}$. We investigate the validity of the following statements.

(M_n). *If the set $H = \{x \in [a, b] : \underline{f}_{(n)}(x) > 0\}$ is of positive outer measure, then f is n-convex on a subset of H having positive outer measure.*

(Z_n). *The set $E_n(f) = \{x \in [a, b] : f_{(n)}(x) = \infty\}$ is of measure zero for every $f : [a, b] \to \mathbb{R}$.*

We prove that (M_n) and (Z_n) are true for $n = 1$ and $n = 2$, but false for $n \geq 3$. More precisely we show that for every $n \geq 3$ there is an $(n - 1)$ times continuously differentiable function f on $[a, b]$ such that $f_{(n)}(x) = \infty$ a.e. on $[a, b]$, and that such a function cannot be n-convex on any set of positive outer measure.

We also show that the category analogue of (Z_n) is false for every n. Moreover, the set $E_n(f)$ can be residual. On the other hand, the category analogue of (M_n) is true for every n. More precisely, if $\{x \in [a, b] : \underline{f}_{(n)}(x) > 0\}$ is of second category, then f is n-convex on a subinterval of $[a, b]$. As a corollary we find that $E_n(f)$ cannot be residual and of full measure simultaneously.

1 Introduction

Let $\underline{f}'$ denote the lower derivative of the function $f : [a, b] \to \mathbb{R}$, and let λ denote Lebesgue outer measure. Our starting point is the following simple fact.

(M_1). *If the set $H = \{x \in [a, b] : \underline{f}'(x) > 0\}$ is of positive outer measure, then there is a subset $A \subset H$ such that $\lambda(A) > 0$ and f is increasing on A.*

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PROOF. Let $H_n = \{x \in H : \frac{f(y)-f(x)}{y-x} > 0$ for every $0 < |y-x| < 1/n\}$. Since $H = \cup_{n=1}^{\infty} H_n$, there is an index n such that $\lambda(H_n) > 0$. For a suitable k the set $A = H_n \cap \left(\frac{k-1}{n}, \frac{k}{n}\right)$ is of positive outer measure, and it is clear that f is increasing on A. $\square$

The following statement is also well-known.

(Z_1). *The set $\{x \in [a,b] : f'(x) = \infty\}$ is of measure zero for every $f : [a,b] \to \mathbb{R}$.*

In this note we shall investigate the generalizations of the statements (M_1) and (Z_1) involving the notions of Peano derivatives and convexity of higher order. Let f be continuous at x, and suppose that there exists a polynomial p such that $f(x+t) = p(t) + o(t^n)$ $(t \to 0)$. Then the number $p^{(k)}(0)$ is called the k^{th} Peano derivative of f at x and is denoted by $f_{(k)}(x)$ for every $0 \le k \le n$. It is easy to see that $f_{(n)}(x)$ equals the limit

$$\lim_{t\to 0} \frac{n!}{t^n} \left(f(x+t) - \sum_{i=0}^{n-1} \frac{f_{(i)}(x)}{i!} t^i \right). \tag{1}$$

If the limit (1) equals infinity, then we write $f_{(n)}(x) = \infty$. Replacing the limit by lim inf in (1) we obtain the n^{th} lower Peano derivative of f at x, denoted by $\underline{f}_{(n)}(x)$. Note that $f_{(n)}(x)$ or $\underline{f}_{(n)}(x)$ are defined only if $f_{(n-1)}(x)$ exists and is finite.

The unilateral derivates $f_{(n)+}(x)$ and $f_{(n)-}(x)$ are defined by taking the corresponding unilateral limits. We shall also consider the derivatives $f_{(n)}(x)$ and $\underline{f}_{(n)}(x)$ in the case when f is defined on a set A and $x \in A$ is a limit point of A. The definitions are the same except that all the limit relations have to be restricted to the set A.

Let f be a real valued function defined on the set $A \subset \mathbb{R}$. The divided differences of f are defined by induction as follows. Let $[x_1; f] = f(x_1)$ for every $x_1 \in A$. If $n \ge 1$ and $[x_1, \ldots, x_n; f]$ is defined whenever $x_1, \ldots, x_n$ are distinct elements of A, then we put

$$[x_1, \ldots, x_n, x_{n+1}; f] = \frac{[x_2, \ldots, x_{n+1}; f] - [x_1, \ldots, x_n; f]}{x_{n+1} - x_1} \tag{2}$$

for every system of distinct points $x_1, \ldots, x_{n+1} \in A$. An easy computation shows that

$$[x_1, \ldots, x_n; f] = \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \ne i}(x_i - x_j)}$$

and thus $[x_1,\dots,x_n;f]$ is independent of the order of $x_1,\dots,x_n$. The function f is said to be p-convex on A if $[x_1,\dots,x_{p+1};f]\ge 0$ for every system of distinct points $x_1,\dots,x_{p+1}\in A$. For $p=0$ this means that f is nonnegative on A. The function f is 1-convex if it is increasing on A; while 2-convexity coincides with the notion of usual convexity. If A is an open interval and $n\ge 2$, then f is n-convex on A if and only if f is $n-2$ times continuously differentiable and $f^{(n-2)}$ is convex on A. (See [2].)

In this note we shall investigate the validity of the following generalizations of (M_1) and (Z_1).

(M_n). *If the set $H=\{x\in[a,b]:\underline{f}_{(n)}(x)>0\}$ is of positive outer measure, then there is a subset $A\subset H$ such that $\lambda(A)>0$ and f is n-convex on A.*

(Z_n). *The set $\{x\in[a,b]:f_{(n)}(x)=\infty\}$ is of measure zero for every $f:[a,b]\to\mathbb{R}$.*

First we consider the case of $n=2$.

