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POROSITY, NOWHERE DENSE SETS AND A THEOREM OF DENJOY

Abstract

In the 1940's, A. Denjoy proved that the typical point of a perfect nowhere dense set in $\mathbb{R}$ is a point of strong porosity for that set. We prove two stronger versions of this for arbitrary metric spaces. Theorem 3 says that if E is any closed nowhere dense set in a metric space, and h is any porosity scale function, then the typical point in E is a point at which E is h-porous. Thus, if the metric space is complete, then "most" points of E are points at which E is very thin in the sense of porosity. Theorem 4 says that if F is closed and h-porous in $\mathbb{R}$, then there exists a closed nowhere dense set E in $\mathbb{R}$ containing F such that F is h-porous in the subspace E. Therefore, in the sense of h-porosity, no nontrivial information about the porosity of a closed set in $\mathbb{R}$ can be inferred from its porosity relative to some closed nowhere dense set in $\mathbb{R}$.

1 Introduction

A. Denjoy [3, pp. 195–196] is credited with the following theorem:

Theorem (Denjoy) *Let P be a perfect nowhere dense subset of $\mathbb{R}$. Then the set of points in P at which P is not strongly bilaterally porous is a first category subset of P.*

This theorem has been used recently to show the existence of a porous (in fact, strongly bilaterally porous) set in $\mathbb{R}$ that is not σ-symmetrically porous (see [4] and [11]). It is also used in 1.4 on p. 27 of [1], in [9], and in [10].

Denjoy's theorem seems not to have been generally known until P. S. Bullen called attention to it at the Fifth Summer Symposium in Real Analysis, held at the University of Missouri-Kansas City in June 1982. The reader is referred to Bullen's article [2] for an exposition of Denjoy's work in connection with

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porosity. The above theorem is stated, but not proved, at the top of p. 90 of [2]. Proofs can be found on p. 417 of [13], p. 188 of [14], and in [9].

A proof of a slightly weaker result (replace "strongly bilaterally porous" with "porosity $\geq \frac{1}{3}$") can be found within the proof of another result on pp. 117–118 of [6].[1] Also, the proof of lemma 1 on pp. 356–357 of [7] shows the existence of a dense set of points at which the perfect nowhere dense set has right porosity $\geq \frac{1}{2}$. Note these papers predate Bullen's contribution ([7] was submitted in April 1982). It seems likely other various less definitive forms of Denjoy's theorem can be found scattered throughout the literature (besides real analysis, dynamical systems theory and ergodic theory come to mind).

Examination of these proofs in $\mathbb{R}$ show that "perfect nowhere dense" may be replaced with "nowhere dense". A proof of this result for nowhere dense subsets of $\mathbb{R}^n$ (with "strongly porous" in place of "strongly bilaterally porous") can be found on p. 195 of [5].[2] L. Zajíček states on p. 321 of [15] that this actually holds for any metric space replacing $\mathbb{R}^n$, but to the author's knowledge no proof currently appears in the literature. Finally, a stronger version of the theorem we stated above for $\mathbb{R}$ which utilizes the notion of h-porosity (see below) can be found on p. 200 of B. S. Thomson's book [14].[3]

In this paper we prove an h-porous strengthening of Zajíček's statement (Theorem 3), as well as a more precise porosity version (Theorem 1) that will imply this strengthening for σ-compact metric spaces. We also include some additional remarks and results relating to these issues. For example, Theorem 4 says that the condition of being nowhere dense relative to some closed nowhere dense subset of $\mathbb{R}$ is no stronger (in the sense of smallness measured by almost any notion of porosity) than simply being nowhere dense relative to $\mathbb{R}$.

For more on porosity, see [10], the appendix of [14] or [15]. We warn the reader, however, that the convention used in [10] regarding primed symbols β', h', etc. is not followed here.

2 Definitions

Throughout this paper, unless otherwise stated, let (X, d) be a metric space, $E \subseteq X$, $x \in X$, and $\mathbf{H}$ be the collection of continuous[4] strictly increasing functions $h(t) : [0, \infty) \to [0, \infty)$ such that $h(0) = 0$. The open ball in X of

[1] In the definition of $E*$ on p. 117 of [6], $\frac{1}{2}$ should be replaced by 1.

