Real Analysis Exchange
Vol. 23(2), 1997–1998, pp. 525–537

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A TYPICAL MEASURE TYPICALLY HAS NO LOCAL DIMENSION

Abstract

We consider local dimensions of probability measure on a complete separable metric space X: $\overline{\alpha}_\mu(x) = \overline{\lim\limits_{r\to 0}} \frac{\log \mu(B_r(x))}{\log r}$, $\underline{\alpha}_\mu(x) = \underline{\lim\limits_{r\to 0}} \frac{\log \mu(B_r(x))}{\log r}$. We show (Theorem 2.1) that for a typical probability measure $\underline{\alpha}_\mu(x) = 0$ and $\overline{\alpha}_\mu(x) = \infty$ for all x except a set of first category. Also $\underline{\alpha}_\mu(x) = 0$ almost everywhere and with some additional conditions on X there is a corresponding result for upper local dimension: in particular, we show that a typical measure on $[0,1]^d$ has $\overline{\alpha}_\mu(x) = d$ almost everywhere (Theorem 2.4).

There are similar results concerning "global" dimensions of probability measures. Theorems 2.2 and 2.3 show in particular that the Hausdorff dimension of a typical measure on any compact separable space equals 0 and the packing dimension of a typical measure on $[0,1]^d$ equals d.

1 Introduction and Preliminaries

Throughout this paper X will denote a separable metric space. Let $\mathcal{P}(X)$ be the space of all Borel probability measures on X. Denote by $B_r(x)$ the open ball with center x and radius r, and by $\overline{B}_r(x)$ the corresponding closed ball.

The Hausdorff dimension of a set A is denoted by $\dim A$, and the packing dimension by $\operatorname{Dim} A$. For definitions see e.g. [5], [8]. We will use the "radius" definition of Dim, as in [8] and [4] as opposed to the "diameter" definition, since it allows to avoid some pitfalls in general metric spaces; for example, (4) is not true in general metric space with the "diameter" definition ([4]).

Upper and lower local dimensions of a measure $\mu \in \mathcal{P}(X)$,

$$\overline{\alpha}_\mu(x) = \overline{\lim_{r\to 0}} \frac{\log \mu(B_r(x))}{\log r} \quad \text{and} \quad \underline{\alpha}_\mu(x) = \underline{\lim_{r\to 0}} \frac{\log \mu(B_r(x))}{\log r},$$

Key Words: measure, dimension, category
Mathematical Reviews subject classification: 28A33, 28A80
Received by the editors June 4, 1997

have been extensively studied (see e.g. [3], [8], [12], [16] and many others). In particular, they are used to construct the so-called multifractal spectrum functions such as

$$f_\mu(\alpha) = \dim\{x \in \operatorname{supp}\mu \mid \alpha_\mu(x) = \alpha\},$$

where $\alpha_\mu(x) = \underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x)$ if they coincide. These limits are the same for closed and open balls.

The dimensions, $\underline{\alpha}_\mu(x)$ and $\overline{\alpha}_\mu(x)$, have been shown to coincide almost everywhere in some particular cases such as the ergodic invariant measures of smooth diffeomorphisms with nonzero Lyapunov exponents ([16]). If this happens, a measure is called regular. The multifractal spectrum was also computed for such constructions as graph-directed fractals and cookie-cutters (see e.g [8], where Olsen applies his general multifractal formalism to both cases). The question arises whether $\alpha_\mu(x)$ and $f(\alpha)$ can be used to describe more general situations. We will show that this is in fact not the case for a typical probability measure (in the sense of category).

Some relations between different notions of regularity are discussed in [14]. There are some results as well describing situations where $\underline{\alpha}_\mu(x) \neq \overline{\alpha}_\mu(x)$. For example, Taylor in [13] shows that this happens for super Brownian motion. Shereshevsky in [11] shows that under some conditions, if μ is an invariant measure of a smooth diffeomorphism, the set where $\underline{\alpha}_\mu(x) \neq \overline{\alpha}_\mu(x)$ is dense and has positive Hausdorff dimension. Finally, Haase in [7] shows that if $x \in X$, then for a typical measure $\mu \in \mathcal{P}(X)$ (that is, all measures up to a set of first category) $\underline{\alpha}_\mu(x) = 0$, and if x is a non-isolated point of X, then for a typical measure $\overline{\alpha}_\mu(x) = \infty$. Theorem 2.1 is basically a generalization of this result.

An interesting question, connected with this, is about the dimension of a typical measure, especially Hausdorff and packing dimension. The dimensions on other spaces have been explored before, such as the dimension of typical compact set or a graph of typical continuous function. For example, Hausdorff dimension of a typical compact subset of $\mathcal{R}^d$ is 0 and the upper entropy dimension is d ([6]). We show later in this paper that the probability measures behave similarly in this matter.