Proposition 1. *Both (M_2) and (Z_2) are true.*

PROOF. We start with (M_2). Suppose that $H=\{x\in[a,b]:\underline{f}_{(2)}(x)>0\}$ is of positive outer measure. Let

$$\begin{aligned}H_n=\{x\in H :&f(y)-f(x)-f'(x)(y-x)>0 \text{ for every } y\in(a,b)\\ &\text{satisfying } 0<|y-x|<1/n\}.\end{aligned}$$

It is clear that $H=\cup_{n=1}^{\infty}H_n$, and thus there is an n such that $\lambda(H_n)>0$. Choose a subinterval I of $[a,b]$ such that $|I|<1/n$ and $\lambda(H_n\cap I)>0$. Let $A=H_n\cap I$. Then $f(y)-f(x)-f'(x)(y-x)>0$ for every $x\in A$ and $y\in I$ with $y\ne x$. Let $x,y,z\in A,\ x<y<z$. Then $f(x)-f(y)-f'(y)(x-y)>0$ and $f(z)-f(y)-f'(y)(z-y)>0$. If we divide the first inequality by $y-x$, the second inequality by $z-y$, and add the resulting inequalities, then we obtain that the second divided difference of f at the points x,y,z is positive. Therefore f is convex on A.

Now we prove (Z_2). Suppose that (Z_2) is false, and let $f:[a,b]\to\mathbb{R}$ be such that $E=\{x\in[a,b]:f_{(2)}(x)=\infty\}$ is of positive measure. By (M_2), there is a set $A\subset E$ such that $\lambda(A)>0$ and f is convex on A. We can select two points, $c,d\in A$ such that $c<d$ and $\lambda(A\cap[c,d])>0$. It is well-known that $f|A\cap[c,d]$ can be extended to $[c,d]$ as a convex function. It is also well-known that every convex function has a finite second Peano derivative almost everywhere. However, if $x\in A\cap(c,d)$, then $f_{(2)}(x)=\infty$ and thus no extension of $f|A\cap[c,d]$ can have a finite second Peano derivative at x. Since $\lambda(A\cap[c,d])>0$, this is a contradiction, and thus (Z_2) is true. $\square$

It was claimed by P. S. Bullen and S. N. Mukhopadhyay [3, (7.24) Corollary] that (Z_n) is true for every n; however, their proof is in error[1]. In fact, as we shall see, both (M_n) and (Z_n) are false for every $n \geq 3$. We remark, however, that the following special case of (Z_n) is true.

Proposition 2. *If $f : A \to \mathbb{R}$ is n-convex on A, then the set $\{x \in A : f_{(n)}(x) = \infty\}$ is of measure zero.*

We shall give the proof in the next section. It is well-known that the following stronger version of (Z_1) is also true. *The set $\{x \in [a,b] : f'_+(x) = \infty\}$ is of measure zero for every $f : [a,b] \to \mathbb{R}$.* (See [5, (4.4) Theorem, p. 270].) We shall prove that this statement does not generalize to $n = 2$.

Theorem 3. *There exists a continuously differentiable function f on $[0,1]$ such that $f_{(2)+}(x) = +\infty$ and $f_{(2)-}(x) = -\infty$ holds a.e. on $[0,1]$.*

For $n \geq 3$ we can prove the following.

Theorem 4. *For every $n \geq 3$ there is an $n-1$ times continuously differentiable function f on $[0,1]$ such that $f_{(n)}(x) = +\infty$ holds a.e. on $[0,1]$.*

Corollary 5. *(M_n) and (Z_n) are false for every $n \geq 3$.*

PROOF. Let $n \geq 3$. It is clear that Theorem 4 contradicts (Z_n). Suppose that (M_n) is true. Let f be a function as in Theorem 4. Since $E = \{x \in [a,b] : f_{(n)}(x) = \infty\}$ is of positive measure, it follows from (M_n) that f is n-convex on a set $A \subset E$ with $\lambda(A) > 0$. By Proposition 2, the set $\{x \in A : f_{(n)}(x) = \infty\}$ is of measure zero. However, this set equals A, a contradiction. $\square$

We shall give the proofs of Theorems 3 and 4 in Section 3.

It is easy to see that the category analogue of (Z_n) fails for every n. Indeed, let $H \subset [0,1]$ be a residual null set. It is well-known that there is an increasing continuous function $f : [0,1] \to \mathbb{R}$ such that $f'(x) = \infty$ for every $x \in H$. That is, the set $\{x : f'(x) = \infty\}$ can be residual. If we take the $(n-1)^{\text{st}}$ integral function of f, then we obtain an n-convex function g such that the set $\{x \in [0,1] : g_{(n)}(x) = \infty\}$ is residual. We shall prove, however, that the category analogue of (M_n) is true for every n.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be such that the set $\{x \in [a,b] : \underline{f}_{(n)}(x) > 0\}$ is of second category. Then f is n-convex on a subinterval of $[a,b]$.*

[1]The error appears in the proof of (7.16) Lemma on p. 274. Here the quantities δ_1, etc. are treated as constants, while δ_1 depends on x_r, δ_2 depends on x_r and x_{r-1}, etc.

We shall give the proof in Section 4. Theorem 6 has the following interesting consequence.

Corollary 7. *The set $E_n(f) = \{x \in [a,b] : f_{(n)}(x) = \infty\}$ cannot be residual and of full measure simultaneously. More precisely, if $E_n(f)$ is of full measure, then it is of first category.*

PROOF. Suppose that $E_n(f)$ is of full measure and of second category. By Theorem 6, there is a subinterval $I \subset [a,b]$ such that f is n-convex on I. Then the n^{th} Peano derivative of f is finite almost everywhere in I. However, $f_{(n)}(x) = \infty$ at each point of $E_n(f) \cap I$, which is a set of positive measure. This contradiction proves the statement. □

We note that $E_n(f)$ can be residual and of positive measure; see Remark 9 following the proof of Theorem 4.

