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ON STRONG PRODUCT MEASURABILITY

Abstract

We consider a special condition implying the strong product measurability of bounded functions with values in a Banach space which all vertical sections are derivatives.

Let \mathbb{R} , \mathbb{Z} , and \mathbb{N} denote the real line, the set of integers, and the set of positive integers, respectively. Let (X, \mathcal{M}) be a measurable space, let (Y, \mathcal{N}, μ) be a measure space, and let Z be a Banach space. We define μ_i as the inner measure generated by μ , i.e., $\mu_i(C) = \sup\{\mu(D) : D \in \mathcal{N}, D \subset C\}$ for each $C \subset Y$. Let $\mathcal{I} \subset \mathcal{M}$ be a proper σ -ideal of subsets of X . We assume that condition (c-c) is fulfilled in $(X, \mathcal{M}, \mathcal{I})$, i.e., that each family of pairwise disjoint elements of $\mathcal{M} \setminus \mathcal{I}$ is at most countable.

We assume that we have a net structure \mathcal{J} in (Y, \mathcal{N}, μ) (see [1]). Recall that a *net* in Y is an at most countable collection of pairwise disjoint measurable sets of positive measure which cover Y . The individual sets in the collection are called *cells*. The family $\mathcal{J} = \bigcup_{n=1}^{\infty} \mathcal{J}_n$, where each \mathcal{J}_n is a net, is called a *net structure*. For each $y \in Y$, there is a sequence (J_n) such that $y \in J_n \in \mathcal{J}_n$ for each n . In this case we write $J_n \Rightarrow y$. Several examples of net structures can be found, e.g., in [5] or [6].

Example 1. For each $n \in \mathbb{N}$, the family \mathcal{J}_n of all intervals of the form

$$\left[\frac{i_1 - 1}{2^n}, \frac{i_1}{2^n} \right) \times \cdots \times \left[\frac{i_m - 1}{2^n}, \frac{i_m}{2^n} \right),$$

where $i_1, \dots, i_m \in \mathbb{Z}$, is a net in \mathbb{R}^m .

A function $g: Y \rightarrow Z$ is called a *derivative* (with respect to the net \mathcal{J}) if g is Bochner-integrable over each $J \in \mathcal{J}$ and

$$\lim_{n \rightarrow \infty} \frac{\int_{J_n} g}{\mu(J_n)} = g(y)$$

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whenever $J_n \Rightarrow y, y \in Y$.

Let $h: X \rightarrow Z$. We say that h is *measurable* if $h^{-1}(U) \in \mathcal{M}$ for every open set $U \subset Z$. We say that h is *nearly simple* if there exists a sequence of pairwise disjoint sets $A_1, A_2, \dots \in \mathcal{M}$ and elements $\alpha_1, \alpha_2, \dots \in Z$, such that $h = \alpha_n$ on A_n for each n , and $X = \bigcup_{n=1}^{\infty} A_n$. We say that h is *strongly measurable* if there exists a sequence of nearly simple functions (h_n) and a set $A \in \mathcal{I}$ such that $h_n \rightarrow h$ on $X \setminus A$. We will see that if h is a strongly measurable function, then there exists a sequence of nearly simple functions uniformly convergent to h outside of some $A \in \mathcal{I}$ (Proposition 5).

First we will investigate the properties of strongly measurable functions. Since each nearly simple function is measurable and the pointwise limit of measurable functions is measurable, each strongly measurable function is measurable as well.

Lemma 1. *If $X = \bigcup_{n \in \mathbb{N}} X_n$, each X_n is measurable, and $h|X_n$ is nearly simple (strongly measurable) for each n , then h is nearly simple (respectively strongly measurable).*

PROOF. For each n let $A_{n1}, A_{n2}, \dots \in \mathcal{M}$ and $\alpha_{n1}, \alpha_{n2}, \dots \in Z$ be such that $h = \alpha_{nk}$ on A_{nk} for each k , and $X_n = \bigcup_{k=1}^{\infty} A_{nk}$. Arrange all sets of the form $A_{nk} \setminus \bigcup_{i < n} X_i$ in a sequence, (B_m) , and let $\beta_m \in Z$ be such that $h = \beta_m$ on B_m . Clearly each B_m is measurable and $X = \bigcup_{m=1}^{\infty} B_m$.