We will use the following well-known relations.

$$\dim(\{x \in X \mid \underline{\alpha}_\mu(x) \leq \alpha\}) \leq \alpha. \tag{1}$$

$$\operatorname{Dim}(\{x \in X \mid \overline{\alpha}_\mu(x) \leq \alpha\}) \leq \alpha. \tag{2}$$

$$\text{If } \mu(A) > 0, A \subseteq \{x \in X \mid \underline{\alpha}_\mu(x) \geq \alpha\}, \text{ then } \dim A \geq \alpha. \tag{3}$$

$$\text{If } \mu(A) > 0, A \subseteq \{x \in X \mid \overline{\alpha}_\mu(x) \geq \alpha\}, \text{ then } \operatorname{Dim} A \geq \alpha. \tag{4}$$

For the proof see e.g. [3], [8]. Note that (4) would not be true for all metric spaces if packing measure were defined using diameters, but true under some fairly general conditions.

A weak* topology on $\mathcal{P}(X)$ is characterized by the following proposition.

Proposition 1.1. *Let $\mu, \{\mu_n\}_{n=1}^{\infty}$ be measures in $\mathcal{P}(X)$. Then the following statements are equivalent.*

(a) $\mu_n \to \mu$ in weak topology.*

(b) $\lim_{n\to\infty} \int f\,d\mu_n = \int f\,d\mu$ for any bounded continuous function f.

(c) $\overline{\lim_{n\to\infty}} \mu_n(F) \le \mu(F)$ for every closed set F.

(d) $\underline{\lim}_{n\to\infty} \mu_n(G) \ge \mu(G)$ for every open set G.

(e) $\lim_{n\to\infty} \mu_n(A) = \mu(A)$ for every Borel set A with boundary of μ-measure 0.

Proof. This is a version of slightly more general Theorem 6.1 in [9]. □

If X is complete and separable, $\mathcal{P}(X)$ is also complete; so we can use an expression "a typical measure" to signify that all measures except a set of first category in $\mathcal{P}(X)$ have the desired properties.

Proposition 1.2. *The probability measures with finite support are dense in $\mathcal{P}(X)$.*

Proof. See Theorem 6.3 in [9]. □

It follows that $\mathcal{P}(X)$ is separable since X is separable.

$\mathcal{P}(X)$ with weak* topology can be metrized in several ways. In particular for X separable a Prokhorov metric p can be used:

$$\begin{aligned} p(\mu,\nu) = \inf\{\epsilon > 0\ |\mu(A) \le \nu(A_\epsilon) + \epsilon \text{ and}& \\ \nu(A) \le \mu(A_\epsilon) + \epsilon \text{ for any Borel set } A\},& \end{aligned}$$

where A_ϵ is the ϵ-neighborhood of A in X.

The inequalities (1)-(4) can also be used to establish a connection between the local and "global" dimensions of a measure. The latter can be computed in a number of ways. We will need the following definitions.

$$\dim^* \mu = \inf\{\dim Y \mid Y \subseteq X,\, \mu(Y) = 1\}.$$

$$\dim_* \mu = \inf\{\dim Y \mid Y \subseteq X,\, \mu(Y) > 0\}.$$

$$\mathrm{Dim}^* \mu = \inf\{\mathrm{Dim}\, Y \mid Y \subseteq X,\, \mu(Y) = 1\}.$$

$$\mathrm{Dim}_* \mu = \inf\{\mathrm{Dim}\, Y \mid Y \subseteq X,\, \mu(Y) > 0\}.$$
$$\overline{C}(\mu) = \lim_{\delta\to 0} \inf\{\overline{\dim}_B(Y) \mid Y \subseteq X,\, \mu(Y) \geq 1-\delta\},$$
$$\underline{C}(\mu) = \lim_{\delta\to 0} \inf\{\underline{\dim}_B(Y) \mid Y \subseteq X,\, \mu(Y) \geq 1-\delta\},$$

where $\dim Y$ and $\mathrm{Dim}\, Y$ denote Hausdorff and packing dimensions correspondingly, and $\overline{\dim}_B Y, \underline{\dim}_B Y$ denote upper and lower box dimensions.

Ledrappier dimensions are defined as follows. Suppose μ is supported on a totally bounded set. Let $N_\mu(\epsilon, \delta)$ be the minimal number of balls of radius ϵ which cover a set of measure greater than $1-\delta$. Then

$$\overline{C}_L(\mu) = \lim_{\delta\to 0} \overline{\lim_{\epsilon\to 0}} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)},$$

$$\underline{C}_L(\mu) = \lim_{\delta\to 0} \underline{\lim_{\epsilon\to 0}} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)}.$$

(Here $\lim\limits_{\delta\to 0}$ is the same as $\sup\limits_{0<\delta<1}$ due to monotonicity.)