[2] It is stated in this paper that the proof originally was for graphs of continuous functions in $\mathbb{R}^2$, and it was a referee's observation that the result continues to hold for any nowhere dense set in $\mathbb{R}^n$.

[3] Our formulation of h-porosity differs slightly from what one finds in [14].

[4] Actually, our theorems hold without the assumption of continuity, including the Borel classification aspects.

radius r and center c is denoted by $B_X(c,r)$; closure and complementation of $E \subseteq X$ relative to X are denoted by $\bar{E}$ and E^c, respectively. By $d(x,E)$ we mean $\sup\{d(x,y) : y \in E\}$.

Definition (h-porosity) *We say that E is h-porous in X at x if $x \notin \bar{E}$ or there exists a sequence $\{B(c_n,r_n)\}$ of open balls lying in E^c such that $c_n \to x$ and $d(x,c_n) < r_n + h(r_n)$ for each n. If E is h-porous in X at x for each $x \in E$, then we will say that E is h-porous in X. Any subset of X which can be written as a countable union of h-porous in X sets is said to be $\sigma - h$-porous in X. If the ambient space X relative to which these various porosities are computed is clear from context, we may omit reference to it.*

Definition (porosity function) *We say that $\mathcal{H} : E \to \mathbf{H}$ is a porosity function for E in X if, for each $e \in E$, E is $\mathcal{H}(e)$-porous in X at e.*

One can view "E is h-porous" (resp., "$\mathcal{H}$ is a porosity function for E") as a generalization of E being uniformly porous (resp., porous). We note that our formulation of h-porosity, unlike some formulations used elsewhere, allows for both stronger and weaker versions of ordinary porosity. Clearly, E is uniformly porous in the usual sense if and only if E is h-porous for some linear function $h \in \mathbf{H}$, and E is h-porous for some $h \in \mathbf{H}$ if and only if E has a constant porosity function. Moreover, if E is porous in the usual sense, then E is h-porous for each $h \in \mathbf{H}$ satisfying $\lim_{t\to 0} \frac{h(t)}{t} = \infty$.

It is easy to see that E is nowhere dense in X if and only if E has a porosity function in X (use first countability of X for the "only if" direction). Moreover, if X is σ-compact and E is nowhere dense in X, then E is h-porous in X for some $h \in \mathbf{H}$.[5] Indeed, E is h-cp in X (see below) for some $h \in \mathbf{H}$, since the closure of any nowhere dense set is nowhere dense. However, this may fail if X is not σ-compact. The author has constructed a nonseparable counterexample and, answering a question posed by the author during the writing of [10], M. Repický [12] has constructed an example in the Hilbert space ℓ_2.[6] Consequently, while our Theorem 1 implies our Theorem 3 for σ-compact metric spaces (e.g. $\mathbb{R}^n$), this is not the case for arbitrary metric spaces.

Definition (closure porosity) *We say that E is h-closure porous (h-cp) in X if E is h-porous in X at x for each $x \in X$ (equivalently, for each $x \in \bar{E}$). Any subset of X which can be written as a countable union of h-cp in X sets is said to be $\sigma - h$-closure porous ($\sigma - h$-cp) in X.*

Because a set is h-porous at x if and only if the closure of that set is h-porous at x, a set is h-cp if and only if it is contained in some *closed* h-porous

[5] These results are proved in Chapter 4 of [10].
[6] Both examples appear in chapter 4 of [10].

set. Therefore, the collection of $\sigma - h$-cp sets is equal to the σ-ideal generated by the closed h-porous sets (equivalently, the σ-ideal generated by the F_σ h-porous sets).

One reason for our introduction of closure porosity is due to the fact that, unlike the case with nowhere dense sets, being a countable union of closed porous sets is in general a stronger notion of *smallness* than simply being a countable union of porous sets. (In this regard, porosity behaves like *Lebesgue measure zero.*) There exists in $\mathbb{R}$ a porous set that is not σ-cp in $\mathbb{R}$ (see [6] and [8]). In fact, given any h, $h' \in \mathbf{H}$, there exists in $\mathbb{R}$ a bilaterally h-porous set that is not σ-h'-cp (see [10]; the proof uses Thomson's strengthening of Denjoy's theorem).

