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A CHARACTERIZATION OF THE SET $\Omega(f) \setminus \omega(f)$ FOR CONTINUOUS MAPS OF THE INTERVAL WITH ZERO TOPOLOGICAL ENTROPY

Abstract

We give a characterization of the set of nonwandering points of a continuous map f of the interval with zero topological entropy, attracted to a single (infinite) minimal set Q. We show that such a map f can have a unique infinite minimal set Q and an infinite set $B \subset \Omega(f) \setminus \omega(f)$ (of nonwandering points that are not ω-limit points) attracted to Q and such that B has infinite intersections with infinitely many disjoint orbits of f.

Let $I = [0, 1]$ be the compact unit interval, let $C(I, I)$ be the class of continuous maps $I \to I$, and let $E_0(I, I) \subset C(I, I)$ be the class of maps with zero topological entropy. A recent paper [3] contains a characterization of the ω-limit sets $\omega_f(x)$ of maps f in $E_0(I, I)$, showing the complexity of maximal infinite ω-limit sets. We recall that there is a map f in $E_0(I, I)$ possessing a maximal ω-limit set $\tilde{\omega} = \omega_f(x)$ of the form $Q \cup P$ where Q is a Cantor set and P a countably infinite set of isolated points in $\tilde{\omega}$ such that P intersects infinitely many (disjoint) orbits and such that $\omega_f(x) = Q$ for any $y \in \tilde{\omega}$ (i.e., Q is a minimal set for f).

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The main aim of this paper is to extend the above quoted results and show that, for any map f in $E_0(I,I)$, the set $\Omega(f) \setminus \omega(f)$ of non-wandering points that are not ω-limit points can have a complicated structure. First, in Theorem 8 below, for a map f in $E_0(I,I)$ we give a characterization of the set of nonwandering points attracted to a given infinite minimal set Q. The subsequent Theorem 9 illustrates ideas from Theorem 8 by an example. More precisely, we exhibit a map f in $E_0(I,I)$ with unique infinite minimal set Q, and with the most complex structure of the set $\Omega(f) \setminus \mathrm{Per}\,(f)$; this set is attracted to $\omega(f) \setminus \mathrm{Per}\,(f) = Q$.

In the sequel, we will use the standard terminology, as, e.g., in [2] or [3]. In particular, given a map f in $C(I,I)$, a is a *nonwandering point* if, for any neighborhood U of a, $f^n(U)$ intersects U for some $n > 0$. The set of nonwandering points of f is denoted by $\Omega(f)$. By $\omega_f(x)$ we denote the ω-limit set of x, and by $\omega(f) = \bigcup\{\omega_f(x); x \in I\}$ the set of ω-limit points of f. Concerning the basic properties of $\Omega(f)$, we refer to [2]. The following three propositions, however, may not be known.

Proposition 1 *If $f \in C(I,I)$, then any point of $\Omega(f) \setminus \omega(f)$ is isolated in $\Omega(f)$.*

Proof. See [5]; cf. also [2, Proposition IV.15]. □

Proposition 2 *If $f \in C(I,I)$, then $\omega(f) = \bigcap_{n=0}^{\infty} f^n(\Omega(f))$. Consequently, there is no sequence $\{a_n\}_{n=1}^{\infty} \subset \Omega(f) \setminus \omega(f)$ such that $f(a_{n+1}) = a_n$, for any n.*

Proof. See [2, Proposition V.10], cf. also [4]. □

Proposition 3 *Let $f \in E_0(I,I)$ and let $a \in \Omega(f) \setminus \omega(f)$. Then $\omega_f(a)$ is an infinite minimal set.*

Proof. See [2, Theorem VI.34]. □

Before stating the next lemma we recall (cf., e.g., [2] or [3]) that if $f \in E_0(I,I)$ and if $\{I_n\}_{n=1}^{\infty}$ is a decreasing sequence of minimal compact periodic intervals such that, for any n, I_n has period 2^n, then the set

$$M = M_f(\{I_n\}) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} f^i(I_n) \tag{1}$$

contains an infinite minimal ω-limit set Q with $\omega_f(x) = Q$ for any $x \in M$ and conversely, any infinite minimal set Q is contained exactly in one set M

of the form (1). In the sequel, we will denote the set M by $M_f(x)$ provided $Q = \omega_f(x)$ and will call it a *maximal simple set for* f.