Proposition 1.3.

$$\dim^* \mu \leq \underline{C}_L(\mu) \leq \underline{C}(\mu) \text{ and}$$
$$\overline{C}_L(\mu) \leq \overline{C}(\mu) = \mathrm{Dim}^* \mu.$$

PROOF. The last equality is proved in [12]; everything else is proved in [16]. □

Let us introduce also a few more global characteristics of a measure.

$$\overline{C}_{L*}(\mu) = \lim_{\delta\to 1} \overline{\lim_{\epsilon\to 0}} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)},$$

$$\underline{C}_{L*}(\mu) = \lim_{\delta\to 1} \underline{\lim_{\epsilon\to 0}} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)},$$

(Here $\lim\limits_{\delta\to 1}$ is the same as $\inf\limits_{0<\delta<1}$.)

$$\overline{C}_*(\mu) = \inf\{\overline{\dim}_B Y \mid Y \subseteq X, \mu(Y) > 0\} \text{ and}$$

$$\underline{C}_*(\mu) = \inf\{\underline{\dim}_B Y \mid Y \subseteq X, \mu(Y) > 0\}.$$

Proposition 1.4.

$$\underline{C}_{L*}(\mu) \leq \overline{C}_{L*}(\mu) \leq \overline{C}_*(\mu) = \mathrm{Dim}_* \mu.$$

PROOF. The first two inequalities are obvious, as is the inequality $\operatorname{Dim}_* \mu \leq \overline{C}_*(\mu)$. For any $D > \operatorname{Dim}_* \mu$ there is a set $Y \subseteq X$ such that $\mu(Y) > 0$ and $\operatorname{Dim} Y < D$. Note that $\operatorname{Dim} Y = \inf\{\sup_i \overline{\dim}_B A_i \mid Y \subseteq \cup A_i\}$ (for proof see e.g. [5], p.48; it is proved for $Y \in \mathbb{R}_n$ there, but the proof works for a general metric space with "radius" definition of packing dimension). Hence there is at most countable collection of sets $\{A_i\}$ such that $\sup_i \overline{\dim}_B A_i < D$ and $\mu(\cup A_i) \geq \mu(Y) > 0$. It follows that there is an A_i with $\mu(A_i) > 0$, $\overline{\dim}_B A_i < D$. Hence $\operatorname{Dim}_* \mu \geq \overline{C}_*(\mu)$. □

Proposition 1.5. *Let μ be a probability measure on a compact space X. If for some d there are numbers $c > 0$ and $R > 0$ such that $\mu(B_r(x)) \leq cr^d$ for all $x \in X, 0 < r < R$, then $\underline{C}_{L*}(\mu) \geq d$.*

PROOF. Suppose that for some δ $\varliminf\limits_{\epsilon \to 0} \frac{\log N_\mu(\epsilon,\delta)}{\log(1/\epsilon)} < d$. Then there are $a < d$ and $\epsilon_l \downarrow 0, \epsilon_l < R$ such that $N_\mu(\epsilon_l, \delta) < \epsilon_l^{-a}$ for all l. Hence for any l there are $N_\mu(\epsilon_l, \delta)$ balls $\{B^i_{\epsilon_l}\}$ with $\mu(\cup_i B^i_{\epsilon_l}) > 1 - \delta$. But then

$$1 - \delta < \mu(\bigcup_i B^i_{\epsilon_l}) \leq cN_\mu(\epsilon_l, \delta)\epsilon_l^d < c\epsilon_l^{d-a} \text{ for all } l,$$

which contradicts the fact that $\epsilon_l^{d-a} \to 0$. □

2 Main Results

Theorem 2.1. *Let X be a complete separable metric space. Then for a typical measure μ in $\mathcal{P}(X)$ there is a residual Borel set A_μ in X such that for any $x \in A_\mu$ we have $\underline{\alpha}_\mu(x) = 0$. If X has no isolated points, then in addition we can have $\overline{\alpha}_\mu(x) = \infty$ for $x \in A_\mu$.*

PROOF. If $\mu_n \to \mu$ in weak* topology, then $\varliminf_{n\to\infty} \mu_n(G) \geq \mu(G)$ for all open G and $\varlimsup_{n\to\infty} \mu_n(F) \leq \mu(F)$ for all closed F (Proposition 1.1). It follows that for fixed x and r the ratio $\frac{\log \mu(\overline{B}_r(x))}{\log r}$ is lower semicontinuous and $\frac{\log \mu(B_r(x))}{\log r}$ is upper semicontinuous with respect to μ.

Consider closed balls first. Then

$$\begin{aligned}\Omega_{a,x,R} &= \left\{\mu \in \mathcal{P}(X) \mid \sup_{r<R} \frac{\log \mu(\overline{B}_r(x))}{\log r} > a\right\} \\ &= \bigcup_{r<R} \left\{\mu \in \mathcal{P}(X) \mid \frac{\log \mu(\overline{B}_r(x))}{\log r} > a\right\}\end{aligned}$$