Another reason for our introduction of closure porosity is that its corresponding σ-ideals are in general much more sensitive to the growth rate of functions from $\mathbf{H}$ than is the case with ordinary porosity. We feel that this is an important consideration for stating our results as precise as we do. Remarks (c) and (g) in [8] imply that given any $0 < \beta' < \beta < \infty$, there exists a closed βt-porous set in $\mathbb{R}$ that is not $\sigma - \beta' t$-cp in $\mathbb{R}$. Moreover, results in chapter 6 of [10] suggest if h, $h' \in \mathbf{H}$ with $\liminf_{t\to 0} \frac{h(t)}{h'(t)} > 1$, then there exists a closed h-porous set in $\mathbb{R}$ that is not $\sigma - h'$-cp in $\mathbb{R}$.

To prevent possible confusion, we point out that "contained in an F_σ $\sigma - h$-porous set" is not in general the same as "$\sigma - h$-cp". (In this regard, porosity does not behave like *Lebesgue measure zero.*) Clearly, every $\sigma - h$-cp set is contained in some F_σ $\sigma - h$-porous set. However, Zajíček has shown if $0 < \beta' < \beta < \infty$, then *any* $\sigma - \beta t$-porous set (F_σ or not) is a $\sigma - \beta' t$-porous set (see 2.15 on p. 319, and the remark following 2.24 on p. 321, of [15])[7]. Zajíček's result, along with the examples mentioned in the previous paragraph, show that (in $\mathbb{R}$, at least) there is a noncommutativity of the conditions (a) "contained in an F_σ such set" and (b) "is a countable union of such sets" in the case of h-porosity. The $\sigma - h$-cp sets arise from imposing on the collection of h-porous sets (a) first, and then (b), whereas subsets of F_σ $\sigma - h$-porous sets arise from imposing on the collection of h-porous sets (b) first, and then (a).

Our theorems actually involve countable unions of closed h-porous sets. Any such set is automatically both a $\sigma - h$-cp set and an F_σ $\sigma - h$-porous set. However, we will call such a set an F_σ $\sigma - h$-cp set (which involves no loss of descriptive strength), thereby describing the σ-ideal to which the set belongs separately from the Borel classification to which the set belongs.

[7] In fact, it is proved in chapter 6 of [10] that given any h, $h' \in \mathbf{H}$ there exists a closed σ-h-symmetrically porous set in $\mathbb{R}$ that is not σ-h'-cp.

3 Theorems

Theorem 1 *Let g and G belong to $\mathbf{H}$ and $\omega > 1$. If E is closed and G-porous in a metric space (X, d), then $E^* = \{e \in E : E$ is not g-porous in (X, d) at $e\}$ is an F_σ $\sigma - H$-cp set in $(E, d\,|_E)$,[8] where $H(t) = G(g^{-1}(\omega t)) + 2g^{-1}(\omega t)$.*

PROOF. For each positive integer k, let

$$E_k^* = \left\{ e \in E : \nexists\, B_X(x, R) \subseteq E^c \text{ with } d(e, x) < \min\{\frac{1}{k}, R + g(R)\} \right\}.$$

Since $E^* = \cup_{k=1}^{\infty} E_k^*$, it will be enough to show that each E_k^* is a closed H-porous set in E . Choose any k which, for the remainder of the proof, we assume to be fixed. It is easy to see that the complement (relative to E) of E_k^* is open in E. We will show that E_k^* is H-porous in E at each point of E_k^* . To this end pick any $e_0 \in E_k^*$, which (along with k) we assume to be fixed for the remainder of the proof.

Suppose that e_0 is isolated in E. Then from the fact that e_0 is not isolated in X (else E would not be nowhere dense in X) it readily follows that $e_0 \notin E_k^*$, a contradiction. Thus, we may assume that e_0 is not isolated in E.

Choose $\delta > 0$. We must show there are $\tilde{e} \in E$ and $\tilde{r} > 0$ such that

(a) $B_E(\tilde{e}, \tilde{r}) \subseteq E \sim E_k^*$

(b) $d(e_0, \tilde{e}) < \min\{\delta,\ \tilde{r} + H(\tilde{r})\}$.