2 Proof of Proposition 2

The statement of Proposition 2 is an immediate corollary of the following theorem by P. S. Bullen and S. N. Mukhopadhyay [3, (6.1) Theorem, p. 267]: *If f is n-convex on a measurable set A on which $f_{(n-1)}$ exists finitely, then $f_{(n),ap}$ (the approximative n^{th} Peano derivative of f) exists finitely almost everywhere on A.* Unfortunately, the proof given by P. S. Bullen and S. N. Mukhopadhyay is not correct [2]. Actually, the proof does not use the n-convexity of f, only the fact that $f_{(n-1)}$ is increasing on A. Under this weaker condition the statement may fail even for $n = 2$. In fact, it is easy to construct a continuously differentiable function f and a perfect set A of positive measure such that $f' = 0$ on A (In particular, f' is increasing on A.), but $f_{(2),ap}$ does not exist at any point of A. The question, whether the statement of [3, (6.1) Theorem] is true or not, remains open.

The following simple lemma is well-known (see [2]). For the sake of completeness, we give the proof.

Lemma 8. *Suppose that f is n-convex on A. If $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ are two sets of distinct elements of A such that $x_i \leq y_i$ $(i = 1, \ldots, n)$, then $[x_1, \ldots, x_n; f] \leq [y_1, \ldots, y_n; f]$.*

PROOF. Let $x_1, \ldots, x_{n-1} \in A$ be distinct elements, and define a function g by $g(x) = [x_1, \ldots, x_{n-1}, x; f]$. Clearly $[x_1, \ldots, x_n, x_{n+1}; f] = [x_n, x_{n+1}; g]$

[2]The error appears in the estimate (6.7), where it is assumed that the relations (6.3) and (6.4) hold *uniformly* in the intervals $(x_i, x_i + \delta_i)$. However, the uniformity of these relations is not proved in Theorem (3.1).

for every $x_n, x_{n+1} \in A \setminus \{x_1, \ldots, x_{n-1}\}$. Since f is n-convex, it follows that every difference quotient of g is nonnegative; that is, g is increasing. Therefore $[x_1, \ldots, x_n; f] \le [x_1, \ldots, x_{n-1}, y_n; f]$ whenever $x_n \le y_n$. Using the fact that the divided differences do not depend on the order of the elements x_i we find that $[x_1, \ldots, x_n; f]$ is an increasing function of each of the variables x_i, from which the statement of the lemma is obvious. □

PROOF OF PROPOSITION 2. Let $f : A \to \mathbb{R}$ be n-convex, and put $E = \{x \in A : f_{(n)}(x) = \infty\}$. We have to prove $\lambda(E) = 0$. Deleting a suitable countable subset, we may assume that every point of A is a bilateral point of accumulation of A. Let $u = \inf A$ and $v = \sup A$. It is enough to show that $\lambda(E \cap (a, b)) = 0$ for every $u < a < b < v$. Fix such an a and b. Since A has no isolated points, the sets $A \cap [u, a)$ and $A \cap (b, v]$ are infinite. We fix distinct elements $u_1, \ldots, u_n \in A \cap [u, a)$ and $v_1, \ldots, v_n \in A \cap (b, v]$, and put $[u_1, \ldots, u_n; f] = L$, $[v_1, \ldots, v_n; f] = M$. Then, by Lemma 8, we have $L \le [x_1, \ldots, x_n; f] \le M$ for every $x_1, \ldots, x_n \in E \cap [a, b]$.

For every interval $J \subset [a, b]$ we shall denote by $C(J)$ the convex hull of the set of numbers $[x_1, \ldots, x_n; f]$, where $x_1, \ldots, x_n$ are arbitrary distinct elements of $A \cap J$. Then $C(J) \subset [L, M]$ for every interval $J \subset [a, b]$. It follows from Lemma 8 that if J_1, J_2 are disjoint subintervals of $[a, b]$, then the intervals $C(J_1)$ and $C(J_2)$ are nonoverlapping.

Let $K > 0$ and let $\mathcal{U}_K$ be the family of all intervals $[c, d] \subset [a, b]$ for which there are elements $c = x_1 < \cdots < x_{n+1} = d$ such that $[x_1, \ldots, x_{n+1}; f] > K$. We show that $|C(J)| > K \cdot |J|$ for every $J \in \mathcal{U}_K$. Indeed, let $J = [c, d]$, and let $c = x_1 < \cdots < x_{n+1} = d$ be such that $[x_1, \ldots, x_{n+1}; f] > K$. Let $[x_1, \ldots, x_n; f] = \alpha$ and $[x_2, \ldots, x_{n+1}; f] = \beta$. Then by (2)

$$K < [x_1, \ldots, x_{n+1}; f] = \frac{\beta - \alpha}{(x_{n+1} - x_1)} = \frac{\beta - \alpha}{|J|}.$$

Since $\alpha, \beta \in C(J)$, we find $|C(J)| \ge \beta - \alpha > K \cdot |J|$, as we stated.