Now suppose that each $h|X_n$ is strongly measurable. For each n choose a sequence of nearly simple functions, (h_{nk}) , and a set $A_n \in \mathcal{I}$ such that $h_{nk} \rightarrow h$ on $X_n \setminus A_n$. For each k define $g_k = h_{nk}$ on $X_n \setminus \bigcup_{i < n} X_i$. Then each g_k is nearly simple and evidently $g_k \rightarrow h$ on $X \setminus \bigcup_{n=1}^{\infty} A_n$. \square

Lemma 2. *Let $h, h_1, h_2, \dots: X \rightarrow Z$. If each of the sets $h_1(X), h_2(X), \dots$ is a separable subspace of Z and $h_n \rightarrow h$, then $h(X)$ is a separable subspace of Z .*

PROOF. Since $Z_0 = \text{cl}(\bigcup_{n=1}^{\infty} h_n(X))$ is a separable subspace of Z , it suffices to prove that $h(X) \subset Z_0$. Let $z \in h(X)$. There exists an $x \in X$ such that $h(x) = z$. Since $h_n(x) \rightarrow z$, we get $z \in \text{cl}\{h_n(x) : n \in \mathbb{N}\} \subset Z_0$. \square

Clearly if h is a nearly simple function, then $h(X)$ is a separable subspace of Z . So by Lemma 2, we obtain the following corollary.

Corollary 3. *Let $h: X \rightarrow Z$. If h is a strongly measurable function, then there exists a set $A \in \mathcal{I}$ such that $h(X \setminus A)$ is a separable subspace of Z .*

PROOF. Let (h_n) be a sequence of nearly simple functions pointwise convergent to h on $X \setminus A$, where $A \in \mathcal{I}$. Then by Lemma 2, $h(X \setminus A)$ is a separable subspace of Z . \square

Lemma 4. *Let $h: X \rightarrow Z$. If h is a measurable function and $h(X)$ is a separable subspace of Z , then h is strongly measurable. Moreover there is a sequence of nearly simple functions uniformly convergent to h on X .*

PROOF. Let $\varepsilon > 0$. Let $\{p_n : n < N\}$ be an at most countable dense subset of $h(X)$. For each $n < N$ define

$$A_n = h^{-1}(B(p_n, \varepsilon)) \setminus \bigcup_{k < n} h^{-1}(B(p_k, \varepsilon)),$$

where $B(p_n, \varepsilon) = \{z \in Z : \|z - p_n\| < \varepsilon\}$. Define $\varphi(x) = p_n$ if $x \in A_n$, $n < N$. Clearly φ is a nearly simple function and $\|\varphi(x) - h(x)\| < \varepsilon$ for each $x \in X$. □

Proposition 5. *If h is a pointwise limit of a sequence of strongly measurable functions, then h is strongly measurable. Moreover there exists a sequence of nearly simple functions (φ_n) and a set $A \in \mathcal{I}$ such that (φ_n) is uniformly convergent to h on $X \setminus A$.*

PROOF. Let (h_n) be a sequence of strongly measurable functions pointwise convergent to h . For each n let $A_n \in \mathcal{I}$ be such that $h_n(X \setminus A_n)$ is a separable subspace of Z . (See Corollary 3.) Then $A = \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{I}$ and by Lemma 2, $h(X \setminus A)$ is a separable subspace of Z . Now our assertion follows from Lemma 4. □

Example 2. Assume Z is nonseparable and let $h: (Z, 2^Z, \{\emptyset\}) \rightarrow Z$ be an arbitrary function. Then evidently h is measurable, but by Corollary 3 and Lemma 4, h is strongly measurable if and only if $h(Z)$ is separable.

We will use the following generalization of Davies' Lemma [3].

Proposition 6. *Assume that a function $h: X \rightarrow Z$ fulfills the following condition:*

$$\text{for each } \varepsilon > 0 \text{ and each } A \in \mathcal{M} \setminus \mathcal{I}, \text{ there is a } B \in \mathcal{M} \setminus \mathcal{I} \text{ with } \tag{1} \\ B \subset A \text{ such that } \text{osc}_B h < \varepsilon,$$

where $\text{osc}_B h = \sup\{\|h(x) - h(t)\| : x, t \in B\}$. Then h is strongly measurable.

PROOF. Fix an $\varepsilon > 0$. It suffices to show that there are a set $K \in \mathcal{I}$ and a nearly simple function φ such that $\|\varphi(x) - h(x)\| < \varepsilon$ for each $x \in X \setminus A$. Let Ω denote the first uncountable ordinal. First we will define by transfinite induction a sequence $\{A_\alpha : \alpha < \Omega\}$, composed of pairwise disjoint elements of \mathcal{M} , such that for each $\alpha < \Omega$, if $A_\alpha \in \mathcal{I}$, then $A_\alpha = \emptyset$.