Because E is G-porous in X at e_0, there exist $\tilde{x} \in X$ and $\tilde{R} > 0$ such that $B_X(\tilde{x}, \tilde{R}) \subseteq E^c$ and $d(e_0, \tilde{x}) < \min\{\frac{1}{3k},\ \frac{\delta}{2},\ G(\tilde{R}) + \tilde{R}\}$. By appropriately expanding any ball satisfying the conditions in the previous sentence, we may assume that $\tilde{R} = d(\tilde{x}, E)$. Choose $\tilde{e} \in E$ so that $d(\tilde{x}, \tilde{e}) = \tilde{R}$ (recall that E is closed in X) and let $\tilde{r} = \min\{d(e_0, \tilde{x}) + \tilde{R}\ ,\ \frac{1}{\omega} \cdot g(\tilde{R})\}$.

To prove (a), choose any $e \in B_E(\tilde{e}, \tilde{r})$. We show that the membership condition for E_k^* fails for e by taking $x = \tilde{x}$ and $R = \tilde{R}$. Clearly, $B_X(\tilde{x}, \tilde{R}) \subseteq E^c$. Moreover,

$$\begin{aligned} d(\tilde{x}, e) \leq& d(\tilde{x}, \tilde{e}) + d(\tilde{e}, e) < \tilde{R} + \tilde{r} \\ &\leq \min\{3 \cdot d(e_0, \tilde{x})\ ,\ \tilde{R} + \frac{1}{\omega} \cdot g(\tilde{R})\} < \min\{\frac{1}{k}\ ,\ \tilde{R} + g(\tilde{R})\}. \end{aligned}$$

To prove (b), we first note that

$$d(e_0, \tilde{e}) \leq d(e_0, \tilde{x}) + d(\tilde{x}, \tilde{e}) = d(e_0, \tilde{x}) + \tilde{R} \leq 2 \cdot d(e_0, \tilde{x}) < 2 \cdot \frac{\delta}{2} = \delta.$$

Now if $\tilde{r} = d(e_0, \tilde{x}) + \tilde{R}$, then we have

$$H(\tilde{r}) + \tilde{r} > \tilde{r} = d(e_0, \tilde{x}) + \tilde{R} = d(e_0, \tilde{x}) + d(\tilde{x}, \tilde{e}) \geq d(e_0, \tilde{e}).$$

[8]By $d|_E$ we mean the restriction of the metric distance function d to the set $E \times E$.

On the other hand, if $\tilde{r} = \frac{1}{\omega} \cdot g(\tilde{R})$, then

$$\begin{aligned}H(\tilde{r}) + \tilde{r} =& H(\frac{1}{\omega} \cdot g(\tilde{R})) + \frac{1}{\omega} \cdot g(\tilde{R}) \\ =& G(g^{-1}(\omega \cdot \frac{1}{\omega} \cdot g(\tilde{R}))) + 2 \cdot g^{-1}(\omega \cdot \frac{1}{\omega} \cdot g(\tilde{R})) + \frac{1}{\omega} \cdot g(\tilde{R}) \\ >& G(\tilde{R}) + \tilde{R} + \tilde{R} > d(e_0, \tilde{x}) + \tilde{R} = d(e_0, \tilde{x}) + d(\tilde{x}, \tilde{e}) \geq d(e_0, \tilde{e}),\end{aligned}$$

where G-porosity of E in X at e_0 is used via $G(\tilde{R}) + \tilde{R} > d(e_0, \tilde{x})$ in the second line of the above. □

It would be interesting to know if Theorem 1 gives the sharpest possible statement in the following sense. Does there exist $H' \in \mathbf{H}$ such that (for any $\omega > 1$) $\liminf_{t\to 0} \frac{H(t)}{H'(t)} > 1$ (i.e. H' approaches 0 faster than H does, as $t \to 0$) and E^* is an F_σ $\sigma - H'$-cp set in $(E, d|_E)$ for all possible X, E, G, and g as above? The author does not see how the proof of Theorem 1 can be sharpened, and so one might conjecture that no such H' exists.[9] On the other hand, for certain choices of g and G the statement of Theorem 1 is not very strong. For example, if $\liminf_{t\to 0} \frac{g(t)}{G(t)} \geq 1$, Theorem 1 simply tells us that the empty set is $\sigma - H$-porous!