is open for any $R > 0, a > 0, x \in X$. Now we want to show that $\Omega_{a,x,R}$ is dense in $\mathcal{P}(X)$. Let $\mu \in \mathcal{P}(X)$ and suppose that $\mu \notin \Omega_{a,x,R}$, that is, $\frac{\log \mu(\overline{B}_r(x))}{\log r} \leq a$ for all $r < R$. Fix any $\epsilon > 0$. If $\mu(\{x\}) \neq 0$, in case X has no isolated points we can find a measure $\nu \in \mathcal{P}(X)$ such that $\rho(\mu, \nu) < \epsilon/2$, and $\nu(\{x\}) = 0$. Otherwise let $\nu = \mu$. We construct $\mu_\epsilon \in \mathcal{P}(X)$ as follows. Pick some $s > a$ and some $r < R$ such that $\nu(B_r(x)) < \epsilon/4$. Let $\mu_\epsilon(B_r(x)) = r^s$ and $\mu_\epsilon(A) = C\nu(A)$ for $A \subseteq X \backslash B_r(x)$, where $C = (1 - r^b)/(1 - \mu(B_r(x)))$ so that $\mu_\epsilon(X) = 1$. Then $\frac{\log \mu(\overline{B}_r(x))}{\log r} > a$ and $\rho(\mu, \mu_\epsilon) \leq \epsilon/2 + \epsilon/4 + (C-1)(1 - \mu(B_r(x))) = 3\epsilon/4 + (1 - r^b) - 1 + \mu(B_r(x)) < \epsilon$. Hence $\Omega_{a,x,R}$ is dense in $\mathcal{P}(X)$.

Let $\{x_i\}_{i=1}^\infty$ be a countable dense subset of X. Then

$$\Omega_{a,R} = \left\{ \mu \in \mathcal{P}(X) \mid \sup_{r<R} \frac{\log \mu(\overline{B}_r(x_i))}{\log r} > a \text{ for all } i \right\}$$

is a countable intersection of open dense sets, i.e. residual for any $a, R > 0$. The same is true for $\Omega_a = \bigcap\limits_n \Omega_{a,1/n}$. Now $\frac{\log \mu(\overline{B}_r(x))}{\log r}$ is also lower semicontinuous with respect to x; so

$$A_{a,\mu,R} = \left\{ x \in X \mid \sup_{r<R} \frac{\log \mu(\overline{B}_r(x))}{\log r} > a \right\}$$

is open for any $a > 0, R > 0, \mu \in \mathcal{P}(X)$. For any fixed $\mu \in \Omega_a$, $A_{a,\mu,1/n}$ is open and dense (since $\{x_i\}_{i=1}^\infty \subseteq A_{a,\mu,1/n}$). Hence

$$A_{a,\mu} = \bigcap_n A_{a,\mu,1/n} = \left\{ x \in X \mid \sup_{r<1/n} \frac{\log \mu(\overline{B}_r(x))}{\log r} > a \text{ for all } n \geq 1 \right\}$$

is residual in X. But

$$A_{a,\mu} \subseteq \left\{ x \in X \mid \overline{\lim_{r\to 0}} \frac{\log \mu(\overline{B}_r(x))}{\log r} \geq a \right\};$$

so

$$\Omega_a \subseteq \left\{ \mu \in \mathcal{P}(X) \mid \overline{\lim_{r\to 0}} \frac{\log \mu(\overline{B}_r(x))}{\log r} \geq a \text{ on a residual subset of } X \right\}.$$

Now take the intersection $\Omega = (\bigcap_{n=1}^\infty \Omega_{1/n})$ and let the corresponding subset of X for a fixed $\mu \in \Omega$ be $A_\mu = \bigcap_{n=1}^\infty A_{1/n,\mu}$. This concludes the proof for the upper local dimension.

The proof is similar for the lower local dimension. Using open balls, we can show that

$$\overline{\Omega}_b = \left\{ \mu \in \mathcal{P}(X) \mid \underline{\lim}_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \le b \text{ on a residual subset of } X \right\}$$

is residual for any $b > 0$. We need only to change sup to inf and reverse inequality signs in the proof above. (Also we will not need to consider the case $\mu(\{x\}) \neq 0$ separately, just take $\mu_\epsilon = r^s$ for some $s < b$ and some small enough $r < R$; so the condition that X has no isolated points is not necessary here).

Take the intersection $\overline{\Omega} = (\bigcap_{n=1}^{\infty} \overline{\Omega}_{1/n})$ to conclude the proof for $\underline{\alpha}_\mu(x)$. Then $\Omega \cap \overline{\Omega}$ gives the residual set of measures for which $\underline{\alpha}_\mu(x) = 0, \overline{\alpha}_\mu(x) = \infty$ for most $x \in X$.

$\square$

The natural question arising next is whether we can take A to be a set of positive measure. By (4), of course, we cannot have $\overline{\alpha}_\mu(x) > \operatorname{Dim} X$ on a set of positive measure; so we can hope only to get $\overline{\alpha}_\mu(x) = \operatorname{Dim} X$. We will show that we can in fact have $\underline{\alpha}_\mu(x) = 0$ and $\overline{\alpha}_\mu(x) = \operatorname{Dim} X$ almost everywhere with some additional conditions on X for the latter. To this end we need first to consider the "global" dimension of a typical measure.

