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SUMMATION OF DIVERGENT SERIES FROM THE NONSTANDARD POINT OF VIEW[†]

Abstract

The aim of this paper is to demonstrate that several non-rigorous methods of mathematical reasoning in the field of divergent series, mostly related to the Euler and Hutton transforms, may be developed in a correct and consistent way by methods of nonstandard analysis.

1 Introduction

1.1 Some Examples

Shift operator. Let us pretend that the "operator" τ, determined by $\tau a_k = a_{k+1}$, is well defined. We may rewrite an arbitrary series

$$S = a_0 + a_1 + a_2 + a_3 + \cdots , \tag{1}$$

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as

$$S = a_0 + \tau a_0 + \tau^2 a_0 + \tau^3 a_0 + \cdots = (1 + \tau + \tau^2 + \tau^3 + \cdots)\, a_0\,, \tag{2}$$

and then as

$$S = \frac{1}{1-\tau}\, a_0\ , \tag{3}$$

assuming that

$$1 + \tau + \tau^2 + \tau^3 + \cdots = \frac{1}{1-\tau}\ . \tag{4}$$

Hutton transform. In the same spirit we may transform (3), say, by

$$\frac{1}{1-\tau} \;=\; \frac{1-\tau+1+\tau}{2-2\tau} \;=\; \frac{1}{2} + \frac{1}{1-\tau}\,\frac{\sigma}{2}\ , \tag{5}$$

where $\sigma = 1 + \tau$ (that is, $\sigma a_k = a_k + a_{k+1}$). Then, since

$$\frac{1}{1-\tau}\,\frac{\sigma}{2} = (1 + \tau + \tau^2 + \cdots)\,\frac{\sigma}{2} = \frac{1+\tau}{2} + \frac{\tau+\tau^2}{2} + \frac{\tau^2+\tau^3}{2} + \cdots\ , \tag{6}$$

we obtain

$$S = \frac{a_0}{2} + \frac{a_0+a_1}{2} + \frac{a_1+a_2}{2} + \frac{a_2+a_3}{2} + \cdots\ . \tag{7}$$

The transform (7) is known as the *Hutton transform*, see Hardy [3].

Euler transform. Another useful transformation is given by

$$\frac{1}{1-\tau} = \frac{1}{1+1-1-\tau} = \frac{1}{2}\,\frac{1}{1-\sigma/2}\ , \tag{8}$$

Then we obtain

$$\frac{1}{1-\sigma/2} = 1 + \frac{\sigma}{2} + \frac{\sigma^2}{4} + \frac{\sigma^3}{8} + \cdots\ , \tag{9}$$

and therefore

$$S = \frac{a_0}{2} + \frac{a_0+a_1}{4} + \frac{a_0+2a_1+a_2}{8} + \frac{a_0+3a_1+3a_2+a_3}{16} + \cdots\ . \tag{10}$$

The equation (10) contains the idea of the Euler summability method. (More exactly, the $(\mathsf{E}, 1)$ method, see Hardy [3].) The right-hand side of (10) is called the *Euler transform* of (1), the initial series.

Euler = iterated Hutton. We observe that formula (9) can be obtained as the result of an infinite reiteration of (5). Indeed,

$$\begin{aligned}\frac{1}{1-\tau} &= \frac{1}{2} + \frac{\sigma}{2}\,\frac{1}{1-\tau} = \frac{1}{2} + \frac{\sigma}{2}\left(\frac{1}{2} + \frac{\sigma}{2}\,\frac{1}{1-\tau}\right)\\ &= \frac{1}{2} + \frac{\sigma}{4} + \frac{\sigma^2}{4}\,\frac{1}{1-\tau} = \frac{1}{2} + \frac{\sigma}{4} + \frac{\sigma^2}{8} + \frac{\sigma^3}{8}\,\frac{1}{1-\tau} = \cdots\end{aligned} \tag{11}$$

In the same sense (10) is the iterated form of (7). Indeed,

$$\begin{aligned} S &= \frac{a_0}{2} + \left(\frac{a_0 + a_1}{2} + \frac{a_1 + a_2}{2} + \frac{a_2 + a_3}{2} + \cdots \right) \\ &= \frac{a_0}{2} + \frac{a_0 + a_1}{4} + \left(\frac{a_0 + 2a_1 + a_2}{4} + \frac{a_1 + 2a_2 + a_3}{4} + \cdots \right) \\ &= \frac{a_0}{2} + \frac{a_0 + a_1}{4} + \frac{a_0 + 2a_1 + a_2}{8} \\ &\quad + \left(\frac{a_0 + 3a_1 + 3a_2 + a_3}{8} + \frac{a_1 + 3a_2 + 3a_3 + a_4}{8} + \cdots \right) = \cdots \end{aligned} \tag{12}$$

1.2 A Classical Interpretation

The only reasonable way to give a precise meaning to τ is to define it as a linear operator acting on the infinite vector representing the sequence of terms. By analogy with the shift operator of symbolic dynamics we refer to this operator as the *shift operator* or the *shift matrix*. Thus one puts

$$\begin{aligned} T &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \\ T^* &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{aligned} \tag{13}$$

and considers T as a candidate to give a precise meaning to τ. Then, assuming that an initial series (1) is fixed, one defines the infinite vector

$$\vec{a}_k = \langle a_k,\, a_{k+1},\, a_{k+2}, \dots \rangle\,, \qquad k = 0, 1, 2, \dots\,,$$

and $\vec{a} = \vec{a}_0$. Let $[M]_k$ denote the kth row of a matrix M; we assume henceforth that enumeration of rows starts with the number 0. Then, since $a_k = [\vec{a}_k]_0$, the equality (2) takes the form

$$S = [U + T + T^2 + T^3 + \cdots]_0\, \vec{a}_0\,, \tag{2.1}$$

where U is the unit matrix. This leads to

$$S = \left[\frac{1}{U - T} \right]_0 \vec{a}_0\,, \tag{3.1}$$

again assuming that

$$(U-T)^{-1} = U + T + T^2 + T^3 + \cdots . \tag{4.1}$$

Next, we transform the fraction $(U-T)^{-1}$ by

$$(U-T)^{-1} = \frac{U}{2} + (U-T)^{-1}\,\frac{\Sigma}{2}\,, \qquad \text{where} \quad \Sigma = U + T\,; \tag{5.1}$$

and, taking $[\ldots]_0$ and applying this to $\vec{a}$, obtain (7).

We may then define the Hutton transform by a special matrix, the *Hutton matrix*

$$H = \frac{U+T^*}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}\,; \tag{14}$$

Thus if $\vec{a}$ denotes the infinite column of terms of (1) and $\vec{h}$ denotes the column of members of (7), then $\vec{h} = H\,\vec{a}$.

Then we may try to proceed the same way with the Euler equations (8), (9), (10), and finally may define the *Euler matrix* E by the condition

$$[E]_k = 2^{-(k+1)}\,[(U+T^*)^k]_k = 2^{-(k+1)}\,[(U+T)^k]_0\,, k = 0, 1, 2, \ldots . \tag{15}$$

But this is still not mathematically rigorous.

Indeed, equality (4.1) is not an algebraic identity. The infinite sum has to be interpreted as a limit. Thus, a vector space and a topology must be chosen. There is no obvious choice ensuring that equality (4.1) holds. Thus it remains dubious in what sense (3.1) could be considered true.

An analogous question is in which sense the "formal" expansion of $1 - \Sigma/2$ could be considered true?

Do these expansions belong to the type of Taylor expansions?

Finally, even (2.1), which has a more or less well defined meaning, can be treated as an equality of *concrete reals* only in the case when (1) converges; otherwise only indirect interpretations in the framework of the theory of summability are available.

1.3 The Aim of This Paper

Thus it is not clear how one can successfully interpret the equations given above in the framework of classical analysis. They are not more than a vague indication of which formulas one should try to find a classical proof for. Our aim is to interpret these equations by methods of nonstandard analysis in precisely the way proposed above.

We shall use the exposition of nonstandard analysis based on **IST**, *internal set theory*, introduced by Nelson [13]. This means that we consider things in the set universe where, in addition to membership, another elementary predicate, st, *standardness*, is assumed to act. In particular, the notion of *real number* includes both standard and nonstandard (in particular, infinitesimal and unlimited) numbers, so that $\mathbb{R}$ denotes the set of all (standard plus nonstandard) real numbers.[1] Section 2 contains a more detailed information of the nonstandard technique we use.