If $X = \mathbb{R}$ (more generally, any convex subset of a normed space), it is easy to see that an H' as above can always be chosen. In this case note that (using the notation from our proof of Theorem 1) $\tilde{e}$ can be chosen as the endpoint of $B_{\mathbb{R}}(\tilde{x}, \tilde{R})$ closest to e_0, so that $d(e_0, \tilde{x}) = d(e_0, \tilde{e}) + d(\tilde{e}, \tilde{x})$. For any (fixed) $g, G \in \mathbf{H}$ and $\omega > 1$, let $H'(t) = G(g^{-1}(\omega t)) + g^{-1}(\omega t)$ and $H(t)$ be as in Theorem 1. Let E^* be the points of non$-g$-porosity as defined above, and let E_b^* be the larger set of points of non-g-bilateral porosity. Then the following result, whose proof we only sketch, is easy to establish by tracing through the proof of Theorem 1 with $\tilde{e}$ chosen to satisfy the condition above.

Theorem 2 *Let $g, G \in \mathbf{H}$, $\omega > 1$, H and H' be as in the paragraph above, and E be a closed G-porous set in $\mathbb{R}$. Then with E^* and E_b^* defined as above, we have the following.*

(a) E^* is an F_σ $\sigma - H'$-cp set in E.

(b) E_b^* is an F_σ $\sigma - H$-cp set in E.

Moreover, if E is assumed to be closed and G-bilaterally porous in $\mathbb{R}$, then E_b^* is an F_σ $\sigma - H'$-cp set in E.

Proof. The remarks preceding the statement of Theorem 2 indicate how (a) is proved. For (b), define sets E_k^{*+} and E_k^{*-} analogous to E_k^*, except the

[9] More to the point, an H' such that there exists in E an F_σ $\sigma - H$-cp set in E failing to be $\sigma - H'$-cp in E.

non-porosity behavior "approximated" is that of non-right porosity for E_k^{*+} and non-left porosity for E_k^{*-}. Then $E_b^* = \cup_{k=1}^{\infty}[E_k^{*+} \cup E_k^{*-}]$ and both E_k^{*+} and E_k^{*-} are closed in E. One shows that E_k^{*+} is H-porous in E by choosing $\tilde{e}$ to be the left endpoint of $B_{\mathbb{R}}(\tilde{x}, \tilde{R})$, and one shows that E_k^{*-} is H-porous in E by choosing $\tilde{e}$ to be the right endpoint of $B_{\mathbb{R}}(\tilde{x}, \tilde{R})$. If E is G-bilaterally porous in $\mathbb{R}$, then there are appropriately sized intervals lying in E^c on both sides of e_0. In this case, one shows that E_k^{*+} is H'-porous in E by choosing $\tilde{e}$ to be the left endpoint of an appropriate interval to the right of e_0, and one shows that E_k^{*-} is H'-porous in E by choosing $\tilde{e}$ to be the right endpoint of an appropriate interval to the left of e_0. □

Of course, one could also ask if Theorem 2 gives the sharpest possible result. In this case there are many such questions that can be asked.

Because $\mathbb{R}$ is σ-compact (see our remarks before the definition of closure porosity), the strengthening of Denjoy's theorem for bilateral porosity given by Thomson on p. 200 of [14] is a corollary of Theorem 2(b). Interestingly, the analogous result for the set E_{sy}^* of points of non-symmetric porosity fails. In [4, p. 261] and [11] (lemma on p. 417), it is shown there exists a closed uniformly symmetrically porous set in $\mathbb{R}$ that fails to have *any* points of strong symmetric porosity.