Next we show that $\mathcal{U}_K$ is a Vitali cover of the set $E \cap (a, b)$. Indeed, let $x \in E \cap (a, b)$ and $\delta > 0$ be given. Since $f_{(n)}(x) = \infty$, it follows from [3, (4.1) Lemma, p. 266] that

$$\lim_{\substack{x_{n+1} \in A \\ x_{n+1} \to x}} \ldots \lim_{\substack{x_1 \in A \\ x_1 \to x}} [x_1, \ldots, x_{n+1}; f] = \infty.$$

Since x is a bilateral point of accumulation of A, we may find distinct elements $x_1, \ldots, x_{n+1} \in A$ such that $[x_1, \ldots, x_{n+1}; f] > K$ and $x - \delta < c \le x \le d < x + \delta$, where $c = \min_{1 \le i \le n+1} x_i$ and $d = \max_{1 \le i \le n+1} x_i$. Then $x \in [c, d] \in \mathcal{U}_K$ and $[c, d] \subset (x - \delta, x + \delta)$, proving that $\mathcal{U}_K$ is indeed a Vitali cover of $E \cap (a, b)$. By

Vitali's covering theorem, there is a sequence J_i of pairwise disjoint elements of $\mathcal{U}_K$ covering a.e. point of $E \cap (a,b)$. Now the intervals $C(J_i)$ are pairwise nonoverlapping and are contained in $[L,M]$. Therefore

$$\lambda(E \cap (a,b)) \le \sum_i |J_i| < \sum_i |C(J_i)|/K \le (M-L)/K.$$

Since K was arbitrary, we conclude that $\lambda(E \cap (a,b)) = 0$. $\square$

3 Proofs of Theorems 3 and 4

The proofs of Theorems 3 and 4 are based on the construction of certain symmetric Cantor sets defined as follows. Let α and β be real numbers satisfying $0 < \beta < 1$ and

$$1 - \frac{\alpha}{1-\beta} > \alpha > 0. \tag{3}$$

We put $I_\emptyset = [0,1]$. Let $n \ge 0,\ i_1, \ldots, i_n \in \{0,1\}$, and suppose that the closed interval $I_{i_1 \ldots i_n}$ has been defined such that

$$|I_{i_1 \ldots i_n}| = \frac{1}{2^n}\left(1 - \sum_{k=0}^{n-1} \alpha \cdot \beta^k\right).$$

(The condition is satisfied for $n = 0$.) Let $J_{i_1 \ldots i_n}$ denote the open interval concentric with $I_{i_1 \ldots i_n}$ and of length $\alpha \cdot (\beta/2)^n$. Since

$$|I_{i_1 \ldots i_n}| > \frac{1}{2^n}\left(1 - \alpha \sum_{k=0}^{\infty} \beta^k\right) = \frac{1}{2^n}\left(1 - \frac{\alpha}{1-\beta}\right) > \frac{\alpha}{2^n} > \alpha \cdot (\beta/2)^n,$$

$J_{i_1 \ldots i_n}$ is a subinterval of $I_{i_1 \ldots i_n}$. Let $I_{i_1 \ldots i_n 0}$ and $I_{i_1 \ldots i_n 1}$ denote the two components of $I_{i_1 \ldots i_n} \setminus J_{i_1 \ldots i_n}$. Then

$$|I_{i_1 \ldots i_n 0}| = |I_{i_1 \ldots i_n 1}| = \frac{1}{2^{n+1}}\left(1 - \sum_{k=0}^{n} \alpha \cdot \beta^k\right).$$

In this way we have defined $I_{i_1 \ldots i_n}$ and $J_{i_1 \ldots i_n}$ for every finite $0-1$ sequence $i_1, \ldots, i_n$. The intervals $J_{i_1 \ldots i_n}$ are pairwise disjoint. Let $G_{\alpha,\beta}$ denote the union of the intervals $J_{i_1 \ldots i_n}$, and let $P_{\alpha,\beta} = [0,1] \setminus G_{\alpha,\beta}$. Then $G_{\alpha,\beta}$ is a dense, open subset of $[0,1]$, $P_{\alpha,\beta}$ is perfect, and

$$\lambda(P_{\alpha,\beta}) = 1 - \sum_{n=0}^{\infty} \alpha \cdot \beta^n = 1 - \frac{\alpha}{1-\beta} > \alpha > 0.$$

PROOF OF THEOREM 3. Let $\frac{1}{2} < \beta < 1$ be fixed and choose an α satisfying (3). Let $P = P_{\alpha,\beta}$. We define $g(x) = 0$ if $x \in P$, $1/(n+1)$ if x is the midpoint of $J_{i_1 \dots i_n}$, and define g linearly in the closures of both halves of $J_{i_1 \dots i_n}$. Then g is nonnegative and continuous on $[0,1]$. We put $\phi(x) = \int_0^x g(t)\,dt$ $(x \in [0,1])$; then ϕ is continuously differentiable and strictly increasing on $[0,1]$.

Let $x, y \in P,\ x < y$ be given. There are $n \geq 0$ and $i_1, \dots, i_n = 0, 1$ such that $x, y \in I_{i_1 \dots i_n},\ x \in I_{i_1 \dots i_n 0}$, and $y \in I_{i_1 \dots i_n 1}$. Then $y - x \leq |I_{i_1 \dots i_n}| \leq 1/2^n$ and

$$\phi(y) - \phi(x) = \int_x^y g(t)\,dt \geq \int_{J_{i_1 \dots i_n}} g(t)\,dt = \frac{1}{2} \cdot \alpha \cdot (\beta/2)^n \cdot \frac{1}{n+1}.$$

If $x \in P$ is fixed, $y \in P,\ y > x$ and $y \to x$, then $n \to \infty$ and hence

$$\frac{\phi(y) - \phi(x)}{(y-x)^2} \geq \frac{1}{2} \cdot \alpha \cdot (\beta/2)^n \cdot \frac{1}{n+1} \cdot 4^n \to \infty,$$

as $\beta > 1/2$. Suppose now that x is a density point of P, and let $x_k > x,\ x_k \to x,\ x_k \in G_{\alpha,\beta}$. If $x_k \in (a_k, b_k)$ where (a_k, b_k) is an interval contiguous to P, then $(b_k - x)/(a_k - x) \to 1$ as $k \to \infty$, and hence

$$\frac{\phi(x_k) - \phi(x)}{(x_k - x)^2} > \frac{\phi(a_k) - \phi(x)}{(b_k - x)^2} = \frac{\phi(a_k) - \phi(x)}{(a_k - x)^2} \cdot \left(\frac{a_k - x}{b_k - x}\right)^2 \to \infty.$$

Therefore $\lim_{y \to x+0} \frac{\phi(y)-\phi(x)}{(y-x)^2} = \infty$ for every density point x of P. Similarly we can prove that $\lim_{y \to x-0} \frac{\phi(y)-\phi(x)}{(y-x)^2} = -\infty$ for every density point x of P. Summing up, we constructed a function ϕ with the following properties.

