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EQUIVALENCE OF $(n+1)$-th ORDER PEANO AND USUAL DERIVATIVES FOR n-CONVEX FUNCTIONS

Abstract

A real-valued function f defined on an interval of $\mathbb{R}$ is said to be n-convex if all its n-th order divided differences are not negative. Let f be such a function defined in a right neighborhood of $t_0 \in \mathbb{R}$ whose usual right derivatives, $f_+^{(r)}$, $1 \le r \le n$, exist in that neighborhood and whose $(n+1)$-th order Peano derivative, $f_{n+1}(t_0)$, exists at t_0. Under these assumptions we prove that f also possesses $(n+1)$-th order usual right derivative $f_+^{(n+1)}(t_0)$ at t_0. This result generalizes the known case for convex (that is 1-convex) functions. The latter appears in works of B. Jessen studying the curvature of convex curves and of J. M. Borwein, M. Fabian, D. Noll studying the second order differentiability of convex functions on abstract spaces.

1 Introduction

Let f be a real function defined in a neighborhood of a point $t_0 \in \mathbb{R}$. In 1891 the Italian mathematician Giuseppe Peano [11] introduced a more general definition of higher order derivatives of f at t_0, while he was studying the Taylor expansion formula for real functions. Peano's paper has to be included in his research giving rigorous foundations to analysis at the end of the 19-th century. Peano never returned to study the contents of his 1891 article. However in this century his idea was pursued in several papers, but not always quoted by the

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name of Peano. The definition of higher order derivative ($n \geq 2$) is more general than the usual one, since it does not require the existence of lower order derivatives in a neighborhood of the point t_0. It is immediate to verify that if the n-th order derivative of f at t_0 exists, also the n-th order Peano derivative exists and they coincide. However the converse does not hold. Many sufficient conditions were established to prove the equivalence between Peano and usual derivative definitions. In particular we recall the sufficient condition proved in [10] that requires the n-th Peano derivative to be bounded (above or below) on an interval. We observe that most of these results require the existence of the n-th Peano derivative on an interval. For a survey on Peano derivatives we refer to [6]. Particularly a classical result due to B. Jessen [9] (for the proof see also [5]) states the equivalence for second Peano and usual derivatives for convex functions. Jessen's proof was a long geometrical one. More recently, without any reference to Jessen's work, a new proof (also with the extension to the setting of abstract spaces) was given by J. M. Borwein and M. Fabian [2] (see also [3]).
In this paper we deal with functions defined on the real line, but there is no difficulty in extending the treatment to abstract spaces. Our aim is to extend the Jessen-Borwein-Fabian equivalence result in two directions, by taking into consideration the $(n+1)$-th order derivatives for n-convex functions. In the case $n = 1$ our result gives a new proof of the Jessen-Borwein-Fabian equivalence result.
Furthermore in section 2 we will prove a result that states equivalence between second Peano and usual derivatives under a Lipschitz-type assumption on the residual in Taylor's formula.

2 Preliminary Concepts

Definition 2.1. Let f be a function defined in a neighborhood of the point $t_0 \in \mathbb{R}$. We say that f has an n-th order right Peano derivative at t_0 when there exist numbers $f_1(t_0), f_2(t_0), \dots, f_n(t_0)$, such that

$$f(t) = f(t_0) + f_1(t_0)(t-t_0) + \frac{f_2(t_0)}{2}(t-t_0)^2 + \cdots + \frac{f_n(t_0)}{n!}(t-t_0)^n + o((t-t_0)^n),$$

as $t \to {t_0}^+$. The number $f_n(t_0)$ is said the n-th order Peano derivative of f at the point t_0.

Obviously the existence of the n-th order Peano derivative $f_n(t_0)$ implies the existence of the lower order Peano derivatives $f_{n-1}(t_0), \dots, f_1(t_0)$. We denote by $f^{(n)}(t_0)$ the usual n-th order derivative of f at t_0 and by $f_+^{(n)}(t_0)$

the usual n-th order right derivative. If $f_+^{(n)}(t_0)$ exists, then obviously $f_n(t_0)$ also exists and $f_n(t_0) = f_+^{(n)}(t_0)$. The converse holds for $n = 1$ but not necessarily for $n \geq 2$, as is shown by the following function.