Theorem 2.2. *If X is a compact separable metric space, a typical measure $\mu \in \mathcal{P}(X)$ has $\dim^* \mu = \underline{C}_L(\mu) = 0$.*

Theorem 2.3. *Let X be a compact separable metric space. Suppose there exists a probability measure $\lambda \in \mathcal{P}(X)$ which is positive on all open sets and has $\overline{C}_{L*}(\lambda) \ge d$. Then a typical measure $\mu \in \mathcal{P}(X)$ has $\operatorname{Dim}_* \mu \ge \overline{C}_{L*}(\mu) \ge d$.*

Note. This is true, in particular, with λ being a Lebesgue measure on $[0,1]^d$. We can also let X be a self-similar fractal set of Hausdorff dimension d, with λ being a Hausdorff measure $\mathcal{H}^d$. See Proposition 1.5.

We will need several lemmas to prove these theorems. In what follows, we will use the open balls in definition of $N_\mu(\epsilon, \delta)$, which does not change the limits.

Lemma 2.1. *For X a compact separable metric space, $N_\mu(\epsilon, \delta)$ is upper semi-continuous with respect to μ.*

Proof. Let $\mu_n \to \mu$ and $N_0(\epsilon, \delta) = \overline{\lim}_{n\to\infty} N_{\mu_n}(\epsilon, \delta)$. Since $N_{\mu_n}(\epsilon, \delta)$ is integer, taking subsequences if needed, we may assume that $N_{\mu_n}(\epsilon, \delta) = N_0(\epsilon, \delta)$. For any $N < N_0(\epsilon,\delta)$ the inequality $N_{\mu_n}(\epsilon,\delta) > N$ means that for any N open balls $\{B_\epsilon(x_i)\}_{i=1}^N$ we have $\mu_n(\cup_{i=1}^N B_\epsilon(x_i)) \le 1-\delta$. Hence

$$\mu\left(\bigcup_{i=1}^N B_\epsilon(x_i)\right) \le \varliminf_{n\to\infty} \mu_n\left(\bigcup_{i=1}^N B_\epsilon(x_i)\right) \le 1-\delta \quad \text{from Proposition 1.1,}$$

so $N_\mu(\epsilon,\delta) > N$. It follows that $N_\mu(\epsilon,\delta) \ge N_0(\epsilon,\delta)$. $\square$

PROOF OF THEOREM 2.2. Since $N_\mu(\epsilon,\delta)$ is upper semicontinuous with respect to μ by Lemma 2.1, so is $\frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)}$. Hence

$$\Omega_{a,\epsilon_0,\delta} = \left\{\mu \in \mathcal{P}(X) \mid \inf_{\epsilon<\epsilon_0} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)} < a\right\}$$

is open for any $a, \epsilon_0 > 0, 0 < \delta < 1$. To show that $\Omega_{a,\epsilon_0,\delta}$ is dense in $\mathcal{P}(X)$, let $\mu \in \mathcal{P}(X)$. For any $\rho > 0$ by Proposition 1.2 there is a measure μ_ρ with finite support such that $p(\mu,\mu_\rho) < \rho$. It means that for any δ, $N_{\mu_\rho}(\epsilon,\delta)$ stays bounded as $\epsilon \to 0$; so for any $a > 0, \epsilon_0 > 0$ there is an $\epsilon < \epsilon_0$ such that $\frac{\log N_{\mu_\rho}(\epsilon,\delta)}{\log(1/\epsilon)} < a$. Now we have

$$\Omega_{a,\delta} = \left\{\mu \in \mathcal{P}(X) \mid \varliminf_{\epsilon\to 0} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)} \le a\right\} \supseteq \bigcap_{n=1}^\infty \Omega_{a,1/n,\delta},$$

and this intersection is a dense G_δ set for any $a > 0, 0 < \delta < 1$. Taking intersections $\cap_{n=1}^\infty \cap_{m=1}^\infty \Omega_{1/n,1/m}$, we get the result. $\square$

Note. The result concerning Hausdorff dimension can also be shown as follows. By Theorem 2.1 and (1) there is a residual Borel set $A \subseteq X$ with $\dim A = 0$. It can be shown (see proof of Lemma 2 in [2]) that for any residual Borel set in X there is a residual set of measures in $\mathcal{P}(X)$ concentrated on this set. But then for a typical measure $\mu \in \mathcal{P}(X)$ we have $\mu(A) = 1$; so $\dim^* \mu = 0$.