(i) ϕ is continuously differentiable on $[0,1]$;

(ii) ϕ is strictly increasing on $[0,1]$;

(iii) $\phi'(x) = 0$ if $x \in P$;

(iv) ϕ is twice differentiable a.e. in $[0,1] \setminus P$;

(v) $\displaystyle\lim_{y \to x \pm 0} \frac{\phi(y) - \phi(x)}{(y-x)^2} = \pm\infty$ if x is a density point of P.

(As for (iv), note that ϕ is locally a quadratic polynomial at each point of $[0,1] \setminus P$ apart from the midpoints of the intervals contiguous to P.)

Let $\mathcal{P}$ denote the family of all subsets of $[0,1]$ that are homothetic with P. Then $\mathcal{P}$ is a Vitali cover of $[0,1]$, and thus there is a sequence (P_k) of pairwise disjoint elements of $\mathcal{P}$ that cover a.e. point of $[0,1]$ (see [5, Chapter IV, Theorem (3.1), p. 109]). Let $c_k = \min P_k$ and $d_k = \max P_k$ $(k = 1, 2, \dots)$. We define

$$f_k(x) = \begin{cases} 0 & \text{if } x \in [0, c_k] \\ \dfrac{d_k - c_k}{2^k}\phi\left(\dfrac{x - c_k}{d_k - c_k}\right) & \text{if } x \in [c_k, d_k] \\ \dfrac{d_k - c_k}{2^k}\phi(1) & \text{if } x \in [d_k, 1] \end{cases}$$

for every k. Then f_k is continuously differentiable on $[0,1]$. Also,

$$0 \le f_k(x) \le \frac{1}{2^k}\phi(1) \text{ and } 0 \le f_k'(x) \le \frac{1}{2^k}\max_{t\in[0,1]}\phi'(t) = \frac{1}{2^k}$$

for every $x \in [0,1]$. We put $f = \sum_{k=1}^{\infty} f_k$. Then f is continuously differentiable on $[0,1]$ and $f' = \sum_{k=1}^{\infty} f_k'$.

We claim that almost every $x \in [0,1]$ has the following properties.

(vi) There is a $k_1 = k_1(x)$ such that x is a density point of P_{k_1};

(vii) There is a $k_2 = k_2(x)$ such that $x \notin [c_k, d_k]$ for every $k > k_2$; and

(viii) If k is such that $x \notin P_k$, then f_k is twice differentiable at x.

Indeed, (vi) is obvious from the fact that $\cup_{k=1}^{\infty} P_k$ covers a.e. point of $[0,1]$. In order to prove (vii), note that each P_k is homothetic with P, and thus $\lambda(P_k)/(d_k - c_k) = \lambda(P)$ for every k. Therefore

$$\sum_{k=1}^{\infty}(d_k - c_k) = \sum_{k=1}^{\infty}\frac{d_k - c_k}{\lambda(P_k)}\lambda(P_k) = \frac{1}{\lambda(P)}\sum_{k=1}^{\infty}\lambda(P_k) = \frac{1}{\lambda(P)} < \infty,$$

and thus the set of points contained by infinitely many of the intervals $[c_k, d_k]$ is null. Finally, (viii) is immediate from (iv). We shall complete the proof by showing that if a point $x \in [0,1]$ satisfies (vi)-(viii), then $f_{(2)\pm}(x) = \pm\infty$. Let x be such a point, and put

$$\frac{f_k(y) - f_k(x) - f_k'(x)(y - x)}{(y - x)^2} = A_k(y)$$

for every $k = 1, 2, \dots$ and $y > x$. Let k_1 and k_2 be as in (vi) and (vii). If $k > k_2$, then $x \notin [c_k, d_k]$ and thus $f_k'(x) = 0$. Since f_k is increasing, we find that $A_k(y) \ge 0$ for every $k > k_2$ and $y > x$. Let $k_3 = \max(k_1, k_2)$. Then for every $y > x$ we have

$$\frac{f(y) - f(x) - f'(x)(y - x)}{(y - x)^2} = \sum_{k=1}^{\infty} A_k(y) \ge \sum_{k=1}^{k_3} A_k(y). \tag{4}$$

Since x is a density point of P_{k_1}, we have

$$\lim_{y\to x+0} A_{k_1}(y) = \lim_{y\to x+0} \frac{f_{k_1}(y) - f_{k_1}(x)}{(y-x)^2} = \infty.$$

If $k \neq k_1$, then $x \notin P_k$ and thus, by (viii), f_k is twice differentiable at x and $\lim_{y\to x+0} A_k(y) = f_k''(x)/2$. Therefore, by (4) we obtain

$$\liminf_{y\to x+0} \frac{f(y) - f(x) - f'(x)(y-x)}{(y-x)^2} \geq \sum_{\substack{k\leq k_3 \\ k\neq k_1}} \frac{f_k''(x)}{2} + \liminf_{y\to x+0} A_{k_1}(y) = \infty,$$

and thus $f_{(2)+}(x) = +\infty$. Similar argument gives $f_{(2)-}(x) = -\infty$. $\square$

PROOF OF THEOREM 4. It is enough to construct a twice continuously differentiable function f such that $f_{(3)} = \infty$ a.e. Indeed, if we take the $(n-3)^{\text{rd}}$ integral function of f, then we obtain a function satisfying the requirements of the theorem.