The idea of our approach is quite typical for nonstandard analysis. We replace series (1), which is assumed to be standard, by a sum of the form

$$S = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_\eta, \tag{16}$$

where η is an unlimited (= infinitely large = nonstandard) natural number. Sums of this type (that is, those of unlimited number of terms) will be called *hyperfinite sums* below. (*Unlimited sum* would be confusing, having a more natural meaning as a sum with unlimited value.)

Definition 1 (Shadows) Let η be an unlimited natural number. A standard series $a_0 + a_1 + a_2 + \cdots$ is the *shadow* of a hyperfinite sum $\alpha_0 + \alpha_1 + \cdots + \alpha_\eta$ iff $a_k \approx \alpha_k$ for all standard k. A standard infinite matrix Z with coefficients Z_{kl} is the *shadow* of a $(\eta+1) \times (\eta+1)$ matrix Z iff $Z_{kl} \approx \mathsf{Z}_{kl}$ for all standard k, l.

(We recall that $a \approx b$ means that the difference $a - b$ is infinitesimal.)

The are two principal questions.

First, which way should (16) relate to (1)? Two conditions can be considered as principal:

a) Series (1) (assumed to be standard) is the shadow of the sum in the right-hand side of (16).

b) The value S of the sum (16) is equal (or, perhaps, $\approx$) to a certain "natural" value of the sum of (1).

The latter condition will be explained below, but we insist that the initial series (1) is not, in general, assumed to be convergent.

[1] However our exposition could equally well be understood in terms of the superstructure approach after obvious changes: what we call *reals* and *integers* should be understood as hyperreals and hyperintegers, $\mathbb{R}$ should be changed to ${}^*\mathbb{R}$, et cetera.

It is not clear whether and how *a*) and *b*) can be realized in the most general case when no restriction on the initial series (1) is imposed.[2] We can, indeed, more or less successfully proceed with the question in the case when (1) is $\mathfrak{E}$ summable[3]. Then (1) is exactly the (formal) Taylor expansion of $f(1)$, where $f(x) = \sum a_k x^k$ in a neighborhood of 0, and we define (16) as the *nonstandard Taylor expansion* of $f(1)$ (this notion is based on Newton's interpolation theorem), see Section 3.

Second, how can we operate with hyperfinite sums like (16) such that the ways of transformation presented in Subsection 1.1 do not face the obstacles mentioned above, and such that the "shadows" of these transformations just coincide with the standard transformations?

We shall give a nonstandard "model" for equalities like $1+2+4+\cdots = -1$, that is, a way to convert the left-hand side to a hyperfinite sum exactly equal to -1 and having $1+2+4+\cdots$ as its shadow (Subsection 3.3).

Then another game is started. We introduce certain linear transformations of hyperfinite sums like (16) which correspond, in some sense, to the shift matrix and the Hutton and Euler matrices. In particular, both Hutton and Euler transforms will be expressed by certain matrices of dimension $(\eta+1)\times(\eta+1)$ whose action does not change the sum value.

We show also that, provided some conditions are satisfied, shadows of transformed hyperfinite sums are equal to the results of standard transformations of shadows of initial sums.

2 On Nonstandard Analysis

This paper does not involve sophisticated parts of nonstandard machinery. A reader having a minimal acquaintance with either the superstructure[4] or an axiomatical[5] version of nonstandard analysis can skip this section at all after a glance at the notation (which fits the axiomatical approach rather than the superstructural one). Writing this we rather had in mind a reader who has only a vague impression of "infinitesimal" methods.

[2] The solution would preassume that one knows definitely the "natural" value of the sum of an arbitrary, in particular divergent, series, which includes at least the Hutton and Euler summability methods and, perhaps, many other summability methods.

[3] We recall that according to Hardy [3], $\mathfrak{E}$ *summability* means that the corresponding power series $\sum a_k x^k$ converges on an interval containing 0 to a function regular in 1. Let f denote (the analytic continuation of) this function. Then $f(1)$ is the $\mathfrak{E}$ *sum* of $\sum a_k$. The letter $\mathfrak{E}$ is in honor of Euler who explicitly expressed the idea of this summability.

[4] We refer to Hurd and Loeb [11], Stroyan and Bayod [15], Lindstrøm [11], or Luxemburg [12].

[5] Nelson [13], *bounded set theory* **BST** of Kanovei [7], or one of external theories of Hrbaček [4] or Kawaï [10]; see also van den Berg [1], Kanovei [8], and Kanovei and Reeken [9].

First of all, usual mathematical reals (real numbers) are from now on called *standard reals*. In addition, new objects, called *nonstandard reals*, come into consideration.

$\mathbb{R} =$ all reals, ${}^{\sigma}\mathbb{R} =$ all standard (i.e. usual mathematical) reals.

Similarly, usual natural numbers will be called *standard natural numbers*; in addition, there are *nonstandard natural numbers*.

$\mathbb{N} =$ all natural numbers, ${}^{\sigma}\mathbb{N} = \mathbb{N} \cap {}^{\sigma}\mathbb{R} =$ all standard natural numbers.

It is postulated that both $\mathbb{R}$ and ${}^{\sigma}\mathbb{R}$ model the same principles of classical analysis. (For instance, $\mathbb{R}$ is an ordered field having $\mathbb{N}$ as a cofinal discrete subset, simply because this is true with respect to ${}^{\sigma}\mathbb{R}$.

To make the picture nontrivial, we suppose that ${}^{\sigma}\mathbb{N}$ is a *proper* initial segment of $\mathbb{N}$.

Numbers in $\mathbb{N} \setminus {}^{\sigma}\mathbb{N}$ (i.e., nonstandard) are called *unlimited*. A real $x \in \mathbb{R}$ is *unlimited* iff $|x| \geq \kappa$ for an unlimited natural κ; otherwise (i.e. when $|x| \leq n$ for a standard natural n) it is *limited*. A limited natural number is necessarily standard, but a limited real may well be nonstandard.

One also shows the existence of *infinitesimal* reals: those $x \neq 0$ in $\mathbb{R}$ which satisfy $|x| < c$ for any standard $c > 0$.

This initial information gives some impression of the nonstandard approach, but does not suffice to present the amount of nonstandard technique we actually need, in particular, because we shall use "hyperfinite sequences" of reals — those of the form $x_1, x_2, \ldots, x_\kappa$, where κ is an unlimited natural number, the treatment of which needs some care. A radical way to correctly apply nonstandard reasoning to everything one may ever need to use, is to work in a nonstandard extension of the whole set universe of "standard" mathematics. This can be handled in the framework of a nonstandard set theory. We present here, in brief, how Nelson's **IST** or bounded set theory **BST** of Kanovei and Reeken [9] (the differences between them are not essential here) works.

Similarly to what is stated above, the sets in the set universe of usual, "standard" mathematics — this universe is now denoted by $\mathbb{S}$ — will be called "standard sets". It is postulated that $\mathbb{S}$ is a proper part of a much larger universe $\mathbb{V}$ which includes both standard and nonstandard sets; $\mathbb{S}$ is distinguished in $\mathbb{V}$ by a a special predicate st of standardness: $\mathbb{S} = \{x : \mathsf{st}\, x\}$.

The considerations at the beginning of this section are accommodated in this general framework as follows:

$$\mathbb{R} \text{ and } \mathbb{N} = \text{resp. reals and natural numbers in } \mathbb{V};$$

$$ {}^{\sigma}\mathbb{R} \text{ and } {}^{\sigma}\mathbb{N} = \text{resp. reals and natural numbers in } \mathbb{S};$$

so that in some sense $\mathbb{S} = {}^{\sigma}\mathbb{V}$.

Interactions between sets in $\mathbb{S}$ and $\mathbb{V}$ are regulated by a few simple "postulates", which we present here informally.

Transfer In particular it implies that both $\mathbb{V}$ and $\mathbb{S}$ model the axioms of Zermelo-Fraenkel theory with the Axiom of Choice (**ZFC**) — the set theoretic basis of current mathematics. It formalizes the idea that nonstandard objects behave like standard objects, an idea which probably, *cum grano salis*, dates back to Leibniz.

Idealization It implies the existence of a variety of nonstandard sets in $\mathbb{V}$, in particular, nonstandard natural and real numbers.

Standardization It ensures that there are no subcollections of elements of a standard set which do not correspond to a standard subset of that set. It also has some important consequences, in particular, implies that ${}^{\sigma}\mathbb{N}$ (standard natural numbers) is an initial segment of $\mathbb{N}$ (all natural numbers).

It is not our aim here to present the exact formulations of the "postulates" or describe in detail how one develops nonstandard mathematics on this basis; we refer the interested reader to the sources mentioned in footnote 5. We think however that the given explanation is quite sufficient to understand the reasoning in the remainder of the paper. Readers acquainted mostly with the "superstructure" version of nonstandard technique will easily interpret our reasoning in their favorite system.