Example 2.1. For the function

$$f(t) = \begin{cases} 0 & \text{if } t \in \mathbb{Q} \\ t^{n+1} & \text{if } t \in \mathbb{R}\backslash\mathbb{Q},\ n \geq 2 \end{cases}$$

one has $f_k(0) = 0, k = 1, \ldots, n$, while $f^{(k)}(0), k = 2, \ldots, n$ do not exist.

Theorem 2.1. *[9] Let f be a convex function defined in a neighborhood of $t_0 \in \mathbb{R}$ and assume that f' exists in a right neighborhood of t_0 and also that $f_2(t_0)$ exists. Then $f''_+(t_0)$ exists and $f_2(t_0) = f''_+(t_0)$.*

In recent years the previous theorem has been extended to abstract spaces by J. M. Borwein, M. Fabian and D. Noll in two papers devoted to the second order differentiability of convex functions on Banach spaces [2] [3].

The previous theorem requires the existence of f' in a right neighborhood of t_0. Generally this existence is not a consequence of the convexity of the function f. Indeed let us consider the family of functions $\{f_n(t)\} = \{\frac{t^4}{n}\}$, for instance for $t \geq 0$. We will define the function f in the following way. Let $f(t) = t^4$ for $t \geq 1 = t_1$; *graph* $f(t) =$ *graph* r_1 in $[t_2, t_1]$, where r_1 is the line joining the point $a_1 = (t_1, 1)$ to the point a_2, and $a_2 = (t_2, \frac{1}{2}t_2^4)$ is the point where the tangent line to $y = t^4$ at a_1 meets $y = \frac{1}{2}t^4$; and so on. It is easy to verify that f is a convex function and that it satisfies definition 2.1 for $t_0 = 0, n = 2, f_1(0) = f_2(0) = 0$. Nevertheless f is not differentiable at $\{t_n\}$ and $t_n = o(1)$.

In the following section we will extend the previous result to the case of the $(n+1)$-th order derivative of a n-convex function.
Let f be a function defined on the interval (a, b) and let $t_0, t_1, \ldots, t_n$ be distinct points of (a, b). We recall that the n-th divided difference of f at these $n+1$ points is given by

$$V_n(f, t_k) = \sum_{j=0}^{n} \frac{f(t_j)}{\prod_{k=0, k\neq j}^{n}(t_j - t_k)}.$$

The following theorems recall some basic properties of divided differences.

Theorem 2.2. *[8] If $p(t)$ is a polynomial of degree n, then for every choice of the points $t_k, 0 \leq k \leq n+1$, one has $V_{n+1}(p, t_k) = 0$.*

Theorem 2.3. *[8] Let $f(t)$ have a continuous n-th derivative in the interval* $\min(z_0, z_1, \ldots, z_n) \leq t \leq \max(z_0, z_1, \ldots, z_n)$. *Then, if the points $z_0, z_1, \ldots, z_n$ are distinct,*

$$V_n(f, z_k) =$$

$$= \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} f^{(n)}\left(t_n(z_n - z_{n-1}) + \cdots + t_1(z_1 - z_0) + z_0\right) dt_n.$$

Definition 2.2. A function $f : (a, b) \to \mathbb{R}$ is said to be n-convex when $V_{n+1}(f, t_k) \geq 0$, for all choices of $n+2$ distinct points $t_0, \ldots, t_{n+1}$ in (a, b).

If $n = 1$, from definition 2.2 we obtain the class of convex functions. If in particular the points $t_0, t_1, \ldots, t_n$ are of the form $t_i = t_0 + ih, (i = 0, 1, \ldots, n;\ h > 0)$, we have

$$V_n(f, t_k) = \frac{1}{n!\,h^n} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(t_0 + ih).$$

We will set

$$\Delta^n f(t_0, h) = \frac{1}{h^n} \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(t_0 + ih).$$

The following results summarize some fundamental properties of n-convex functions.

Theorem 2.4. *[1] [4] If $f : (a, b) \to \mathbb{R}$ is continuous, then f is n-convex if and only if $\Delta^{n+1} f(t, h) \geq 0$, for every $t \in (a, b)$ and for every h such that $t + (n + 1)h \in (a, b)$.*

Theorem 2.5. *[4] Let $f : (a, b) \to \mathbb{R}$ be a n-convex function.*

(i) If $a \leq t_0 < \cdots < t_n \leq b$, $a \leq z_0 < \cdots z_n \leq b$, $z_k \leq t_k$, $0 \leq k \leq n$, then $V_n(f, z_k) \leq V_n(f, t_k)$.