Let $\frac{1}{\sqrt{2}} < \beta < 1$ be fixed, and choose an α satisfying (3). Let $P = P_{\alpha,\beta}$. We define a function g as follows. First, if $x \in P$, then we put $g(x) = 0$. Next we define g on the intervals contiguous to P. Let $J_{i_1\ldots i_n} = (a,b)$ be such an interval, and let $c = a + (b-a)/3$ and $d = a + 2(b-a)/3$. We put $g(c) = 1/(n+1)$, $g(d) = -1/(n+1)$, and let g be linear on each of the closed intervals $[a,c]$, $[c,d]$, and $[d,b]$. In this way we defined g on $[0,1]$. It is easy to see that g is continuous.

Let $h(x) = \int_0^x g(t)\,dt$ $(x \in [0,1])$. Then h is continuously differentiable on $[0,1]$. Since the integral of g over each interval contiguous to P equals zero, it follows that $h(x) = 0$ if $x \in P$. It is easy to see that $h(x) > 0$ if $x \in [0,1] \setminus P$. It is also easy to check that

$$\int_{J_{i_1\ldots i_n}} h(x)\,dx = \frac{1}{9(n+1)}\,|J_{i_1\ldots i_n}|^2 = \frac{\alpha^2}{9(n+1)} \cdot (\beta/2)^{2n}.$$

Now we put $\phi(x) = \int_0^x h(t)\,dt$ $(x \in [0,1])$. Then ϕ is twice continuously differentiable and $\phi'' = g$.

Let $x, y \in P$, $x < y$ be given. There are $n \geq 0$ and $i_1, \ldots, i_n = 0, 1$ such that $x, y \in I_{i_1\ldots i_n}$, $x \in I_{i_1\ldots i_n 0}$, and $y \in I_{i_1\ldots i_n 1}$. Then $y - x \leq |I_{i_1\ldots i_n}| \leq 1/2^n$ and

$$\phi(y) - \phi(x) = \int_x^y h(t)\,dt \geq \int_{J_{i_1\ldots i_n}} h(t)\,dt = \frac{\alpha^2}{9(n+1)} \cdot (\beta/2)^{2n}.$$

Therefore

$$\frac{\phi(y) - \phi(x)}{(y-x)^3} \geq \frac{\alpha^2}{9(n+1)} \cdot (\beta/2)^{2n} \cdot 8^n \to \infty$$

if $y \to x+0$ and $y \in P$. The same is true if $y \to x-0$ and $y \in P$. If x is a density point of P, then in a similar way as in the proof of Theorem 3 we conclude that $\lim_{y \to x} \frac{\phi(y)-\phi(x)}{(y-x)^3} = \infty$.

Summing up, we constructed a function ϕ with the following properties.

(ix) ϕ is twice continuously differentiable on $[0,1]$;

(x) ϕ is strictly increasing on $[0,1]$;

(xi) $\phi'(x) = \phi''(x) = 0$ if $x \in P$;

(xii) ϕ is three times differentiable a.e. in $[0,1] \setminus P$;

(xiii) $\lim\limits_{y \to x} \dfrac{\phi(y)-\phi(x)}{(y-x)^3} = \infty$ if x is a density point of P.

Now we repeat the argument of the proof of Theorem 3. Let P_k, c_k, d_k be as in the proof of Theorem 3. We define

$$f_k(x) = \begin{cases} 0 & \text{if } x \in [0, c_k], \\ \dfrac{(d_k-c_k)^2}{2^k} \phi\left(\frac{x-c_k}{d_k-c_k}\right) & \text{if } x \in [c_k, d_k], \\ \dfrac{(d_k-c_k)^2}{2^k} \phi(1) & \text{if } x \in [d_k, 1] \end{cases}$$

for every k. Then f_k is continuously differentiable on $[0,1]$, and

$$0 \le f_k(x) \le \frac{1}{2^k}\phi(1), \quad 0 \le f_k'(x) \le \frac{1}{2^k} \quad \text{and} \quad 0 \le f_k''(x) \le \frac{1}{2^k}$$

for every $x \in [0,1]$. We put $f = \sum_{k=1}^{\infty} f_k$. Then f is twice continuously differentiable on $[0,1]$, $f' = \sum_{k=1}^{\infty} f_k'$ and $f'' = \sum_{k=1}^{\infty} f_k''$.

Suppose that a point $x \in [0,1]$ satisfies (vi) and (vii) (See the proof of Theorem 3.) as well as the following property.

(xiv) If k is such that $x \notin P_k$, then f_k is three times differentiable at x.

We shall prove that in this case $f_{(3)}(x) = \infty$. Since (vi), (vii) and (xiv) hold for a.e. point, this will complete the proof. Let x be a point satisfying (vi), (vii) and (xiv), and for every $k = 1, 2, \ldots$ and $y \ne x$ put

$$\frac{f_k(y) - f_k(x) - f_k'(x)(y-x) - \frac{f_k''(x)}{2}(y-x)^2}{(y-x)^3} = B_k(y).$$

Let k_1 and k_2 be as in (vi) and (vii). If $k > k_2$, then $x \notin [c_k, d_k]$ and thus $f_k'(x) = f_k''(x) = 0$. Since f_k is increasing, we find that $B_k(y) \ge 0$ for every $k > k_2$ and $y \ne x$. Let $k_3 = \max(k_1, k_2)$. Then for every $y \ne x$ we have

$$\frac{f(y) - f(x) - f'(x)(y-x) - \frac{f''(x)}{2}(y-x)^2}{(y-x)^3} = \sum_{k=1}^{\infty} B_k(y) \ge \sum_{k=1}^{k_3} B_k(y). \tag{5}$$