We conclude with a special definition: sums of the form (16) in Subsection 1.3, where η is an unlimited natural number, will be called *hyperfinite sums*.

3 Nonstandard Taylor Expansion

Thus the first problem is to find a hyperfinite sum (16) which corresponds, in the sense mentioned above, to the initial series (1), which is assumed to be standard. The idea is to define (16) as a nonstandard Taylor expansion. Thus we have to take some space to present the relevant definitions in the domain of interpolation theory.

By the way, the exposition is based on the original idea, due to Newton and Taylor, of the definition of Taylor series, not that which is currently detailed in textbooks on calculus (and which was developed, probably, by Lagrange). It is very close to what we now call nonstandard analysis (in a narrow sense).

Thus we first introduce the nonstandard version of Taylor expansion, then show that, provided certain conditions are satisfied, the standard expansion is the shadow of the nonstandard one, and finally consider an example.

3.1 Definition of the Expansion

Actually what is called "nonstandard Taylor expansion" here is a very well known tool in numerical analysis: Newton's interpolation formula, which we intend to apply in the case when the sequence of the points of interpolation has an unlimited number of points and all elements with standard indices are infinitesimal.

We recall some definitions.

Divided differences are defined as follows:

$$\mathbf{D}F[x_0,\ldots,x_k] = \sum_{m=0}^{k} F(x_m) \prod_{l=0,\, l\neq m}^{k} \frac{1}{x_m - x_l}\,,$$

and separately $\mathbf{D}F[x_0] = F(x_0)$ in the case $k=0$; here $x_0,\ x_1,\ x_2,\ \ldots,\ x_k$ are arbitrary real numbers, F an arbitrary function defined at each x_i and taking values in a vector space over $\mathbb{R}$ (say in $\mathbb{R}$ itself or in the space of square matrices of a certain dimension). The definition is permutation invariant: $\mathbf{D}F[x_0,\ldots,x_k] = \mathbf{D}F[x_0',\ldots,x_k']$, whenever $x_0',\ldots,x_k'$ is a permutation of $x_0,\ldots,x_k$.

Divided differences can also be introduced by induction on k:

$$\mathbf{D}F[x_0,x_1,\ldots,x_{k+1}] = \frac{\mathbf{D}F[x_1,\ldots,x_{k+1}] - \mathbf{D}F[x_0,\ldots,x_k]}{x_{k+1}-x_0}\,. \tag{17}$$

In particular,

$$\mathbf{D}F[x_i,x_j] = \frac{F(x_j)-F(x_i)}{x_j - x_i}\,.$$

The *generalized power* is defined by

$$\xi^{(k)} = (\xi - x_0)\,(\xi - x_1)\cdots(\xi - x_{k-1})\,; \quad \text{separately} \quad \xi^{(0)} = 1\,. \tag{18}$$

Proposition 2 (Newton's interpolation theorem, see e.g. Singer [14]) *Let* η *be a natural number,* $x_0,\ldots,x_\eta$ *be pairwise different, and* $F(x_i)$ *defined for all* $i=0,\ldots,\eta$*. Then*

$$F(x_\eta) = \sum_{k=0}^{\eta} \mathbf{D}F[x_0,x_1,\ldots,x_k]\, x_\eta^{(k)}.$$

Definition 3 (Nonstandard Taylor expansion) *The right-hand side in the equality of Proposition* 2 *is the nonstandard Taylor expansion of* $F(x_\eta)$ *with respect to points* $x_0, \ldots, x_\eta$.

3.2 Shadow of Nonstandard Expansion

In this subsection we prove that under certain conditions the standard Taylor expansion of a given function f is the shadow of the nonstandard expansion.

Theorem 4 *Assume that* (1) *is standard and* $\mathfrak{E}$ *summable. Let* η *be an unlimited natural number,* $x_k \approx 0$ *for all limited* k*, and* $x_\eta = 1$*. Then the nonstandard Taylor expansion of the function* $f(x) = \sum a_k x^k$ *with respect to* x_k*,* $k = 0, 1, \ldots, \eta$*, has a value equal to the* $\mathfrak{E}$ *sum of* (1) *and also has* (1) *as its shadow.*

PROOF. Thus let the corresponding power series $\sum a_k x^k$ converge on an open interval containing 0 to a function regular in 1. Let f denote (the analytic continuation of) $\sum a_k x^k$. We put $S = f(1)$, the $\mathfrak{E}$ sum of (1). In this case f is standard and (1) is the Taylor expansion of $f(1)$ (which may not converge to S, of course), so that the members of (1) satisfy

$$a_k = \frac{f^{(k)}(0)}{k\,!} \quad \text{for all } k\,.$$

It is asserted that the series (16), defined by

$$\alpha_k = \mathbf{D} f[x_0, \ldots, x_k]\, 1^{(k)}\,, \quad k = 0,\, 1,\, \ldots, \eta\,,$$

satisfies the theorem. Newton's theorem implies that the sum of (16) is equal to the $\mathfrak{E}$ sum of (1). It remains to check that (1) is the shadow of (16), that is, $a_k \approx \alpha_k$ for all limited k.

It is known (see e.g. Gel'fond [2] or Jordan [6]) that for all k there is a ϑ such that

$$\mathbf{D}[x_0, \ldots, x_k] = f^{(k)}(\vartheta)\,(k\,!)^{-1}$$

where ϑ is a number between the least and the largest among $x_0, \ldots, x_k$. But all standard order derivatives of f are standard continuous functions. Furthermore ϑ is infinitely close to 0 for any standard k. Finally $1^{(k)} \approx 1$ for standard k. □

The assumption of $\mathfrak{E}$ summability is rather restrictive, of course; indeed, very interesting *rapidly divergent* series like $\sum (-1)^n\, n\,!$ are excluded from consideration. However the remainder of the exposition, related to nonstandard transformations of (16), (when a connection with (1) is not assumed) does not actually depend on the analytical properties of f.

3.3 An Example of a Nonstandard Expansion

Let f be defined by $f(x) = (h - tx)^{-1}$, for fixed h and t. Then

$$\mathbf{D}f[x_0, \dots, x_k] = f(x_0)\, f(x_1)\, \dots\, f(x_k)\, t^k, \tag{19}$$

which can be easily proved by induction on k, using (17), for all pairwise different $x_0, \dots, x_k$. Then Newton's theorem takes the form

$$\frac{1}{h - tx_\eta} = \sum_{k=0}^{\eta} f(x_0)\, \dots\, f(x_k)\, t^k\, x_\eta^{(k)}.$$

Assume that $x_k = k\,\eta^{-1}$ for all k (an *equidistant* system of points) and $h = 1$. Then

$$\frac{1}{1-t} = \sum_{k=0}^{\eta} \frac{1^{(k)}\, t^k}{1\,(1 - t\,\eta^{-1})\, \dots\, (1 - kt\,\eta^{-1})}\,, \tag{20}$$

The standard version is as follows

$$(1-t)^{-1} = 1 + t + t^2 + t^3 + \cdots, \qquad \text{provided } |t| < 1. \tag{21}$$

Assume that t is standard. If η is unlimited then every term with limited index k in (20) is infinitely close to the corresponding term in (21) — thus (21) is the shadow of (20). Take notice that (21) holds (in the sense of convergence of the series) in the domain $|t| < 1$, while (20) requires only that $k\,t\,\eta^{-1} \neq 1$ for $k = 0, \dots, \eta$.

To demonstrate how this works we take $t = 2$. Then

$$\frac{1}{1-2} = -1 = \sum_{k=0}^{\eta} \frac{1(1-\eta^{-1}))(1-2\,\eta^{-1})\dots(1-[k-1]\,\eta^{-1})}{1(1-2\,\eta^{-1})(1-4\,\eta^{-1})\dots(1-2k\,\eta^{-1})}\, 2^k. \tag{22}$$

The infinitely large η must be odd, of course, to avoid division by zero. Let α_k denote the k th member of the sum (22). Then $\alpha_k \approx 2^k$ for finite k, hence (22) may be considered as a nonstandard version of the equation $1+2+4+8+\cdots = -1$, which, in turn, is the shadow of (22).

Numerically, in this case, the values of α_k increase exponentially (even more rapidly) for unlimited $k < \eta/2$. Then the signs of α_k begin to oscillate, but the absolute values continue to increase in the domain $k < 2\eta/3$ exponentially. In the domain $2\eta/3 < k < 3\eta/4$ the absolute values increase more slowly. Finally in the domain $3\eta/4 < k < \eta$ the absolute values decrease, but remain unlimited since $|\alpha_k| > 2^k$ for all k.