By omitting "closed", we can do much better. Let $g \in \mathbf{H}$ with $\lim_{t\to 0} \frac{g(t)}{t} = \infty$. By "$E$ is g-semi-symmetrically porous at x", we mean

$$\limsup_{\delta\to 0^+} \frac{1}{\delta} \cdot \min\left\{g\left(\gamma^-(E,x,\delta)\right),\ g\left(\gamma^+(E,x,\delta)\right)\right\} > 0$$

for each $x \in E$, where $\gamma^+(E,x,\delta)$ denotes the length of the largest open interval in $(x,\, x+\delta) \cap E^c$, and similarly for $\gamma^-(E,x,\delta)$. (Note that interchanging "lim sup" and "min" above gives essentially g-bilateral porosity.) A straightforward modification of the construction on p. 260 of [4] (see chapter 5 of [10]) shows that for *any* such $g \in \mathbf{H}$, there exists a nowhere dense set in $\mathbb{R}$ having *no* points (much less, a residual set of points) of g-semi-symmetric porosity (i.e. $E_{sy}^* = E$ for that set).[10]

Theorem 3 *Let $g \in \mathbf{H}$ and N be nowhere dense in a metric space (X,d). Denoting the closure of N in X by E, then $E^* = \{e \in E : E$ is not g-porous in (X,d) at $e\}$ is an F_σ first category set in $(E, d|_E)$. Moreover, $N^* = E^* \cap N$ is a first category set in $(N, d|_N)$.*[11]

[10]The nowhere dense set in question is G_δ and not closed. It is not known whether a closed such set exists, but note that the set as a subspace of $\mathbb{R}$ is a second category (Baire, in fact) space. Thus, having $E_{sy}^* = E$ for this set still contradicts the statement that the *typical* point of E is a point of g-semi-symmetric porosity of E.

[11]$N^* \equiv \{e \in N : N$ is not g-porous in X at $e\} = E^* \cap N$ because N is g-porous at e if and only if E is g-porous at e.

Proof. Because E is nowhere dense in X, there exists a porosity function $\mathcal{G} : E \to \mathbf{H}$ for E in X. Define the sets E_k^* exactly as we did in the proof of Theorem 1. Each E_k^* is closed in E (same proof), and so it will be enough to prove that each E_k^* is nowhere dense in E. Fixing k for the remainder of the proof, we will show that $\mathcal{H} : E_k^* \to \mathbf{H}$ defined by $(\mathcal{H}(e))(t) = (\mathcal{G}(e) \circ g^{-1})(2t) + 2g^{-1}(2t)$ for $e \in E_k^*$ is a porosity function for E_k^* in E. To this end, choose any $e_0 \in E_k^*$, which (along with k) we assume to be fixed for the remainder of the proof. Now simply retrace the proof of Theorem 1, beginning after e_0 was chosen, replacing ω, G, and H by 2, $\mathcal{G}(e_0)$, and $\mathcal{H}(e_0)$, respectively. To prove the statement involving N, use the fact that the notions *nowhere dense* and *first category* are hereditary with respect to dense subsets. □

In the theorems above, we have been considering *small* (first category, nowhere dense, σ-porous, etc.) subsets of *small* (nowhere dense) sets. It seems natural to ask how *small* a *small* set in a *small* set must be. More specifically, suppose we know that a set F is a nowhere dense set in some closed nowhere dense set E in $\mathbb{R}$.[12] What can one say about the size of F relative to $\mathbb{R}$? For instance, must F be porous in $\mathbb{R}$? Does F even have to have Lebesgue measure zero? The following theorem says that any nowhere dense subset of $\mathbb{R}$ is nowhere dense relative to some closed nowhere dense set in $\mathbb{R}$. In fact, little about the porosity in $\mathbb{R}$ of a closed nowhere dense set F can be said that does not already hold relative to some closed nowhere dense set in $\mathbb{R}$.

Theorem 4 *Let $h \in \mathbf{H}$ and F be closed and h-porous in $\mathbb{R}$. Then there exists a closed set E that is nowhere dense in $\mathbb{R}$ and such that F is h-porous in E.*

Proof. We construct E by adding to F a countable collection of points in the following manner. For each isolated point of F, choose any convergent sequence having that isolated point as its only cluster point. To avoid possible clustering of the "heads" of these sequences, arrange for each sequence to wander no further from its associated isolated point than one half of that isolated point's "radius of isolation".[13] Now include the midpoints of all the bounded components of the complement (relative to $\mathbb{R}$) of F. If the bounded component lies between two isolated points, the midpoint may have already

[12] Such "nested nowhere denseness" of sets plays a crucial role in one of Denjoy's best known accomplishments. Denjoy's reconstruction of a primitive of a function (Denjoy's *totalization*) involves a transfinite well-ordered sequence of perfect sets in $\mathbb{R}$, each set nowhere dense in the next set.