Since x is a density point of P_{k_1}, we have

$$\lim_{y \to x} B_{k_1}(y) = \lim_{y \to x} \frac{f_{k_1}(y) - f_{k_1}(x)}{(y-x)^3} = \infty.$$

If $k \ne k_1$, then $x \notin P_k$ and thus, by (xiv), f_k is three times differentiable at x and $\lim_{y \to x} B_k(y) = f_k'''(x)/6$. Therefore, by (5) we obtain

$$\begin{aligned}\liminf_{y \to x} &\frac{f(y) - f(x) - f'(x)(y-x) - \frac{f''(x)}{2}(y-x)^2}{(y-x)^3} \\ &\ge \sum_{\substack{k \le k_3 \\ k \ne k_1}} \frac{f_k'''(x)}{6} + \liminf_{y \to x} B_{k_1}(y) = \infty\end{aligned}$$

and thus $f_{(3)}(x) = \infty$. □

Remark 9. The function ϕ constructed in the proof of Theorem 4 satisfies (ix)-(xiii), and thus $\phi_{(3)}(x) = \infty$ holds at each density point x of P. Let $d(x)$ denote the distance of x from P. It is clear that whenever a function ψ satisfies $\phi \le \psi \le \phi + d^3$ on $[0,1]$, then $\psi_{(3)}(x) = \infty$ at each density point x of P. As we mentioned in the introduction, there is a continuous function α on $[0,1]$ such that $\alpha_{(3)}(x) = \infty$ holds at the points of a residual set. It is easy to see that sticking together suitable affine copies of the graph of α we can construct a function ψ satisfying $\phi \le \psi \le \phi + d^3$. This function has the property that $E_3(\psi)$ is residual (because it is residual in each interval contiguous to P), and of positive measure (because it contains the density points of P).

4 Proof of Theorem 6

In the proof of Theorem 6 we shall apply some ideas of [1, Section 3].

Lemma 10. *Let f be locally bounded and right continuous in (c,d), and let μ be a finite Borel measure on $[0,1]$ such that $\mu([1-\delta,1]) > 0$ for every $\delta > 0$. Suppose that A is a dense subset of (c,d), and for every $x \in A$ there is a real number $c(x)$ such that $\int_0^1 [f(x+th) - f(x) - c(x)th]\, d\mu(t) \ge 0$ for every $h \in (c-x, d-x)$. Then f is convex in (c,d).*

PROOF. We may assume that $d - c \leq 1$. Let C denote the set of points of continuity of f. Since f is right continuous, $(c, d)\setminus C$ is countable. In particular, C is dense in (c, d). It is enough to show that f is convex on C; since f is right continuous, this will imply that f is convex in (c, d).

Suppose f is not convex on C. Then there are points $u, v, w \in C$, $u < v < w$ such that $f(v)$ lies above the chord joining $(u, f(u))$ and $(w, f(w))$. Subtracting a linear function from f we may assume that $f(u) = f(w) = 0$ and $f(v) > 0$. Let $m = \sup\{f(x) : x \in [u, w]\}$. Then $m \geq f(v) > 0$. Since f is locally bounded, we have $m < \infty$. Let $\delta > 0$ be such that $f(x) < m/4$ for every $x \in [u, u + \delta] \cup [w - \delta, w]$. Let $0 < \eta < m/4$ be fixed. There is a point $x_0 \in (u, w)$ such that $f(x_0) > m - \eta$. Since A is dense in (u, w) and f is right continuous at x_0, it follows that there is a point $x \in (u, w) \cap A$ such that $f(x) > m - \eta$. We show that choosing η small enough, both $c(x) \geq 0$ and $c(x) \leq 0$ provide a contradiction.

Suppose $c(x) \geq 0$, and let $h = w - x$. Then $0 < h < 1$, and

$$\begin{aligned} 0 &\leq \int_0^1 [f(x + th) - f(x) - c(x)th]\, d\mu(t) \\ &\leq \int_0^1 [f(x + th) - f(x)]\, d\mu(t) = \int_0^{1-\delta} + \int_{1-\delta}^1 = I_1 + I_2. \end{aligned}$$

If $t \in [0, 1 - \delta]$, then $f(x + th) - f(x) \leq m - (m - \eta) = \eta$ and therefore $I_1 \leq \eta \cdot \mu([0, 1])$. If $t \in [1 - \delta, 1]$, then $x + th \in [w - \delta, w]$ and hence $f(x + th) < m/4$ and

$$I_2 < \left[\frac{m}{4} - (m - \eta)\right] \mu([1 - \delta, 1]) < -\frac{m}{2}\mu([1 - \delta, 1]).$$

Thus if $\eta < \frac{m}{2} \cdot \frac{\mu([1-\delta,1])}{\mu([0,1])}$, then

$$I_1 + I_2 < \eta \cdot \mu([0, 1]) - (m/2)\mu([1 - \delta, 1]) < 0.$$

If $c(x) \leq 0$, then take $h = u - x < 0$ to obtain a similar contradiction. □

Lemma 11. *Suppose f is k times differentiable in a neighborhood of x, and the Peano derivative $f_{(k+1)}(x)$ exists. If the right hand derivative $\left(f^{(k)}\right)'_+$ of $f^{(k)}$ exists at x, then $\left(f^{(k)}\right)'_+(x) = f_{(k+1)}(x)$.*

PROOF. By k applications of L'Hôpital's rule we obtain

$$\begin{aligned} f_{(k+1)}(x) &= \lim_{t \to +0} \frac{(k + 1)!}{t^{k+1}} \left(f(x + t) - \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!} t^i \right) \\ &= \lim_{t \to +0} \frac{f^{(k)}(x + t) - f^{(k)}(x)}{t} = \left(f^{(k)}\right)'_+(x). \end{aligned}$$ □

PROOF OF THEOREM 6. The proof is by induction on n. Let $n = 1$, and suppose that the set $H = \{x \in [a,b] : \underline{f}_{(1)}(x) > 0\}$ is of second category. Let $H_k = \{x \in H : (f(y) - f(x))/(y - x) > 0$ for every $0 < |y - x| < 1/k\}$. Since $H = \cup_{k=1}^{\infty} H_k$, there is an index k such that H_k is dense in a subinterval $I \subset [a,b]$ with $|I| < 1/k$. It is clear that f is increasing (= 1-convex) in I.