But in the case $t = -2$ the last terms are infinitesimal.

4 The Shift Operator

The aim of this section is to give a nonstandard interpretation for the equations (1), (2), (3), (4). More precisely, we want to figure out how we can operate with hyperfinite sums (16) in analogy to the transformations (2)–(4) of the standard series (1).

4.1 Unsuccessful Approach

It is a natural idea to consider the matrix

$$\mathsf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{of dimension } (\eta + 1) \times (\eta + 1)$$

as something which is expected to act like τ. We suppose that a hyperfinite sum (16) is fixed and define the $(\eta + 1)$vector

$$\vec{\alpha}_k = \langle \alpha_k, \alpha_{k+1}, \dots, \alpha_\eta, 0, 0, \dots, 0 \rangle \quad (k \text{ zeros}) .$$

We remember that $[M]_l$ denotes the lth row of a matrix M and rows are enumerated from top to bottom starting with 0 as the number of the top row. Then $\vec{\alpha}_k = \mathsf{T}^k \, \vec{\alpha}_0$ and $[\vec{\alpha}_k]_0 = \alpha_k$; therefore,

$$[\mathsf{T}^k]_0 \, \vec{\alpha}_0 = [\mathsf{T}^k \, \vec{\alpha}_0]_0 = [\vec{\alpha}_k]_0 = \alpha_k .$$

The sum (16) takes the form

$$\mathsf{S} = [\mathsf{U} + \mathsf{T} + \mathsf{T}^2 + \dots + \mathsf{T}^\eta \,]_0 \, \vec{\alpha}_0 , \tag{23}$$

(U denotes the unit matrix of the corresponding dimension). Moreover, in the case of finite matrices the following equality is an algebraic identity

$$(\mathsf{U} - \mathsf{T})^{-1} = \mathsf{U} + \mathsf{T} + \mathsf{T}^2 + \dots + \mathsf{T}^\eta , \tag{24}$$

and equality (23) could be considered as a nonstandard counterpart of (2).

There is, however, a problem. As soon as we are committed to treat what is supposed to be a nonstandard version of (2) as a nonstandard expansion of what is supposed to be a nonstandard version of $(1 - \tau)^{-1}$, we should demonstrate that T^k is equal to the kth member of the nonstandard expansion of the left-hand side, or at least that the 0th rows coincide. It is not likely that a good solution for this problem can be obtained in the frameworks of the approach based on (24) and (23): indeed, if one defines $F(x) = (\mathsf{U} - x\mathsf{T})^{-1}$ then $\mathbf{D}F[x_0, x_1] \neq \mathsf{T}$ provided $x_0 = 0$ and $x_1 \neq x_0$.

4.2 Successful Approach

Thus the idea is to forget about (24) and (23) and, first of all, consider (16) in the form which is implied by Newton's interpolation theorem, that is, as

$$\mathsf{S} = \mathsf{a}_0 + \mathsf{a}_1\, 1^{(1)} + \mathsf{a}_2\, 1^{(2)} + \cdots + \mathsf{a}_\eta\, 1^{(\eta)}\ . \tag{$1'$}$$

The following is assumed:

1*. η is an unlimited natural number.
2*. The points x_k, $0 \le k \le \eta$, are fixed.
3*. $\xi^{(k)} = (\xi - x_0)(\xi - x_1)\cdots(\xi - x_{k-1})$ in accordance with (18).

The factors a_k are, however, arbitrary (say we do not assume that they are explicitly connected with a certain function in the sense of a nonstandard Taylor expansion). Then we put

$$\vec{\mathsf{a}}_k = \langle \mathsf{a}_k, \mathsf{a}_{k+1}, \ldots, \mathsf{a}_\eta, 0, 0, \ldots, 0\rangle \qquad (k \text{ zeros}),$$

separately $\vec{\mathsf{a}} = \vec{\mathsf{a}}_0$, so that $\vec{\mathsf{a}}_k = \mathsf{T}^k\,\vec{\mathsf{a}}_0$ and $[\vec{\mathsf{a}}_k]_0 = \mathsf{a}_k$; hence

$$[\mathsf{T}^k]_0\,\vec{\mathsf{a}}_0 = [\mathsf{T}^k\,\vec{\mathsf{a}}_0]_0 = [\vec{\mathsf{a}}_k]_0 = \mathsf{a}_k.$$

The equality ($1'$) takes the form

$$\mathsf{S} = [\mathsf{U} + \mathsf{T}\, 1^{(1)} + \mathsf{T}^2\, 1^{(2)} + \cdots + \mathsf{T}^\eta\, 1^{(\eta)}]_0\, \vec{\mathsf{a}}_0\,, \tag{$2'$}$$

which we consider as the nonstandard counterpart of (2).

To see that now the problem at the end of Subsection 4.1 can be solved, we replace the unit matrix U by the matrix $\mathbb{1} = \mathsf{U} + \mathsf{X}\mathsf{T}$, where

$$\mathsf{X} = \begin{pmatrix} x_0 & 0 & 0 & 0 & \ldots \\ 0 & x_1 & 0 & 0 & \ldots \\ 0 & 0 & x_2 & 0 & \ldots \\ 0 & 0 & 0 & x_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad \begin{array}{l} \text{of dimension} \\ (\eta+1)\times(\eta+1)\ , \end{array}$$

so that

$$\mathbb{1} = \begin{pmatrix} 1 & x_0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & x_1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & x_2 & 0 & \ldots \\ 0 & 0 & 0 & 1 & x_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \quad \begin{array}{l} \text{of dimension} \\ (\eta+1)\times(\eta+1)\ . \end{array}$$

Then we remark that $(\mathbb{1} - x\mathsf{T})^{-1}$ exists in the case of finite matrices, put

$$\mathsf{F}(x) = (\mathbb{1} - x\mathsf{T})^{-1} \;=\; \frac{1}{\mathbb{1} - x\mathsf{T}}\,, \tag{25}$$

and demonstrate that the nonstandard Taylor expansion of $\mathsf{F}(1)$ leads to the right-hand side of $(2')$. This is based on the following technical theorem.

Theorem 5 *Assume that* $\kappa \le \eta$ *and points* $x_0, \dots, x_\kappa$ *are pairwise different. Then*

$$[\mathsf{F}(x_\kappa)]_0 = \left[\frac{1}{\mathbb{1} - x_\kappa \,\mathsf{T}}\right]_0 = \sum_{k=0}^{\kappa} [\mathsf{T}^k]_0 \; {x_\kappa}^{(k)} \,.$$

PROOF. It is sufficient to verify the following equality:

$$[\mathsf{T}^k]_0 = [\,\mathbf{D}\mathsf{F}[x_0, \dots, x_k]\,]_0 \;; \tag{$\star$}$$

then the result would follow by Newton's theorem. To check $(\star)$, we prove

Lemma 6 $\mathbf{D}\mathsf{F}[\xi_0, \xi_1, \xi_2, \dots, \xi_k]{=}\mathsf{F}(\xi_0)\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2)\dots\mathsf{T}\mathsf{F}(\xi_k)$, *provided* $k \le \kappa$ *and* $\xi_0, \xi_1, \dots, \xi_k$ *are arbitrary pairwise different real numbers.*

PROOF. This is evident for $k = 0$ and easy for $k = 1$. Indeed,

$$\begin{aligned}\mathbf{D}\mathsf{F}[\xi, \zeta] &= \frac{\mathsf{F}(\zeta) - \mathsf{F}(\xi)}{\zeta - \xi} = \frac{1}{\zeta - \xi}\; \mathsf{F}(\xi)\,[\mathsf{F}(\xi)^{-1} - \mathsf{F}(\zeta)^{-1}]\,\mathsf{F}(\zeta) \\ &= \frac{1}{\zeta - \xi}\; \mathsf{F}(\xi)\,(\mathbb{1} - \xi\mathsf{T} - \mathbb{1} + \zeta\mathsf{T})\; \mathsf{F}(\zeta) = \mathsf{F}(\xi)\;\mathsf{T}\,\mathsf{F}(\zeta)\,.\end{aligned}$$

We proceed with the induction step. To shorten formulas we consider the step from 3 to 4. Thus let $\mathbf{D}\mathsf{F}[\xi_0, \xi_1, \xi_2] = \mathsf{F}(\xi_0)\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2)$. Then

$$\mathbf{D}\mathsf{F}[\xi_1, \xi_2, \xi_3] = \mathbf{D}\mathsf{F}[\xi_3, \xi_1, \xi_2] = \mathsf{F}(\xi_3)\;\mathsf{T}\,\mathsf{F}(\xi_1)\;\mathsf{T}\,\mathsf{F}(\xi_2)\,.$$

Hence

$$\begin{aligned}\mathbf{D}\mathsf{F}[\xi_0,\xi_1, \xi_2, \xi_3] &= \frac{1}{\xi_3 - \xi_0}(\mathbf{D}\mathsf{F}[\xi_1, \xi_2, \xi_3] - \mathbf{D}\mathsf{F}[\xi_0, \xi_1, \xi_2]) \\ &= \frac{1}{\xi_3 - \xi_0}[\mathsf{F}(\xi_3) - \mathsf{F}(\xi_0)]\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2) \\ &= \mathbf{D}\mathsf{F}[\xi_0, \xi_3]\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2) \\ &= \mathsf{F}(\xi_0)\mathsf{T}\mathsf{F}(\xi_3)\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2) = \mathsf{F}(\xi_0)\mathsf{T}\mathsf{F}(\xi_1)\mathsf{T}\mathsf{F}(\xi_2)\mathsf{T}\mathsf{F}(\xi_3),\end{aligned}$$

as required. (We have used (17), the result obtained for $k = 1$, and the permutation invariance.)