[13] More generally, we may use any closed nowhere dense set in $\mathbb{R}$ in place of such a sequence, as long as that closed nowhere dense set lies within the same interval we restricted the sequence to and it has the corresponding isolated point of F as a limit point. Note that any limit point of the union of these closed nowhere dense sets must also be a limit point of these isolated points. Hence their union, together with the closed set F, is closed.

been included during the previous step (and if not, note that its inclusion will not affect the properties of the previous construction). Clearly, E is closed and nowhere dense in $\mathbb{R}$. To see that F is h-porous in E, choose any point of F that is not isolated in F (h-porosity at the isolated points follows from the convergent sequences we included). Because F is h-porous in $\mathbb{R}$, there exists an appropriate "h-sequence" of open intervals belonging to F^c converging to that point. This same sequence of intervals can be used to verify that F is h-porous in E at this point.[14] Note that our placement of certain points of $E \cap F^c$ at the *midpoints* of the bounded components of F^c was necessary for the "intervals in E" to be of the appropriate size. $\square$

We have actually proved the following more precise result. Let F be closed and nowhere dense in $\mathbb{R}$ and $\mathcal{G}$ be a porosity function for F in $\mathbb{R}$. Then there exists a closed nowhere dense set E in $\mathbb{R}$ such that $\mathcal{G}$ is also a porosity function for F in E. Moreover, whatever "bilateral" or "symmetric" porosity conditions F has relative to $\mathbb{R}$, F will have the same conditions relative to E. However, it is not clear to the author that the same can be said for any *lim inf* porosity conditions F has relative to $\mathbb{R}$.

It seems that far more subtle geometric reasoning is needed to generalize Theorem 4 to $\mathbb{R}^n$, even with "nowhere dense" replacing "h-porous". We leave as an open question whether any such generalizations are possible.

References

[1] M. Balcerzak, *Some Properties of Ideals of Sets in Polish Spaces*, Acta Universitatis Łodziensis, Łódź, 1991.

[2] P. S. Bullen, *Denjoy's index and porosity*, Real Analysis Exchange, **10** (1984-85), 85–144.

[3] A. Denjoy, *Leçons sur le Calcul des Coefficients d'une Série Trigonométrique* (Part II), Gauthier-Villars, 1941.

[4] M. J. Evans, P. D. Humke and K. Saxe, *A symmetric porosity conjecture of Zajíček*, Real Analysis Exchange, **17** (1991-92), 258–271.

[5] J. Foran, *Continuous functions need not have σ-porous graphs*, Real Analysis Exchange, **11** (1985-86), 194–203.

[6] J. Foran and P. D. Humke, *Some set theoretic properties of σ-porous sets*, Real Analysis Exchange, **6** (1980-81), 114–119.

[14] If the fact that E has isolated points seems to be "cheating", then we can arrange for E to be a perfect set by placing appropriate Cantor sets in each bounded contiguous interval of F. All we require is that each of these Cantor sets contains the midpoint of the contiguous interval it is placed in.

[7] G. Petruska, *Derivatives take every value on the set of approximate continuity points*, Acta Math. Hung., **42** (1983), 355–360.

[8] D. L. Renfro, *On various porosity notions in the literature*, Real Analysis Exchange, **20** (1994-95), 63–65.

[9] D. L. Renfro, *The Lebesgue density theorem fails for Baire category*, manuscript.

[10] D. L. Renfro, *On some various porosity notions*, in preparation.

[11] M. Repický, *An example which discerns porosity and symmetric porosity*, Real Analysis Exchange, **17** (1991-92), 416–420.

[12] M. Repický, personal communication to author (Feb. 27, 1995).

[13] B. S. Thomson, *Differentiation bases on the real line, II*, Real Analysis Exchange, **8** (1982-83), 278–442.

[14] B. S. Thomson, *Real Functions*, Lecture Notes in Math., **1170**, Springer-Verlag, 1985.

[15] L. Zajíček, *Porosity and σ-porosity*, Real Analysis Exchange, **13** (1987-88), 314–350. (Correction in RAE, **14**, p. 5)