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ON THE MEASURABILITY OF FUNCTIONS $f : \mathbb{R}^2 \to \mathbb{R}$ HAVING PAWLAK'S PROPERTY IN ONE VARIABLE

Abstract

In this article we present a condition on the sections f^y of a function $f : \mathbb{R}^2 \to \mathbb{R}$ having Lebesgue measurable sections f_x and quasicontinuous sections f^y which implies the measurability of f. This condition is more general than the Baire$_1^{**}$ property introduced by R. Pawlak in [7]. Some examples of quasicontinuous functions satisfying this condition and discontinuous on the sets of positive measure are given.

Let $\mathbb{R}$ be the set of all reals. In the lecture [7] R. J. Pawlak introduced the following definition:.

Denoting by $D(g)$ the set of all discontinuities of a function $g : \mathbb{R} \to \mathbb{R}$ we say that g has the property $\mathcal{B}_1^{**}$ if the restricted function $g \upharpoonright D(g)$ is continuous.

The family $\mathcal{B}_1^{**}$ is a very interesting subclass of the class $\mathcal{B}_1$ of all functions of Baire class one. It contains also some functions g for which $D(g)$ is of positive (Lebesgue) measure, for example the characteristic functions of closed nowhere dense sets of positive measure.

Let $A \subset \mathbb{R}^2$ be a Sierpiński nonmeasurable set such that for every straight line $l \subset \mathbb{R}^2$, $\text{card}(l \cap A) \leq 2$ ([9]). Then the characteristic function f of the set A is nonmeasurable (in the sense of Lebesgue) and all sections $f_x(t) = f(x,t)$ and $f^y(u) = f(u,y)$, $t, u, x, y \in \mathbb{R}$, have Pawlak's property $\mathcal{B}_1^{**}$ and are continuous almost everywhere.

Let $D \subset \mathbb{R}$ be a nonempty set. Recall that a function $h : D \to \mathbb{R}$ is quasicontinuous ([5, 6]) at a point $x \in D$ if for every positive real η and for

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every open interval I containing x there is an open interval $J \subset I$ such that $J \cap D \neq \emptyset$ and $f(J \cap D) \subset (f(x) - \eta, f(x) + \eta)$.

Denote by $\mathcal{A}$ the family of all functions $g : \mathbb{R} \to \mathbb{R}$ for which $D(g)$ are nowhere dense and for each nonempty set $E \subset D(g)$ belonging to the density topology ([1, 10]) the restricted function $g \restriction E$ is quasicontinuous.

Evidently, $\mathcal{B}_1^{**} \subset \mathcal{A}$ and $\mathcal{B}_1^{**} \neq \mathcal{A}$, since all almost everywhere continuous functions g having nowhere dense sets $D(g)$ and such that $g \restriction D(g)$ are discontinuous belong to $\mathcal{A} \setminus \mathcal{B}_1^{**}$.

There are also non Borel functions belonging to $\mathcal{A}$. For example, for every non Borel set B containing in the Cantor ternary set the characteristic function of the set B belongs to $\mathcal{A}$ and is not Borel.

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that all sections f_x, $x \in \mathbb{R}$, are measurable. If all sections f^y, $y \in \mathbb{R}$, are quasicontinuous and belong to the family $\mathcal{A}$, then f is measurable as the function of two variables.*

In the proof of this theorem we apply the following Lemma which is a particular case of Davies Lemma from [2].

Lemma 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If for every positive real η and for each measurable set $A \subset \mathbb{R}^2$ of positive measure there is a measurable set $B \subset A$ of positive measure such that $\operatorname{osc}_B f \leq \eta$, then the function f is measurable.*

PROOF OF THEOREM 1. We will show that the function f satisfies the assumptions of the above Lemma. Let $A \subset \mathbb{R}^2$ be a set of positive measure and let η be a positive real. For $x, y \in \mathbb{R}$ let $A_x = \{u \in \mathbb{R}; (x, u) \in A\}$ be the vertical section of the set A corresponding to x and respectively let $A^y = \{t \in \mathbb{R}; (t, y) \in A\}$ be the horizontal section of the set A corresponding to y. Moreover let

$$K = \{(x, y) \in A; x \text{ is a density point of } A^y\},$$
$$E = \{(x, y) \in K; x \in D(f^y)\}$$

and let $H = K \setminus E$. Denote by μ (μ_2) Lebesgue measure in $\mathbb{R}$ ($\mathbb{R}^2$) and observe that by a well known theorem from Saks' monograph ([8], pp. 130–131) $\mu_2(A \setminus K) = 0$.

Now we will consider two cases.

Case I. The set H is not of measure 0.

Then for every point $(x, y) \in H$ there are open intervals $I(x, y)$ and $J(x, y)$ with rational endpoints such that $\mu(I(x, y) \cap A^y) > 0$ and $d(J(x, y)) < \frac{\eta}{4}$ ($d(J(x, y))$ denotes the length of the interval $J(x, y)$) and $f^y(I(x, y)) \subset J(x, y)$.