To continue the proof of $(\star)$, we note that

$$[\mathsf{T}^k\; \mathsf{F}(x_k)]_0 = [\mathsf{T}^k]_0\,, \qquad k = 0, 1, 2, \dots, \eta\,. \tag{26}$$

Indeed, first, $[\mathsf{T}^k\ \mathsf{F}(x_k)]_0 = [\mathsf{F}(x_k)]_k = [(\mathbb{1} - x_k\mathsf{T})^{-1}]_k$. On the other hand, we have $[(\mathbb{1} - x_k\mathsf{T})]_k = [\mathsf{U}]_k$, hence $[(\mathbb{1} - x_k\mathsf{T})^{-1}]_k = [\mathsf{U}]_k$. Therefore $[\mathsf{T}^k\ \mathsf{F}(x_k)]_0 = [\mathsf{U}]_k = [\mathsf{T}^k]_0$.

Applying (26) consecutively for $k = 0, 1, 2, \ldots, \kappa$ to the right–hand side of the equality of the Lemma 6 and having in mind that $[AB]_0 = [A]_0\, B$, we get $(\star)$ and accomplish the proof of Theorem 5. $\square$

The next corollary is related to the case $\kappa = \eta$. Thus, in addition to the requirements 1^*, 2^*, 3^*, stated above we assume hereafter that

4^*. Points $x_0, \ldots, x_\eta$ are pairwise different and $x_\eta = 1$.

Corollary 7

$$[\mathsf{F}(1)]_0 = \left[\frac{1}{\mathbb{1} - \mathsf{T}}\right]_0 = \sum_{k=0}^{\eta} [\mathsf{T}^k]_0\, 1^{(k)}\,, \tag{4'}$$

and in addition $[\mathsf{T}^k]_0 = [\mathbf{D}\mathsf{F}[x_0, \ldots, x_k]\,]_0$, *so that* (4′) *is the nonstandard Taylor expansion of the expression* $(\mathbb{1} - x\mathsf{T})^{-1}$ *for* $x = 1$.

The equality (4′) is the nonstandard counterpart of (4). But (4′) has the precise meaning of a sum and a complete proof which is based on the nonstandard Taylor expansion of the function (25) and does not depend on any special assumption about the nature of the initial series (1′).

Now we multiply both sides of (4′) with $\vec{\mathsf{a}}_0$ and, using (2′) obtain the following counterpart of (3):

$$\mathsf{S} = \left[\frac{1}{\mathbb{1} - \mathsf{T}}\right]_0 \vec{\mathsf{a}}_0\,, \quad \text{hence} \;\; \mathsf{S} = [\mathsf{F}(1)]_0\, \vec{\mathsf{a}}_0\ . \tag{3'}$$

Remark An interesting result was obtained in passing by: $\mathsf{F}(\xi)\,\mathsf{T}\,\mathsf{F}(\zeta) = \mathsf{F}(\zeta)\,\mathsf{T}\,\mathsf{F}(\xi)$. (Apply Lemma 6 in the case $k = 2$.) This is quite surprising because the matrices $\mathsf{F}(\xi)$ do not in general commute with T and with each other. This clarifies the connection between the lemma and (19) — take notice that $\mathsf{F}(x)$ coincides with $(h - tx)^{-1}$ when $h = \mathbb{1}$ and $t = \mathsf{T}$. Putting in (19) factors t between the factors $f(x_k)$ we achieve the same formal structure in both formulas. But unlike the case of real-valued functions, relevant to (19), one cannot, in general, commute factors in this setting.

5 Nonstandard Hutton Transform

Thus the similarity between (1), (2), (3), (4) on the one hand and (1′), (2′), (3′), (4′) on the other, is based on the identification of τ with T, 1 (in the denominator $1-\tau$) with $\mathbb{I}$, and on the adjoining of factors $1^{(k)}$ and taking the zero row. The aim of this section is to develop the Hutton transform from this standpoint. We introduce an $(\eta+1)\times(\eta+1)$matrix H, having the Hutton matrix H of Subsection 1.2 as shadow, and then demonstrate that the nonformal treatment of the Hutton transform (equations (5), (6), and (7)) can be carried out both adequately and in a mathematically precise way in the framework of the nonstandard approach.

An unlimited natural number η, a system of points of interpolation x_k, $k\le\eta$, satisfying conditions 1^*, 2^*, 3^*, 4^* of Section 4, a hyperfinite sum

$$\mathsf{S}=\mathsf{a}_0+\mathsf{a}_1\,1^{(1)}+\mathsf{a}_2\,1^{(2)}+\cdots+\mathsf{a}_\eta\,1^{(\eta)}, \tag{1'}$$

and the corresponding vector $\vec{\mathsf{a}}=\vec{\mathsf{a}}_0=\langle\mathsf{a}_0,\dots,\mathsf{a}_\eta\rangle$, are again given.

5.1 Nonstandard Hutton Matrix

First of all we define a matrix.

Definition 8 (Nonstandard Hutton matrix) The *Hutton matrix* H of dimension $(\eta+1)\times(\eta+1)$ is introduced by

$$\mathsf{H}=\frac{1}{2}\begin{pmatrix} 1+x_0 & 0 & 0 & 0 & 0 & \dots\\ 1 & 1+x_1 & 0 & 0 & 0 & \dots\\ 0 & 1 & 1+x_2 & 0 & 0 & \dots\\ 0 & 0 & 1 & 1+x_3 & 0 & \dots\\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

The *Hutton transform* $\vec{\mathsf{h}}$ of $\vec{\mathsf{a}}$ (both $\vec{\mathsf{h}}$ and $\vec{\mathsf{a}}=\vec{\mathsf{a}}_0$ are vectors containing $\eta+1$ elements) is defined by $\vec{\mathsf{h}}=\mathsf{H}\,\vec{\mathsf{a}}$, that is,

$$\mathsf{h}_0=\frac{1+x_0}{2}\,\mathsf{a}_0\,,\quad\text{and}\quad \mathsf{h}_k=\frac{\mathsf{a}_{k-1}+(1+x_k)\,\mathsf{a}_k}{2}\quad\text{for}\quad 1\le k\le\eta\,.$$

Thus, provided all x_k, k standard, are infinitely small, the standard matrix H introduced in Subsection 1.2 is the shadow of H in the sense of Definition 1. Therefore, the matrix H, acting on $\vec{\mathsf{a}}$, converts the hyperfinite sum $(1')$ to

$$\mathsf{S}=\sum_{k=0}^{\eta}\mathsf{h}_k\,1^{(k)}=\frac{1+x_0}{2}\,\mathsf{a}_0+\sum_{k=1}^{\eta}\frac{\mathsf{a}_{k-1}+(1+x_k)\,\mathsf{a}_k}{2}\,1^{(k)}\,. \tag{7'}$$

Proposition 9 *Assume that a standard series* (1) *is the shadow of* $(1')$, η *is unlimited, and* $x_k\approx0$ *for all limited numbers* k. *Then* (7) *is the shadow of* $(7')$.

Equality (7′) is easily verifiable; the coefficient of a_k in the right-hand side is, indeed, equal to $1^{(k)}$, in accordance with (1′).

The question is how (7′) can be obtained by a transformation similar to (5), (6). This is explained in the remainder of this section.