Lemma 2.2. *For X a compact separable metric space, $N_\mu(\epsilon,\delta)$ is left continuous with respect to ϵ.*

PROOF. Let ϵ be a discontinuity point of $N_\mu(\epsilon,\delta)$ and let $\epsilon_n \uparrow \epsilon$. Let $N_0 = \lim_{\epsilon_n\to\epsilon} N_\mu(\epsilon_n,\delta)$ (which exists since $N_\mu(\epsilon,\delta)$ is a decreasing function of ϵ). Since $N_\mu(\epsilon,\delta)$ is integer, for large n we have $N_\mu(\epsilon_n,\delta) = N_0$. Pick any $N < N_0$ and any N open balls $\{B_\epsilon(x_i)\}_{i=1}^N$. Then for large n we have $\mu(\cup_{i=1}^N B_{\epsilon_n}(x_i)) \le 1-\delta$. Hence $\mu(\cup_{i=1}^N B_\epsilon(x_i)) = \lim_{n\to\infty} \mu(\cup_{i=1}^N B_{\epsilon_n}(x_i)) \le 1-\delta$; so $N_\mu(\epsilon,\delta) > N$. By monotonicity $N_\mu(\epsilon,\delta) = N_0$. $\square$

Lemma 2.3. *If X is a compact separable metric space, for any $\mu, \{\mu_n\}_{n=1}^{\infty}$ with $\mu_n \to \mu$, any $\epsilon > 0, 0 < \delta < 1, 0 < \nu < min(\delta, 1-\delta)$ we have*

$$N_\mu(\epsilon + \nu, \delta + \nu) \leq \varliminf_{n\to\infty} N_{\mu_n}(\epsilon, \delta).$$

Proof. Let $N_0(\epsilon, \delta) = \varliminf_{n\to\infty} N_{\mu_n}(\epsilon, \delta)$. Since $N_{\mu_n}(\epsilon, \delta)$ is integer, taking subsequences we can assume that $N_{\mu_n}(\epsilon, \delta) = N_0(\epsilon, \delta)$. This means that for each n there are $N_0(\epsilon, \delta)$ balls $B_\epsilon(x_i^n)$ such that $\mu_n(\cup_{i=1}^{N_0} B_\epsilon(x_i^n)) > 1 - \delta$.

Pick any ν as in the statement of the lemma. For large enough n we have $p(\mu_n, \mu) < \nu$, where p is a Prokhorov metric. For any Borel set A we have then $\mu_n(A) \leq \mu(A_\nu) + \nu$. Let $A = \bigcup_{i=1}^{N_0} B_\epsilon(x_i^n)$. Then $A_\nu = \bigcup_{i=1}^{N_0} B_{\epsilon+\nu}(x_i^n)$; so we have

$$\mu(\bigcup_{i=1}^{N_0} B_{\epsilon+\nu}(x_i^n)) \geq \mu_n(\bigcup_{i=1}^{N_0} B_\epsilon(x_i^n)) - \nu > 1 - \delta - \nu.$$

Hence $N_\mu(\epsilon + \nu, \delta + \nu) \leq N_0(\epsilon, \delta)$. □

Lemma 2.4. *Let X be a compact metric space. Suppose there exists a probability measure $\lambda \in \mathcal{P}(X)$ which is positive on all open sets and has $\overline{C}_{L*}(\lambda) \geq d$. Then the set of measures with $\overline{C}_{L*}(\mu) \geq d$ is dense in $\mathcal{P}(X)$.*

Proof. Pick any $\mu \in \mathcal{P}(X)$ and $\eta > 0$. Since λ is positive on balls, using finite cover of X by balls of radius $\eta/2$ we can construct an η-partition $\{I_k^\eta\}_{k=1}^{K(\eta)}$ of X with $|I_k^\eta| < \eta$ and $\lambda(I_k^\eta) > 0$ for all k. Construct a new measure $\mu_\eta = \sum_k c_k^\eta \lambda_k^\eta$, where λ_k^η is λ restricted to I_k^η, and $c_k^\eta = \mu(I_k^\eta)/\lambda(I_k^\eta)$. Then $p(\mu, \mu_\eta) \leq \eta$. Suppose for some δ

$$\varlimsup_{\epsilon\to 0} \frac{\log N_{\mu_\eta}(\epsilon, \delta)}{\log(1/\epsilon)} < d.$$

Then there are $a < d$ and $\epsilon_0 > 0$ such that $N_{\mu_\eta}(\epsilon, \delta) < \epsilon^{-a}$ for all $\epsilon < \epsilon_0$. Hence for any such ϵ there are $N_{\mu_\eta}(\epsilon, \delta)$ balls $\{B_\epsilon^i\}$ with $\mu_\eta(\cup_i B_\epsilon^i) > 1 - \delta$. But then

$$1 - \delta < \mu_\eta(\bigcup_i B_\epsilon^i) = \sum_k c_k^\eta \lambda_k^\eta(\bigcup_i B_{\epsilon_l}^i) \leq C(\eta)\lambda(\bigcup_i B_\epsilon^i),$$

where $C(\eta) = \max c_k^\eta$. It means that $N_\lambda(\epsilon, \delta_0) \leq \epsilon^{-a}$ for all $\epsilon < \epsilon_0$, where $\delta_0 = 1 - (1-\delta)/C(\eta)$. Hence we have

$$\varlimsup_{\epsilon\to 0} \frac{\log N_\lambda(\epsilon, \delta_0)}{\log(1/\epsilon)} \leq a < d,$$

which contradicts the assumption that $\overline{C}_{L*}(\lambda) \geq d$. □

PROOF OF THEOREM 2.3. By Lemmas 2.1,2.3 for any $\epsilon, \delta, \nu, \mu, \mu_n \to \mu$ as in the statements of the lemmas we have

$$N_\mu(\epsilon+\nu, \delta+\nu) \leq \varliminf_{n\to\infty} N_{\mu_n}(\epsilon, \delta) \leq \varlimsup_{n\to\infty} N_{\mu_n}(\epsilon, \delta) \leq N_\mu(\epsilon, \delta)$$