(ii) For $1 \leq r \leq n - 1$ the derivative $f^{(r)}$ exists and is continuous in (a, b).

(iii) $f_+^{(n)}$ exists in (a, b) and is monotonic increasing.

(iv) If $a \leq t_0 < \cdots < t_n \leq t \leq y_0 < \cdots < y_n \leq b$, then

$$n!\, V_n(f, t_k) \leq f_+^{(n)}(t) \leq n!\, V_n(f, y_k).$$

(v) The function $\Delta^n f(t, h)$ is increasing in h.

(vi) *If f admits an $(n+1)$-th order right Peano derivative $f_{n+1}(t)$ in (a,b), then $f_{n+1}(t) \geq 0$.*

We end this section with the equivalence result we mentioned in the introduction.

Theorem 2.6. *Let f be a function defined in a right neighborhood of the point $t_0 \in \mathbb{R}$ and assume*

(i) *f' exists in a neighborhood of t_0;*

(ii) *f admits second right Peano derivative at $t_0, f_2(t_0)$; that is,*

$$f(t_0+h) = f(t_0) + f'(t_0)h + \frac{1}{2}f_2(t_0)h^2 + g(h)h^2,$$

where $g(h) \to 0$, as $h \to 0^+$;

(iii) *for every s, h "sufficiently small" $|g(h+s) - g(h)| \leq |s|k(h)$, for some nonnegative function k such that $k(h)h \to 0^+$, for $h \to 0^+$.*

Then $f''_+(t_0)$ exists and coincides with $f_2(t_0)$.

PROOF. Without loss of generality we can assume $f'(t_0) = 0$. We have

$$\frac{f'(t_0+h) - f'(t_0)}{h} = \lim_{s\to 0^+} \frac{f(t_0+h+s) - f(t_0+h)}{hs}.$$

Since $f_2(t_0)$ exists, we obtain

$$\lim_{s\to 0^+} \frac{\frac{1}{2}f_2(t_0)(h+s)^2 + g(h+s)(h+s)^2 - \frac{1}{2}f_2(t_0)h^2 - g(h)h^2}{hs}$$

$$= \lim_{s\to 0^+} \Big(f_2(t_0)\Big(\frac{s}{2h} + 1\Big) + g(h+s)\frac{h}{s} - g(h)\frac{h}{s} + g(h+s)\frac{s}{h} + 2g(h+s)\Big).$$

Now, from hypothesis iii) we can easily see that the limit of the last quantity exists as $h \to 0^+$ and that this last limit equals $f_2(t_0)$. □

3 The Main Result

Theorem 3.1. *Let f be a n-convex function defined in a right neighborhood of the point $t_0 \in \mathbb{R}$ and assume that in this neighborhood f admits the usual right derivatives $f_+^{(r)}$, $1 \leq r \leq n$. If the $(n+1)$-th order right Peano derivative of f at t_0, $f_{n+1}(t_0)$ exists, also $f_+^{(n+1)}(t_0)$ exists and $f_+^{(n+1)}(t_0) = f_{n+1}(t_0)$.*

Proof. Without loss of generality we suppose that the function f is defined on $[0,1)$ and that in this right neighborhood of $t_0 = 0$ there exist derivatives up to order n. Furthermore we observe that Theorem 2.2 ensures that the function f is n-convex if and only if this property is satisfied by the function $F(t) = f(t) - f(0) - \sum_{k=1}^{n} \dfrac{t^k}{k!} f_+^{(k)}(0)$. Hence we may suppose that $f(0) = f'_+(0) = \cdots = f_+^{(n)}(0) = 0$. Then we get $f(t) = \frac{1}{(n+1)!} f_{n+1}(0)\, t^{n+1} + g(t)$, where $g(t) = o(t^{n+1}), t \to 0^+$. Now let $t \in (0,1)$ and consider a real number $m \geq 1$. Obviously $t^{1+1/m} < t < t^{1-1/m} \leq 1$. Theorem 2.5 gives the inequalities

$$\Delta^n f\left(t^{1+1/m}, \frac{t - t^{1+1/m}}{n}\right) \leq f_+^{(n)}(t) \leq \Delta^n f\left(t, \frac{t^{1-1/m} - t}{n}\right).$$