5.2 The Hutton Matrix as the Result of a Taylor Expansion

The right way of transformation is a bit tricky. We first observe that

$$[\mathsf{U}]_0 = [\mathbb{1} - x_0 \mathsf{T}]_0 = 2^{-1} \left[(1 + x_0) (\mathbb{1} - \mathsf{T}) + (1 - x_0) (\mathbb{1} + \mathsf{T}) \right]_0 .$$

Therefore

$$\left[\frac{1}{\mathbb{1} - \mathsf{T}} \right]_0 = [\mathsf{U}]_0 \, \frac{1}{\mathbb{1} - \mathsf{T}} = \frac{1 + x_0}{2} \, [\mathsf{U}]_0 + \frac{1 - x_0}{2} \left[\Sigma \, \frac{1}{\mathbb{1} - \mathsf{T}} \right]_0 , \qquad (5')$$

where $\Sigma = \mathbb{1} + \mathsf{T}$. This is the nonstandard counterpart of (5).

To go ahead, we have to expand the fraction $(\mathbb{1} - \mathsf{T})^{-1}$ at the right-hand side by Corollary 7. This does not seem to be easy; indeed, equality (4′) of Corollary 7 is relevant to the 0th row of the matrix $(\mathbb{1} - \mathsf{T})^{-1}$, which is insufficient since the fraction is placed in (5′) as a right-hand factor. Thus we have to figure out the commutation rules in this case.

We let T^* be the transpose of T, so that

$$\mathsf{T}^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{of dimension } (\eta + 1) \times (\eta + 1) \, ,$$

(then by the way $\mathsf{H} = 2^{-1}(\mathsf{U} + \mathsf{T}^* + \mathsf{X})$) and put ${}^k\mathsf{X} = \mathsf{T}^k \, \mathsf{X} \, \mathsf{T}^{*k}$, so that

$${}^k\mathsf{X} = \begin{pmatrix} x_k & 0 & 0 & 0 & \dots \\ 0 & x_{k+1} & 0 & 0 & \dots \\ 0 & 0 & x_{k+2} & 0 & \dots \\ 0 & 0 & 0 & x_{k+3} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad \text{of dimension } (\eta + 1) \times (\eta + 1) \, ,$$

having numbers $x_k, x_{k+1}, \dots, x_\eta, 0, \dots, 0$ (k zeros) on the diagonal. Then we define

$${}^k\mathbb{1} = \mathsf{U} + {}^k\mathsf{X} \, \mathsf{T} \, , \qquad {}^k\Sigma = {}^k\mathbb{1} + \mathsf{T} \, , \qquad \text{and} \qquad {}^k\mathsf{F}(x) = ({}^k\mathbb{1} - x\mathsf{T})^{-1}.$$

In particular, ${}^0\mathsf{X} = \mathsf{X}$, ${}^0\mathbb{1} = \mathbb{1}$, ${}^0\Sigma = \Sigma$, and ${}^0\mathsf{F} = \mathsf{F}$.

Lemma 10 (Commutation rules)

$$\mathsf{T}\,{}^k\mathsf{X} = {}^{k+1}\mathsf{X}\,\mathsf{T}\,, \quad \mathsf{T}\,{}^k\mathbb{1} = {}^{k+1}\mathbb{1}\,\mathsf{T}\,, \quad \textit{and} \quad \mathsf{T}\,{}^k\Sigma = {}^{k+1}\Sigma\,\mathsf{T}\,; \tag{27}$$

$$\mathsf{T}\,{}^k\mathsf{F}(\xi) = {}^{k+1}\mathsf{F}(\xi)\,\mathsf{T}\,; \tag{28}$$

$${}^k\mathbb{1}\;{}^k\mathsf{F}(\xi) = {}^{k+1}\mathsf{F}(\xi)\;{}^{k+1}\mathbb{1} \quad \textit{and} \quad {}^k\Sigma\;{}^k\mathsf{F}(\xi) = {}^{k+1}\mathsf{F}(\xi)\;{}^{k+1}\Sigma\,. \tag{29}$$

PROOF. We start with (27). By definition,

$$\mathsf{T}\,{}^k\mathsf{X} = \mathsf{T}^{k+1}\,\mathsf{X}\,\mathsf{T}^{*k} = \mathsf{T}^{k+1}\,\mathsf{X}\,\mathsf{T}^{*(k+1)}\,\mathsf{T} = {}^{k+1}\mathsf{X}\,\mathsf{T}\,.$$

The principal second equality is implied by the fact that, since the matrix $\mathsf{X}' = \mathsf{T}^{k+1}\,X\,\mathsf{T}^{*k}$ has only zeros in the first column, $\mathsf{X}'\,\mathsf{T}^*\,\mathsf{T} = \mathsf{X}'$. Therefore,

$$\mathsf{T}\,{}^k\mathbb{1} = \mathsf{T}\mathsf{U} + \mathsf{T}\,{}^k\mathsf{X}\,\mathsf{T} = \mathsf{U}\mathsf{T} + {}^{k+1}\mathsf{X}\,\mathsf{T}\,\mathsf{T} = {}^{k+1}\mathbb{1}\,\mathsf{T}\,.$$

The equality for ${}^k\Sigma$ immediately follows from the one for ${}^k\mathbb{1}$.

We prove (28). Multiplying both sides of (28) by ${}^k\mathbb{1} - \xi\mathsf{T}$ as the right-hand factor and by ${}^{k+1}\mathbb{1} - \xi\mathsf{T}$ as the left-hand factor, one obtains

$$({}^{k+1}\mathbb{1} - \xi\mathsf{T})\,\mathsf{T} = \mathsf{T}\,({}^k\mathbb{1} - \xi\mathsf{T})\,,$$

which follows from (27). Finally the equalities (29) are reduced, by the same transformation, to (27). □

Corollary 11 $$\Sigma\,\frac{1}{\mathbb{1} - \mathsf{T}} = \frac{1}{{}^1\mathbb{1} - \mathsf{T}}\;\;{}^1\Sigma = {}^1\mathsf{F}(1)\;\;{}^1\Sigma\,.$$

PROOF. Use (29). □

Thus the fraction is set up as the left factor. The next step is to compute the expansion of ${}^1\mathsf{F}(1)$ with respect to the "reduced" system of points $x_1, \dots, x_\eta$. According to Theorem 5 (the case $\kappa = \eta - 1$), the required expansion has the form

$$[{}^1\mathsf{F}(1)]_0 = [{}^1\mathsf{F}(x_\eta)]_0 = \left[\frac{1}{{}^1\mathbb{1} - \mathsf{T}}\right]_0 = \sum_{k=0}^{\eta-1}\,[\mathsf{T}^k]_0\;1^{[k]}\,,$$

where $1^{[k]} = (1 - x_1)\cdots(1 - x_k) = 1^{(k+1)}(1 - x_0)^{-1}$. Thus

$$\left[\Sigma\,\frac{1}{\mathbb{1} - \mathsf{T}}\right]_0 = \left[\frac{1}{{}^1\mathbb{1} - \mathsf{T}}\;\;{}^1\Sigma\right]_0 = (1 - x_0)^{-1}\,\left[\sum_{k=1}^{\eta}\,[\mathsf{T}^{k-1}]_0\;1^{(k)}\right]\;{}^1\Sigma\,, \tag{6$'$}$$

which is the counterpart of (6). Therefore, multiplying (5′) by $\vec{\mathsf{a}} = \vec{\mathsf{a}}_0$, one obtains

$$\mathsf{S} = \left[\frac{1}{\mathbb{1} - \mathsf{T}}\right]_0 \vec{\mathsf{a}}_0 = \left[\frac{1+x_0}{2}\,[\mathsf{U}]_0 \,+\, \frac{1}{2}\,\sum_{k=1}^{\eta} [\mathsf{T}^{k-1}\ \ {}^{1}\Sigma]_0\, 1^{(k)}\right] \vec{\mathsf{a}}\,.$$

We want to demonstrate that the right-hand side of this equality is equal to (7′). This is the content of the following lemma.

Lemma 12 $2^{-1}(1+x_0)\,[\mathsf{U}]_0 = [\mathsf{H}]_0$, *and* $2^{-1}\ [\mathsf{T}^{k-1}\ \ {}^{1}\Sigma]_0 = [\mathsf{H}]_k$ *for all* $1 \le k \le \eta$.

PROOF. Notice that by definition $\mathsf{H} = 2^{-1}\,[\mathsf{U} + \mathsf{X} + \mathsf{T}^*]$; therefore $[\mathsf{H}]_0 = 2^{-1}(1+x_0)\,[\mathsf{U}]_0$. The other equality needs more efforts.