As ϵ or δ increase, $N_\mu(\epsilon, \delta)$ decreases. Fix $\mu \in \mathcal{P}(X)$. Consider all lines $\delta = \epsilon + \alpha$ on the plane with α rational. On each of these lines $N_\mu(\epsilon, \delta)$ is monotone; so it has countably many discontinuities there. All but countably many lines $\delta = \delta_0$ do not pass through any of these discontinuities. Denote the set of such δ_0's by D_μ. For each $\delta_0 \in D_\mu$ there is a dense set $E_\mu(\delta) = \{\delta_0 - \alpha \mid \alpha \in \mathbb{Q}\}$ such that for $\delta \in D_\mu, \epsilon \in E_\mu(\delta)$ we have $N_\mu(\epsilon+\nu, \delta+\nu) \to N_\mu(\epsilon, \delta)$ as $\nu \to 0$. It follows that for $\delta \in D_\mu, \epsilon \in E_\mu(\delta)$ all the inequalities above become equalities, which means $N_\mu(\epsilon, \delta) = \lim_{n\to\infty} N_{\mu_n}(\epsilon, \delta)$ for any $\mu_n \to \mu$, making $N_\bullet(\epsilon, \delta)$ continuous at μ.

By Lemma 2.4 we can choose a countable dense set of measures

$$\mathcal{M} \subseteq \left\{ \mu \in \mathcal{P}(X) \mid \inf_\delta \varlimsup_{\epsilon\to 0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} \geq d \right\}.$$

Let $D = \cap\{D_\mu \mid \mu \in \mathcal{M}\}$. D contains all but countably many points of $(0,1)$. Pick any $\delta \in D$. Fix $\eta > 0, \epsilon_0 > 0$. For any $\mu \in \mathcal{M}$ there is $\epsilon_1 < \epsilon_0$ such that

$$\frac{\log N_\mu(\epsilon_1, \delta)}{\log(1/\epsilon_1)} > d - \frac{\eta}{2}. \tag{5}$$

By Lemma 2.2 $N_\mu(\epsilon, \delta)$ is left continuous. Hence there is an interval $(\epsilon_2, \epsilon_1]$ on which $N_\mu(\epsilon, \delta) > d - \eta/2$. But $E_\mu(\delta)$ is dense in $(0,1)$; so the set $E_\mu(\delta) \cap (0, \epsilon_1)$ is not empty. For any ϵ in this set, $\frac{\log(N_\bullet(\epsilon,\delta))}{\log(1/\epsilon)}$ is continuous at μ; so there is an open neighborhood of μ such that for any measure ν in this neighborhood we have

$$\left| \frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} - \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} \right| < \frac{\eta}{2}. \tag{6}$$

Hence by (5) and (6)

$$\frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} > d - \eta;$$

so

$$\sup_{\epsilon<\epsilon_0} \frac{\log N_\nu(\epsilon, \delta)}{\log(1/\epsilon)} > d - \eta.$$

It follows that for any $\delta \in D, \eta, \epsilon_0 > 0$

$$\Omega_{\epsilon_0,\delta,\eta} = \left\{ \mu \in \mathcal{P}(X) \mid \sup_{\epsilon<\epsilon_0} \frac{\log(N_\mu(\epsilon, \delta))}{\log(1/\epsilon)} > d - \eta \right\}$$

is an open dense set in $\mathcal{P}(X)$. Then

$$\Omega_{\delta,\eta} = \left\{ \mu \in \mathcal{P}(X) \mid \varlimsup_{\epsilon \to 0} \frac{\log(N_\mu(\epsilon,\delta))}{\log(1/\epsilon)} \geq d - \eta \right\} \supseteq \bigcap_{n=1}^{\infty} \Omega_{1/n,\delta,\eta}$$

is residual in $\mathcal{P}(X)$. Taking intersections $\cap_{n=1}^{\infty} \cap_{m=1}^{\infty} \Omega_{1/n,1/m}$ concludes the proof. □

Theorem 2.4. *If X is a compact separable metric space, then a typical measure $\mu \in \mathcal{P}(X)$ has $\underline{\alpha}_\mu(x) = 0$ a.e. with respect to μ. If there is a probability measure $\lambda \in \mathcal{P}(X)$ which is positive on all open sets and has $\overline{C}_{L*}(\lambda) \geq d$, then for a typical measure $\mu \in \mathcal{P}(X)$ also $\overline{\alpha}_\mu(x) \geq d$ a.e.*

Proof. If $\dim \mu = 0$, by (3) we have $\underline{\alpha}_\mu(x) = 0$ a.e.