Let $\alpha < \Omega$ and suppose we have already defined the set A_β for each $\beta < \alpha$. Set $K = X \setminus \bigcup_{\beta < \alpha} A_\beta$. If $K \in \mathcal{I}$, then we define $A_\alpha = \emptyset$. In the opposite

case observe that $K \in \mathcal{M}$. (Recall that $\alpha < \Omega$.) Use condition (1) to find a set $A_\alpha \in \mathcal{M} \setminus \mathcal{I}$ such that $A_\alpha \subset K$ and $\text{osc}_{A_\alpha} f < \varepsilon$.

By (ccc) there is a $\xi < \Omega$ such that $A_\alpha = \emptyset$ for each $\alpha \geq \xi$. We may assume that $A_\alpha \neq \emptyset$ for each $\alpha < \xi$. Define $K = X \setminus \bigcup_{\alpha < \xi} A_\alpha$. For each $\alpha < \xi$ choose an $x_\alpha \in A_\alpha$. Define $\varphi(x) = f(x_\alpha)$ if $x \in A_\alpha$ for some $\alpha < \xi$, and $\varphi(x) = f(x_0)$ if $x \in K$. Since $A_\xi = \emptyset$, we have $K \in \mathcal{I}$. So, φ is a nearly simple function. (We use the fact that $\xi < \Omega$.) Clearly $\|\varphi(x) - h(x)\| < \varepsilon$ for each $x \in X \setminus A$. \square

Let $f: X \times Y \rightarrow Z$. In the space $X \times Y$, we consider the product σ -field $\mathcal{M} \times \mathcal{N}$ (i.e., the σ -field generated by the family of all sets of the form $M \times N$, where $M \in \mathcal{M}$ and $N \in \mathcal{N}$), and the σ -ideal $\{A \times B : A \in \mathcal{I}, B \subset Y\}$. So, f is nearly simple if there are elements $\alpha_1, \alpha_2, \dots \in Z$ and a sequence of pairwise disjoint sets $A_1, A_2, \dots \in \mathcal{M} \times \mathcal{N}$, such that $f = \alpha_n$ on A_n for each n , and $X \times Y = \bigcup_{n=1}^{\infty} A_n$; f is strongly measurable if there exists a sequence of nearly simple functions (f_n) and a set $A \in \mathcal{I}$ such that $f_n \rightarrow f$ on $(X \times Y) \setminus (A \times Y)$. For each $x \in X$, we define the section $f_x: Y \rightarrow Z$ by $f_x(y) = f(x, y)$. Analogously, for each $y \in Y$, the section $f^y: X \rightarrow Z$ is defined by $f^y(x) = f(x, y)$.

Theorem 7. *Assume that a bounded function $f: X \times Y \rightarrow Z$, which all sections f_x are derivatives, satisfies the following condition:*

for every $\varepsilon > 0$ and every set $A \in \mathcal{M} \setminus \mathcal{I}$, there exist a $B \in \mathcal{M} \setminus \mathcal{I}$ with $B \subset A$ and a $C \subset Y$ such that $\mu_i(Y \setminus C) < \varepsilon$ and $\text{osc}_B f^y < \varepsilon$ for every $y \in C$. (2)

Then f is strongly measurable.

PROOF. Let M be such that $\|f(x, y)\| < M/4$ for all $(x, y) \in X \times Y$. Fix a $J \in \mathcal{J}$, and put $h_J(x) = \int_J f_x$. We will prove that h_J is strongly measurable. We will use Proposition 6. Fix an $\varepsilon > 0$ and an $A \in \mathcal{M} \setminus \mathcal{I}$. By (2), there are a $B \in \mathcal{M} \setminus \mathcal{I}$ with $B \subset A$ and a $C \subset Y$ such that

$$\mu_i(Y \setminus C) < \varepsilon/M \quad (3)$$

and

$$\text{osc}_B f^y < \frac{\varepsilon}{4\mu(J)} \quad \text{for every } y \in C. \quad (4)$$

Fix points $u, v \in B$. Define

$$D = \left\{ y \in J : \|f(u, y) - f(v, y)\| < \frac{\varepsilon}{4\mu(J)} \right\}.$$

Since the sections f_u and f_v are derivatives, the set D is \mathcal{N} -measurable. By (4), we get $D \supset C$. So by (3), $\mu(J \setminus D) < \varepsilon/M$. Consequently,

$$\begin{aligned} \|h_J(u) - h_J(v)\| &= \left\| \int_J f_u - \int_J f_v \right\| \leq \int_D \|f_u - f_v\| + \int_{J \setminus D} \|f_u - f_v\| \\ &\leq \frac{\varepsilon}{4\mu(J)} \cdot \mu(J) + \frac{\varepsilon}{M} \cdot \frac{M}{2} = \frac{3\varepsilon}{4} \end{aligned}$$

and $\text{osc}_B h_J < \varepsilon$. By Proposition 6, we conclude that h_J is strongly measurable.