Since $[M]_k = [\mathsf{T}^k\, M]_0$, the problem is reduced to the equality

$$[\mathsf{T}^{k-1}\ \ {}^{1}\Sigma]_0 = [\mathsf{T}^k\ (\mathsf{U} + \mathsf{X} + \mathsf{T}^*)]_0\,.$$

The left–hand side is equal to $[{}^{k}\Sigma\ \mathsf{T}^{k-1}]_0$ by (27). As far as the right–hand side is concerned, we first note that $[\mathsf{T}^k\ \mathsf{T}^*]_0 = [\mathsf{T}^{k-1}]_0$ in the case $k > 1$, and $[\mathsf{T}\,\mathsf{T}^*]_0 = [\mathsf{U}]_0$ in the case $k = 1$. Thus the right–hand side is equal to

$$[\mathsf{T}^k +\ {}^{k}\mathsf{X}\,\mathsf{T}^k + \mathsf{T}^{k-1}]_0 = [(\mathsf{U} +\ {}^{k}\mathsf{X}\,\mathsf{T} + \mathsf{T})\ \mathsf{T}^{k-1}]_0 = [{}^{k}\Sigma\ \mathsf{T}^{k-1}]_0\,,$$

as required. □

6 Nonstandard Euler Transform

In this section we concentrate on the nonstandard interpretation of the Euler transform. The plan is approximately the same as in the previous section. We define a nonstandard version E of the Euler matrix E of Subsection 1.2, prove that the latter is the shadow of the former (under certain conditions) and that its action commutes with the shadow operation (this will require a bit more efforts than in the case of Hutton transform), then prove that E can be obtained by a nonstandard "simulation" of equations (8), (9), (10), and finally demonstrate that a certain reiteration of the Hutton transform of Section 5 leads to E as well.

A natural number η, a system of points of interpolation x_k, $k \le \eta$, satisfying conditions 1^*–4^* of Section 4, a hyperfinite sum (1′), and the corresponding vector $\vec{\mathsf{a}} = \vec{\mathsf{a}}_0$ continue to be fixed.

6.1 Nonstandard Euler Matrix

We recall that the standard Euler matrix E was introduced in Subsection 1.2 by $[E]_k = 2^{-(k+1)}\,[(U+T^*)^k]_k = 2^{-(k+1)}\,[(U+T)^k]_0$, $k = 0,1,2,\dots$. The nonstandard version takes the following form.

Definition 13 (Nonstandard Euler transform) We set ${}^{(k)}\mathsf{\Sigma} = {}^{k}\mathsf{\Sigma}^{k-1}\mathsf{\Sigma}\dots{}^{1}\mathsf{\Sigma}$ for all $k \le \eta$. The *Euler matrix* E is introduced by the equalities

$$[\mathsf{E}]_k = 2^{-(k+1)}\,(1+x_k)\,[{}^{(k)}\mathsf{\Sigma}]_0 \quad - \quad \text{for all } k, \quad 0 \le k \le \eta\,.$$

The *Euler transform* $\vec{\mathsf{e}}$ of $\vec{\mathsf{a}}$ (both $\vec{\mathsf{e}}$ and $\vec{\mathsf{a}} = \vec{\mathsf{a}}_0$ are vectors containing $\eta+1$ elements) is defined by $\vec{\mathsf{e}} = \mathsf{E}\,\vec{\mathsf{a}}$.

This determines E uniquely, of course, although not by a compact formula like the one by which H was introduced, see Definition 8. On the other hand, we are able now to obtain the nonstandard counterpart of (10), similar to (7′) in Section 5. It is as follows.

$$\left.\begin{aligned} \mathsf{S} = \sum_{k=0}^{\eta} \mathsf{e}_k\, 1^{(k)}\,, \quad &\text{where} \quad \langle \mathsf{e}_0,\dots,\mathsf{e}_\eta\rangle = \vec{\mathsf{e}} = \mathsf{E}\,\vec{\mathsf{a}} = \mathsf{E}\,\vec{\mathsf{a}}_0\,, \\ &\text{so that} \quad \mathsf{e}_k = 2^{-(k+1)}\,(1+x_k)\,[{}^{(k)}\mathsf{\Sigma}]_0\,\vec{\mathsf{a}}\,. \end{aligned}\right\} \tag{10′}$$

The difference between this and (7′) is that, first, we do not have a compact formula which defines e_k in terms of a_k, second, it is not evident that (10′) really follows from (1′), and third, it is not at once clear that the following "shadow" proposition holds.

Proposition 14 *Assume that* (1) *is the shadow of* (1′), η *is unlimited, and* $x_k \approx 0$ *for limited* k. *Then* (10) *is the shadow of* (10′).

PROOF. We say that a matrix M is *almost triangular* iff there exists a limited natural number n_0 such that $l \le k + n_0$ whenever $M_{kl} \ne 0$ (M_{kl} denotes the $\langle k,l\rangle$th element of M) (take notice that the relation $\ne$ rather than $\not\approx$, is used in the definition). Thus an almost triangular matrix can contain nonzero elements above the diagonal only in a very restricted way.

The matrices T, T^*, U, E, H, and ${}^{k}\mathsf{X}$, ${}^{k}\mathbb{I}$, ${}^{k}\mathsf{\Sigma}$, ${}^{(k)}\mathsf{\Sigma}$ for all *limited* k are almost triangular; the composition of a limited number of almost triangular matrices is almost triangular as well.

It is also clear that if Z is an almost triangular $(\eta+1)\times(\eta+1)$ matrix having a standard twice infinite matrix Z as shadow, and $\vec{\mathsf{v}}$ an $(\eta+1)$vector, having a standard infinite vector $\vec{v}$ as shadow, then $\mathsf{Z}\,\vec{\mathsf{v}}$ has $Z\,\vec{v}$ as shadow. (We recall that the notion of shadow was introduced by Definition 1 in Subsection 1.3.)

Take notice that, in the assumptions of Proposition 14, every ${}^k\mathsf{X}$ (k limited) has the zero matrix as shadow. Since the property of being the shadow is preserved under addition and multiplication (for almost triangular matrices), all matrices ${}^k\mathsf{\Sigma}$ have the standard matrix Σ introduced in Subsection 1.2 as shadow, and ${}^{(k)}\mathsf{\Sigma}$ has Σ^k as shadow (provided k is limited). Therefore E is the shadow of E. This completes the proof of Proposition 14. □

To illustrate Proposition 14 we present the first few values of e_k.

$$\begin{aligned}
\mathsf{e}_0 &= 2^{-1}\,\mathsf{a}_0\,;\\
\mathsf{e}_1 &= 2^{-2}(1+x_1)\,[\mathsf{a}_0 + (1+x_1)\,\mathsf{a}_1]\,;\\
\mathsf{e}_2 &= 2^{-3}(1+x_2)\,[\mathsf{a}_0 + (2+x_1+x_2)\,\mathsf{a}_1 + (1+x_2)^2\,\mathsf{a}_2]\,;\\
\mathsf{e}_3 &= 2^{-4}(1+x_3)\,\{\mathsf{a}_0 + (3+x_1+x_2+x_3)\,\mathsf{a}_1\\
&\quad + [(1+x_2)^2 + (1+x_2)(1+x_3) + (1+x_3)^2]\,\mathsf{a}_2 + (1+x_3)^3\,\mathsf{a}_3\}\,;\\
&\quad \text{and so on}\,.
\end{aligned}$$

6.2 Euler Matrix as the Result of Taylor Expansion

This subsection gives an exact model for equations (8), (9), and (10), based on the nonstandard Taylor expansion. This will result in the matrix E, as required.

The counterpart of (8) is as follows:

$$\frac{1}{\mathbb{1}-\mathsf{T}} = \frac{1}{2\mathbb{1}-\mathsf{\Sigma}} = \frac{\Phi(1/2)}{2}, \quad \text{where} \quad \Phi(y) = \frac{1}{\mathbb{1}-y\mathsf{\Sigma}}\,, \tag{8'}$$

The next equation (9) in Subsection 1.1 contains the expansion of $(1-\sigma/2)^{-1}$. This leads us to nonstandard expansion of the function $\Phi(y)$ for the value $y = 1/2$. To find a suitable system $y_0, \ldots, y_\eta$ of points of interpolation, we transform $\Phi(y)$ in the following way:

$$\Phi(y) = \frac{1}{\mathbb{1}-y\,(\mathbb{1}+\mathsf{T})} = \frac{1}{1-y}\;\frac{1}{\mathbb{1}-\frac{y}{1-y}\mathsf{T}} = \frac{1}{1-y}\,\mathsf{F}\!\left(\frac{y}{1-y}\right).$$

Thus the natural connection between the variables x and y can be given by

$$x = \frac{y}{1-y}\,, \quad \text{hence} \quad y = \frac{x}{1+x}\,.$$

This gives a hint to introduce the points y_k by

$$y_k = \frac{x_k}{1+x_k}\,, \quad \text{where} \quad k = 0, 1, 2, \ldots, \eta\,. \tag{30}$$

In particular, $y_\eta = 1/2$, and, by (18), $(1/2)^{(k)} = \prod_{i=0}^{k-1}(y_\eta - y_i)$.