If $\mathrm{Dim}_* \mu \geq d$, by (2) we have $\overline{\alpha}_\mu(x) \geq d$ a.e. Theorems 2.3 and 2.2 now give the desired result. □

3 Example

Let us now construct a simple example of a measure μ with $\underline{\alpha}_\mu(x) < \overline{\alpha}_\mu(x)$ a.e. The technique used here is common with this type of problem; see, for instance, Example 5.1 in [15] or a very detailed account of a similar example in [3].

Consider a Cantor-type set $F \subset (0,1)$ constructed as follows. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence of positive integers. Let p_1, p_2 be positive integers and $r_1, r_2 > 0$ such that $p_1 r_1 < 1$, $p_2 r_2 < 1$. Replace $I_0 = [0,1]$ by p_1 disjoint intervals of length r_1. Repeat this with each of the resulting intervals and so on. Do this a_1 times; then do the same with parameters p_2, r_2 a_2 times; then again with p_1, r_1 a_3 times and so on. Let μ be a probability measure which is equally divided between all intervals on each step. Let $c_k = \sum_{i=1}^{k} a_i, b_k = \sum_{i=1}^{k} a_{2i}, d_k = \sum_{i=1}^{k} a_{2i-1}$. We will denote by I_n any interval of construction after c_n steps.

Then for the length of intervals we have

$$|I_{2k}| = r_1^{d_k} r_2^{b_k}, \quad |I_{2k+1}| = r_1^{d_{k+1}} r_2^{b_k},$$

and for the measure

$$\mu(I_{2k}) = p_1^{-d_k} p_2^{-b_k}, \quad \mu(I_{2k+1}) = p_1^{-d_{k+1}} p_2^{-b_k}.$$

Hence

$$\alpha_{2k} := \frac{\log \mu(I_{2k})}{\log |I_{2k}|} = \frac{-d_k \log p_1 - b_k \log p_2}{d_k \log r_1 + b_k \log r_2}$$
$$= -\frac{\log p_2}{\log r_2} \frac{\frac{d_k}{b_k}\frac{\log p_1}{\log p_2} + 1}{\frac{d_k}{b_k}\frac{\log r_1}{\log r_2} + 1},$$
$$\alpha_{2k+1} := \frac{\log \mu(I_{2k+1})}{\log |I_{2k+1}|} = -\frac{\log p_2}{\log r_2} \frac{\frac{d_{k+1}}{b_k}\frac{\log p_1}{\log p_2} + 1}{\frac{d_{k+1}}{b_k}\frac{\log r_1}{\log r_2} + 1}.$$

Suppose we choose $\{a_i\}$ so that $\overline{\lim\limits_{k\to\infty}}(d_{k+1}/b_k) > \underline{\lim\limits_{k\to\infty}}(d_k/b_k)$ (that is, $\overline{\lim\limits_{k\to\infty}}(a_{2k+1}/\sum\limits_{i=1}^{k} a_{2i}) > 0$). Then we have

$$\underline{\alpha} = \underline{\lim_{k\to\infty}} \frac{\log \mu(I_{2k})}{\log |I_{2k}|} < \overline{\lim_{k\to\infty}} \frac{\log \mu(I_{2k+1})}{\log |I_{2k+1}|} = \overline{\alpha}$$

For $x \in F$ let now $\{I_n\}_{n=0}^{\infty}$ be a sequence of intervals converging to x. Let $\epsilon_k = |I_{2k}|$. We have $B_{\epsilon_k}(x) \supseteq I_{2k}$; so

$$\frac{\log \mu(B_{\epsilon_k}(x))}{\log \epsilon_k} \le \frac{\log \mu(I_{2k})}{\log |I_{2k}|}.$$

Hence $\underline{\lim\limits_{\epsilon\to 0}} \frac{\log \mu(B_\epsilon(x))}{\log \epsilon} \le \underline{\alpha}$. After c_{2k+1} steps the distance between two intervals of construction is at least $c|I_{2k+1}|$, where c is the minimal distance between intervals after the first step. Let $\delta_k = c|I_{2k+1}|$. Then $F \cap B_{\delta_k}(x) \subseteq I_{2k+1}$; so

$$\frac{\log \mu(B_{\delta_k}(x))}{\log \delta_k} \ge \frac{\log \mu(I_{2k+1})}{\log c|I_{2k+1}|}.$$

Hence $\overline{\lim\limits_{\epsilon\to 0}} \frac{\log \mu(B_\epsilon(x))}{\log \epsilon} \ge \overline{\alpha}$.

The author thanks Professor G. Edgar for introducing her to this topic and for his patient guidance of her research.

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