Now we will repeat the Lipiński's argument [4] to show the strong measurability of f . For each $n \in \mathbb{N}$ define

$$f_n(x, y) = h_J(x)/\mu(J)$$

if $y \in J \in \mathcal{J}_n$. By Lemma 1, each function f_n is strongly measurable. We will show that $f_n \rightarrow f$, whence by Proposition 5, we will obtain the assertion of the theorem.

Fix a point $(x, y) \in X \times Y$. Let $J_n \ni y$. Using the fact that f_x is a derivative, we obtain

$$f(x, y) = f_x(y) = \lim_{n \rightarrow \infty} \int_{J_n} f_x / \mu(J_n) = \lim_{n \rightarrow \infty} h_{J_n}(x) / \mu(J_n) = \lim_{n \rightarrow \infty} f_n(x, y).$$

This completes the proof. □

Remark 1. By Remark 2 of [2], if each set $A \subset \mathbb{R}$ of cardinality less than that of continuum has Lebesgue measure zero, then there is a Lebesgue measurable function $f: \mathbb{R}^2 \rightarrow [0, 1]$ which all sections f_x and f^y are derivatives and which does not satisfy condition (2). So, condition (2) is not necessary for the measurability of a bounded function f which all sections f_x and f^y are derivatives.

The following theorem is proved in [2].

Theorem 8. *Assume that a locally bounded function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ which all sections $f_x, x \in \mathbb{R}$, are derivatives, satisfies the following condition:*

$$\begin{aligned} &\text{for every } \varepsilon > 0 \text{ and every open interval } I, \text{ there exist an } x \in I \\ &\text{and a linear set } D \text{ such that } \mu_i(\mathbb{R} \setminus D) < \varepsilon \text{ and for every } y \in D \\ &\text{there is an open interval } I_y \ni x \text{ such that } \text{osc}_{I_y} f^y < \varepsilon. \end{aligned} \tag{5}$$

Then f has the Baire property.

Observe that in general condition (5) does not imply condition (2).

Example 3. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f(x, y) = xy$. Then f is continuous, so f fulfills condition (5). Notice that if $a < b$ and $|f^y(a) - f^y(b)| = |y(b-a)| < 1$, then $|y| < (b-a)^{-1}$. So, for each set $B \subset \mathbb{R}$ with cardinality greater than 1 the set $\{y \in \mathbb{R} : \text{osc}_B f^y < 1\}$ is bounded. Consequently, condition (2) (where \mathcal{M} denotes the σ -field of all sets with the Baire property and \mathcal{I} is the σ -ideal of meager sets) fails.

Despite of that we will show that Theorem 8 is actually a particular case of Theorem 7.

Theorem 9. *Assume that f satisfies the assumptions of Theorem 8. Then for each $m \in \mathbb{N}$, the restricted function $f|_{[-m, m]^2}$ fulfills condition (2). So, by Theorem 7 and Lemma 1, f has the Baire property.*

PROOF. Fix an $\varepsilon > 0$ and an $A \in \mathcal{M} \setminus \mathcal{I}$. There is an open interval I such that $I \setminus A \in \mathcal{I}$. Use condition (5) to find an $x \in I$ and a $D \subset [-m, m]$ such that $\mu_i([-m, m] \setminus D) < \varepsilon$ and for every $y \in D$ there is an open interval $I_y \ni x$ such that $\text{osc}_{I_y} f^y < \varepsilon$. For each $n \in \mathbb{N}$ define

$$D_n = \{y \in D : I_y \supset (x - n^{-1}, x + n^{-1})\}.$$

Then $D = \bigcup_{n \in \mathbb{N}} D_n$. Since (D_n) is an ascending sequence of sets,

$$\mu_i([-m, m] \setminus D_n) < \varepsilon$$

for some $n \in \mathbb{N}$. Put $B = (x - n^{-1}, x + n^{-1}) \cap I \cap A$ and $C = D_n$. Evidently $B \subset A$, $B \in \mathcal{M} \setminus \mathcal{I}$, and $\mu_i([-m, m] \setminus C) < \varepsilon$. Moreover

$$\text{osc}_B f^y \leq \text{osc}_{I_y} f^y < \varepsilon$$

for every $y \in C$. □

Example 4. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Then function f has the Baire property. It fulfills condition (2) if we define \mathcal{M} and \mathcal{I} as in Example 3. Meanwhile one can readily verify that for each $y \in \mathbb{R}$ and each open interval J we have $\text{osc}_J f^y = 1$. So, condition (5) fails.

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