Proposition 15 $(1/2)^{(k)} = \dfrac{1^{(k)}}{2^k\,(1+x_0)\,(1+x_1)\,\ldots\,(1+x_{k-1})}\,.$

Proof. By (30), we have

$$y_\eta - y_i = \frac{x_\eta - x_i}{(1+x_\eta)\,(1+x_i)} = \frac{1-x_i}{2(1+x_i)}\,. \qquad \square$$

By Newton's theorem, $\Phi(1/2)$ is expanded with respect to $y_0,\ldots,y_\eta$ as

$$\Phi(1/2) = \sum_{k=0}^{\eta} \mathbf{D}\Phi[y_0,\ldots,y_k]\ (1/2)^{(k)}\,. \tag{31}$$

where the generalized powers $(1/2)^{(k)}$ are computed by Proposition 15. The next step is to find the corresponding divided differences. This will lead to the Euler matrix.

Lemma 16 $[\,\mathbf{D}\Phi[y_0,\ldots,y_k]\,]_0 = (1+x_0)\ \ldots\ (1+x_k)\ [\,{}^{(k)}\Sigma\,]_0\,.$

Proof. We first claim that

$$\mathbf{D}\Phi[y_0,\ldots,y_k] = \Phi(y_0)\ \Sigma\ \Phi(y_1)\ \ldots\ \Sigma\ \Phi(y_k)\,.$$

The proof does not differ from the proof of Lemma 6.

Therefore, since $\Phi(y_k) = (1+x_k)\ \mathsf{F}(x_k)$, we obtain

$$\begin{aligned}\mathbf{D}\Phi[y_0,\ldots,y_k] =& (1+x_0)\ldots(1+x_k)\mathsf{F}(x_0)\Sigma\mathsf{F}(x_1)\ldots\Sigma\mathsf{F}(x_k)\\ =& (1+x_0)\cdots(1+x_k)\mathsf{F}(x_0)\,{}^{1}\mathsf{F}(x_1)\cdots{}^{k}\mathsf{F}(x_k)\,{}^{(k)}\Sigma\end{aligned} \tag{32}$$

by Lemma 10. It is asserted that $[{}^{l}\mathsf{F}(x_l)]_0 = [\mathsf{U}]_0$ for all l.

Indeed, ${}^{l}\mathsf{F}(x_l) = ({}^{l}\mathbb{I} - x_l\,\mathsf{T})^{-1}$. On the other hand,

$${}^{l}\mathbb{I} - x_l\,\mathsf{T} = \mathsf{U} + {}^{l}\mathsf{X}\,\mathsf{T} - x_l\,\mathsf{T}\,,$$

which shows that $[{}^{l}\mathbb{I} - x_l\,\mathsf{T}]_0 = [\mathsf{U}]_0$. This implies the required equality $[{}^{l}\mathsf{F}(x_l)]_0 = [\mathsf{U}]_0$.

Coming back to (32) (2nd line), we end the proof of the lemma. $\square$

Corollary 17

$$\left[\frac{1}{\mathbb{I}-\mathsf{T}}\right]_0 = \frac{1}{2}\left[\frac{1}{\mathbb{I}-\Sigma/2}\right]_0 = \sum_{k=0}^{\eta} \frac{1+x_k}{2^{k+1}}\ [\,{}^{(k)}\Sigma\,]_0\ 1^{(k)}\,. \tag{9$'$}$$

(The right equality is the nonstandard counterpart of (9), of course.)
Proof. Use Proposition 15, Lemma 16, and formula (31). □

Corollary 18

$$\mathsf{S} = \left[\frac{1}{\mathbb{1}-\mathsf{T}}\right]_0 \vec{\mathsf{a}}_0 = \frac{1}{2}\left[\frac{1}{\mathbb{1}-\Sigma/2}\right]_0 \vec{\mathsf{a}}_0 = \sum_{k=0}^{\eta} \frac{1+x_k}{2^{k+1}}\ [\,{}^{(k)}\Sigma\,]_0\ \vec{\mathsf{a}}_0\, 1^{(k)}\,.$$

Proof. Use (3′) and Corollary 17. □

Thus we have got (10′).

6.3 Euler Matrix via Reiteration of Hutton Transform

We finally demonstrate that the formula of Corollary 17 can be obtained as reiteration of the Hutton transform of Section 5, similarly to equations (11) and (12).

We observe first of all that

$$[\mathsf{U}]_0 = [{}^{k}\mathbb{1} - x_k \mathsf{T}]_0 = 2^{-1}\,[\,(1+x_k)\,({}^{k}\mathbb{1}-\mathsf{T}) + (1-x_k)\,({}^{k}\mathbb{1}+\mathsf{T})\,]_0\,.$$

Therefore using Lemma 10 (commutation rules), we obtain

$$\begin{aligned}\left[\frac{1}{{}^{k}\mathbb{1}-T}\right]_0 = [\mathsf{U}]_0\,\frac{1}{{}^{k}\mathbb{1}-T} &= \frac{1+x_k}{2}\,[\mathsf{U}]_0 + \frac{1-x_k}{2}\left[{}^{k}\Sigma\,\frac{1}{{}^{k}\mathbb{1}-T}\right]_0\\ &= \frac{1+x_k}{2}\,[\mathsf{U}]_0 + \frac{1-x_k}{2}\left[\frac{1}{{}^{k+1}\mathbb{1}-T}\right]_0\ {}^{k+1}\Sigma \end{aligned} \tag{33}$$

which is precisely the equality of the Hutton transform (5′) (together with the transformation given by the first equality of (6′)) in a generalized form.

We apply (33) consecutively, starting from $[(\mathbb{1}-\mathsf{T})^{-1}]_0$, for the values $k = 0, 1, \ldots, \eta$. The first two steps are as follows:

$$\begin{aligned}\left[\frac{1}{\mathbb{1}-\mathsf{T}}\right]_0 =& \frac{1+x_0}{2}\,[\mathsf{U}]_0 + \frac{1}{2}\left[\frac{1}{{}^{1}\mathbb{1}-\mathsf{T}}\right]_0\ {}^{1}\Sigma\ 1^{(1)}\\ =& \frac{1+x_0}{2}\,[\mathsf{U}]_0 + \frac{1+x_1}{4}\,[\,{}^{(1)}\Sigma\,]_0\,1^{(1)} + \frac{1}{4}\left[\frac{1}{{}^{2}\mathbb{1}-\mathsf{T}}\right]_0\ {}^{(2)}\Sigma\ 1^{(2)}\,.\end{aligned}$$

After η steps, the rightmost, "singular" addendum turns out to be

$$\begin{aligned}2^{-\eta}\left[\frac{1}{{}^{\eta}\mathbb{1}-\mathsf{T}}\right]_0\ {}^{(\eta)}\Sigma\ 1^{(\eta)} =& (1+x_\eta)\ 2^{-(\eta+1)}\ [\,\mathsf{U}\,]_0\,{}^{(\eta)}\Sigma\ 1^{(\eta)}\\ =& (1+x_\eta)\ 2^{-(\eta+1)}\ [\,{}^{(\eta)}\Sigma\,]_0\ 1^{(\eta)}\,,\end{aligned}$$

since, first, $x_\eta = 1$ and second (that follows from the first), $[{}^\eta\mathbb{1} - \mathsf{T}]_0 = [({}^\eta\mathbb{1} - \mathsf{T})^{-1}]_0 = [\mathsf{U}]_0$. Finally we obtain after η "Hutton" steps the following:

$$\left[\frac{1}{\mathbb{1} - \mathsf{T}}\right]_0 = \sum_{k=0}^{\eta} 2^{-(k+1)}\,(1 + x_k)\,[\,{}^{(k)}\Sigma\,]_0\,1^{(k)}\,,$$

which is exactly (9′) while the transformation itself can be observed as the nonstandard counterpart of (11).

To model (12) one has to multiply all steps of the reasoning by $\vec{\mathsf{a}} = \vec{\mathsf{a}}_0$ as the right-hand factor. Then the process takes the form of successive application of the Hutton transform to shorter and shorter remainders, so that the Hutton matrix of dimension $(\eta - k + 1) \times (\eta - k + 1)$, generated by $x_k, \dots, x_\eta$ as points of interpolation, is applied at the kth step, $k = 0, 1, \dots, \eta$.

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