Let $p(t) = \frac{1}{(n+1)!} t^{n+1}$. We will show that $\Delta^n p(t,h) = t + \frac{n}{2} h$. Indeed, by using the integral representation of the n-th divided difference (Theorem 2.3)

$$\begin{aligned}
\Delta^n p(t,h) &= n! \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} p^{(n)}\Big(\sum_{k=1}^{n} t_k h + t\Big)\, dt_n \\
&= n! \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} \Big(\sum_{k=1}^{n} t_k h + t\Big)\, dt_n \\
&= n! \left(\frac{h}{(n+1)!} + \sum_{k=1}^{n-1} \frac{h}{\prod_{i=2, i\neq k+1}^{n+1} i} + \frac{1}{n!} t \right) \\
&= t + n! \frac{\sum_{k=1}^{n} k}{(n+1)!} h = t + \frac{n}{2} h.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Delta^n f\left(t^{1+1/m}, \frac{t - t^{1+1/m}}{n}\right) \\
&= \frac{1}{2}\, f_{n+1}(0)\, t(1 + t^{1/m}) + \Delta^n g\left(t^{1+1/m}, \frac{t - t^{1+1/m}}{n}\right) \\
&\leq f_+^{(n)}(t) \leq \Delta^n f\left(t, \frac{t^{1-1/m} - t}{n}\right) \\
&= \frac{1}{2} f_{n+1}(0)\, t(t^{-1/m} + 1) + \Delta^n g\left(t, \frac{t^{1-1/m} - t}{n}\right).
\end{aligned}$$

Since $f_+^{(n)}(0) = 0$, we have

$$-\frac{1}{2}\, f_{n+1}(0)\,(1 - t^{1/m})$$

$$
\begin{aligned}
&+ t^{-1}\left(\frac{t-t^{1+1/m}}{n}\right)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}g\left(t^{1+1/m}+\frac{i}{n}(t-t^{1+1/m})\right)\\
\le&\frac{f_+^{(n)}(t)-f_+^{(n)}(0)}{t}-f_{n+1}(0)\\
\le&\frac{1}{2}\,f_{n+1}(0)\,(t^{-1/m}-1)\\
&+ t^{-1}\left(\frac{t^{1-1/m}-t}{n}\right)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}g\left(t+\frac{i}{n}(t^{1-1/m}-t)\right).
\end{aligned}
$$

Now fix $0<\varepsilon<1$ and choose $m=m(t)$ such that $1-t^{1/m}=\varepsilon$; that is, $m=\frac{\ln t}{\ln(1-\varepsilon)}$. We get

$$
\begin{aligned}
&-\frac{1}{2}f_{n+1}(0)\varepsilon+t^{-n-1}\left(\frac{\varepsilon}{n}\right)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}g\left(t(1-\varepsilon)+\frac{i}{n}t\varepsilon\right)\\
\le&\frac{f_+^{(n)}(t)-f_+^{(n)}(0)}{t}-f_{n+1}(0)\\
\le&\frac{1}{2}f_{n+1}(0)\frac{\varepsilon}{(1-\varepsilon)}+t^{-n-1}\left(\frac{\varepsilon}{n(1-\varepsilon)}\right)^{-n}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}g\left(t+\frac{i}{n}t\frac{\varepsilon}{1-\varepsilon}\right).
\end{aligned}
$$

From the hypotheses we obtain that the left member in these inequalities converges to $-\frac{1}{2}f_{n+1}(0)\varepsilon$ as $t\to 0^+$, while the right member converges to $\frac{1}{2}f_{n+1}(0)\frac{\varepsilon}{1-\varepsilon}$ as $t\to 0^+$. So we have

$$
\begin{aligned}
-\frac{1}{2}f_{n+1}(0)\varepsilon &\le \liminf_{t\to 0^+}\frac{f_+^{(n)}(t)-f_+^{(n)}(0)}{t}-f_{n+1}(0)\\
&\le \limsup_{t\to 0^+}\frac{f_+^{(n)}(t)-f_+^{(n)}(0)}{t}-f_{n+1}(0)\le\frac{1}{2}f_{n+1}(0)\frac{\varepsilon}{1-\varepsilon}.
\end{aligned}
$$

Since ε is arbitrary we get the assertion. $\square$

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