Let $I_1, I_2, \dots, I_n, \dots$ be a sequence of all open intervals with rational endpoints, let $J_1, \dots, J_n, \dots$ be an enumeration of all open intervals with $d(J_n) < \frac{\eta}{4}$ and for $n, m = 1, 2, \dots$ let

$$A_{n,m} = \{(x, y) \in H; I(x, y) = I_n \text{ and } J(x, y) = J_m\}.$$

Then $H = \bigcup_{n,m=1}^{\infty} A_{n,m}$, and consequently there is a pair of positive integers j, k for which the set $A_{j,k}$ is not of measure zero. Let

$$\begin{aligned} V =& \{y; \exists_x (x, y) \in A_{j,k}\}, \\ U =& \{y; y \text{ is an outer density point of the set } V\} \end{aligned}$$

and $X = K \cap (I_j \times U)$. The set X is measurable and by Fubini's Theorem it is of positive measure. Find a point $w \in I_j$ such that the section X_w is measurable and the linear Lebesgue measure $\mu(X_w)$ is positive. Since the section f_w is measurable and consequently almost everywhere approximately continuous, there is a nonempty measurable set $G \subset X_w$ of finite measure belonging to the density topology ([1]) such that $f(w, u) \in J_k$ for $u \in G$. Put $F = K \cap (I_j \times G)$ and $M = (K \cap (I_j \times G)) \cap f^{-1}(L_k)$, where L_k is the closed interval having the same center as J_k and length equal η. By Fubini's Theorem the set F is measurable and of positive measure. We will prove that the set $F \setminus M$ is of measure zero.

In reality, if the set $F \setminus M$ is not of measure zero, then by the quasicontinuity of the sections f^y for each point $(x, y) \in F \setminus M$ there is an open interval $K(x, y) \subset I_j$ with rational endpoints such that $f(t, y) \in \mathbb{R} \setminus J_k$ for $t \in K(x, y)$. So there is an open interval $I \subset I_j$ such that the set

$$Z = \{(x, y) \in F \setminus M; K(x, y) = I\}$$

is not of measure zero. Let $W = \{y \in \mathbb{R}; \exists_x (x, y) \in Z\}$ and let $v \in I$ be a point. Then for $y \in W$ we have $f(v, y) \in \mathbb{R} \setminus L_k$ and for $y \in G \cap V$ the relation $f(v, y) \in J_k \subset L_k$ holds. Since the section f_v is measurable, we obtain a contradiction.

Let $B = F \setminus M$. Then the set $B \subset A$ is measurable, $\mu_2(B) > 0$ and $\operatorname{osc}_B f \le \eta$.

Case II. The set H is of measure 0.

In this case we put

$$K_1 = \{(x, y) \in E; x \text{ is a density point of } E^y\}.$$

Since all sections $f^y \in \mathcal{A}$, $y \in \mathbb{R}$, as in Case I, we find open intervals I, J and a set $P \subset \mathbb{R}$ such that $d(J) < \frac{\eta}{4}$, P is not of measure zero, $I \cap (K_1)^y \neq \emptyset$ for $y \in P$ and $f(x, y) \in J$ for $(x, y) \in K_1 \cap (I \times P)$. Let

$$Z = \{y; \text{ the outer density of the section } P_x \text{ at } y \text{ is } 1\}$$

and $S = K_1 \cap (I \times Z)$. Then S is measurable and by Fubini's Theorem $\mu_2(S) > 0$. Put

$$U = \{(x,y) \in S; f(x,y) \in \mathbb{R} \setminus L\},$$

where L is the open interval of length η having the same center as J. We will prove that $\mu_2(U) = 0$. If not, then there are an open interval $I_1 \subset I$ and a set $B_1 \subset Z$ which is not of measure zero such that $\mu(I \cap S_x) > 0$ for $y \in B_1$, and $f(x,y) \in \mathbb{R} \setminus J$ for $(x,y) \in S \cap (I_1 \times B_1)$. If $x \in I_1$ is a point such that $\mu(S_x) > 0$, then we obtain a contradiction to the measurability of the section f_x. So, $\mu_2(U) = 0$, the set $B = S \setminus U \subset K \subset A$ is measurable, $\mu_2(B) > 0$ and $\operatorname{osc}_B f \leq \eta$. □

Corollary 1. *If all sections f_x, $x \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are measurable and all sections f^y, $y \in \mathbb{R}$, are quasicontinuous and almost everywhere continuous, then f is measurable.*

Remark 1. It is obvious to observe that if a function $g : \mathbb{R} \to \mathbb{R}$ has the Darboux property and belongs to the family $\mathcal{B}_1^{**}$, then $g \in \mathcal{A}$ and g is quasicontinuous. So, if all sections f^y of a function $f : \mathbb{R}^2 \to \mathbb{R}$ have the Darboux property and belong to $\mathcal{B}_1^{**}$ and if all sections f_x are measurable, then f is measurable.

In articles [3, 4] some other conditions are given implying the measurability of the function $f : \mathbb{R}^2 \to \mathbb{R}$ having the measurable sections f_x. One is called strong approximate quasicontinuity ([3]) and the second is denoted by (H) ([4]). However, each of these conditions implies the continuity at almost all points. The next example shows that there are Darboux functions g with $D(g)$ of positive measure in the class $\mathcal{B}_1^{**}$.

Example 1. Let $C \subset [0,1]$ be a Cantor set of positive measure. In every component (a,b) of the open set $(0,1) \setminus C$ we find a closed interval $I(a,b) = [c(a,b), d(a,b)] \subset (a,b)$ and a continuous function $f_{(a,b)} : (a,b) \to (0,1)$ such that

$$f_{(a,b)}(I(a,b)) = [0,1] \text{ and } f_{(a,b)}((a,b) \setminus I(a,b)) = \{0\}.$$

Putting

$$g(x) = \begin{cases} f_{(a,b)}(x) & \text{for } x \in (a,b), \text{ where } (a,b) \text{ is a component of } (0,1) \setminus C \\ 0 & \text{otherwise} \end{cases}$$

we obtain a function satisfying all required properties.

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