Let $n \geq 1$ and suppose that the statement is true for n. Let $f : [a,b] \to \mathbb{R}$ be such that the set $H = \{x \in [a,b] : \underline{f}_{(n+1)}(x) > 0\}$ is of second category. Since $f_{(n)}(x)$ exists and is finite for every $x \in H$, there exists a positive integer K such that $f_{(n)}(x) > -K$ at the points of a second category subset of H. Adding $K \cdot x^n/n!$ to f (which does not affect the value of $\underline{f}_{(n+1)}$ or the $(n+1)$-convexity of f in any interval), we may assume that $f_{(n)}(x) > 0$ and $\underline{f}_{(n+1)}(x) > 0$ at each point of H. Let I be a subinterval of $[a,b]$ such that H is of the second category in each subinterval of I (see [4, §10, V. p. 85]). By the induction hypothesis, f is n-convex in a subinterval $J \subset I$. Let H_k denote the set of points $x \in H \cap J$ such that

$$\frac{(n+1)!}{h^{n+1}} \left(f(x+h) - \sum_{i=0}^{n} \frac{f_{(i)}(x)}{i!} h^i \right) > 0 \tag{6}$$

for every $0 < |h| < 1/k$. Since $H \cap J = \cup_{k=1}^{\infty} H_k$, there is a k such that H_k is of the second category. Let (c,d) be a subinterval of J such that $d - c < 1/k$ and H_k is dense in (c,d). Putting $A = (c,d) \cap H_k$ we find that

(i) f is n-convex in (c,d), and

(ii) at each point of the dense subset $A \subset (c,d)$ the Peano derivative $f_{(n)}(x)$ exists finitely, and (6) holds for every $h \neq 0$ such that $x + h \in (c,d)$.

We shall prove that f is $(n+1)$-convex in (c,d). First suppose $n = 1$. Then f is increasing by (i). If $x \in A$, then $f(x+h) - f(x) - f'(x)h > 0$ for every $h \in (c-x, d-x)$, $h \neq 0$. As we saw in the proof of Proposition 1, this implies that f is convex on A. Since f is increasing, we conclude that f is convex in (c,d).

Next suppose $n > 1$. Since f is n-convex in (c,d), f is $n-2$ times differentiable and $f^{(n-2)}$ is convex in (c,d). Then the right hand side derivative $\left(f^{(n-2)}\right)'_+ = g$ exists everywhere, and is increasing and right continuous in (c,d). It follows from Lemma 11 that $f_{(n-1)}(x) = g(x)$ at each point of A. Since $f^{(n-2)}$ is convex, it is absolutely continuous, and thus $f^{(n-2)}$ equals the integral function of g. We shall prove that g is convex in (c,d). This will complete the proof. Indeed, if g is continuous, then $f^{(n-2)}$ is a primitive of g. Therefore f is $n-1$ times differentiable and $f^{(n-1)} = g$ is convex in (c,d); that is, f is $(n+1)$-convex in (c,d).

It is enough to show that g satisfies the conditions of Lemma 10. Since g is increasing, it is locally bounded in (c,d). If $x \in A$, then by (ii), we have

$$\frac{(n+1)!}{h^{n+1}}\left(f(x+h)-\sum_{i=0}^{n-2}\frac{f^{(i)}(x)}{i!}h^i-\frac{g(x)}{(n-1)!}h^{n-1}-\frac{f_{(n)}(x)}{n!}h^n\right)>0 \quad (7)$$

for every $h \neq 0$ such that $x+h \in (c,d)$. It is well-known (and easy to prove by induction on k) that if $f^{(k-1)}$ is absolutely continuous on $[x,x+h]$, then

$$f(x+h)-\sum_{i=0}^{k-1}\frac{f^{(i)}(x)}{i!}h^i=\frac{h^k}{(k-1)!}\int_0^1 f^{(k)}(x+th)(1-t)^{k-1}dt.$$

Therefore, by (7), we have

$$\begin{aligned}
0 <& \frac{(n+1)!}{h^{n+1}}\left(\frac{h^{n-1}}{(n-2)!}\int_0^1 g(x+th)(1-t)^{n-2}dt-\frac{g(x)}{(n-1)!}h^{n-1}-\frac{f_{(n)}(x)}{n!}h^n\right)\\
=&\frac{(n+1)n(n-1)}{h^2}\left(\int_0^1 g(x+th)(1-t)^{n-2}dt-\frac{g(x)}{(n-1)}-\frac{f_{(n)}(x)}{n(n-1)}h\right)\\
=&\frac{(n+1)n(n-1)}{h^2}\int_0^1\left[g(x+th)-g(x)-f_{(n)}(x)th\right](1-t)^{n-2}dt
\end{aligned}$$

for every $h \in (c-x,d-x),\ h \neq 0$. Putting $\mu(B)=\int_B(1-t)^{n-2}dt$, we can see that the conditions of Lemma 10 are satisfied with $f=g$. Therefore g is convex on (c,d), which completes the proof. $\square$

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