In fact, M is a simple set, according to the following inductive definition. A compact set $X \subset I$ is a *simple set* for f, if f maps X onto X and if either X is a singleton or X admits a decomposition $S \cup T$ into compact portions that are exchanged by f and such that each of S and T is a simple set for f^2. In particular, a periodic orbit is *simple* if it is a simple set; it follows that each simple periodic orbit has period 2^n, for some $n \geq 0$. A map restricted to a simple set is a *simple map*.

Now it is easy to see that if $\omega_f(x)$ is a minimal set and if $M_f(x)$ has representation (1), then, for each n, the trajectory of x is eventually in $\mathrm{Orb}\,_f(I_n)$, the orbit of I_n. Thus if $\Omega_f(x)$ denotes the set of points y in $\Omega(f)$ with $\omega_f(y) = \omega_f(x)$ then we have the following

Lemma 4 *Let $f \in E_0(I,I)$ and let $\omega_f(x)$ be an infinite minimal set. Then $\Omega_f(x) \subset M_f(x)$.*

Lemma 5 *Let $f \in E_0(I,I)$, let M be given by (1) and let $g \in C(I,I)$ be a continuous extension of $f|M$. Let J be an interval intersecting two different connected components of M. Then $g^k(J) \supset I_n$, for some k and n.*

Proof. Let M_0 and M_1 be disjoint components of M intersecting J. Then, by (1), M_0 and M_1 are contained in two different components of $Orb_f(I_n)$, for some n. Denote this components by J_0 and J_1, respectively. Now note that J_0 contains just two component intervals J', J'' from $\mathrm{Orb}\,_f(I_{n+1})$ and that both these intervals are exchanged by f^{2^n}. Since J_1 is invariant with respect to f^{2^n}, we easily get that $f^{2^n}(J)$ contains one of the intervals J', J'', say J'. Consequently, since $f|M = g|M$, we get $g^{2^n}(J) \supset J'$ and since J' is periodic, the result follows. □

The following lemma is useful when changing a map $f \in E_0(I,I)$, possessing $\omega(f)$ with isolated points attracted to an infinite minimal set Q (like a map constructed in [3]) to a map $g \in E_0(I,I)$ with an infinite set of points in $\Omega(g) \setminus \omega(g)$ attracted to Q.

Lemma 6 *Let $f \in E_0(I,I)$, let $Q = \omega_f(x)$ be an infinite minimal set and let $\{a_n\}_{n=1}^{\infty} \subset \Omega_f(x) \setminus Q$.*

(i) There is a sequence $\{U_n\}_{n=1}^{\infty}$ of pairwise disjoint compact intervals such that, for any n, $U_n \cap M_f(x) = \{a_n\}$.

(ii) Let the points $\{a_n\}_{n=1}^{\infty}$ have pairwise disjoint orbits. For each n, let $V_n \neq U_n$ be a compact subinterval of U_n containing a_n and let g be a map with

the following properties:

$$g(y) = f(y) \quad \text{for} \quad y \notin \bigcup_{n=1}^{\infty} U_n, \tag{2}$$

$$g(V_n) = f(a_n), \quad g(U_n) = f(U_n), \tag{3}$$

and, for any interval W_n containing a_n,

$$g(U_n \setminus W_n) \supset f(U_n \setminus W_n). \tag{4}$$

Then $a_n \notin \Omega(g)$ while $g(a_n) \in \Omega(g)$, for any n.

Proof. (i) This property must be known but, since we cannot give a reference, we include the argument. By Lemma 4, $a_n \in M_f(x)$. Let M_n be the connected component of $M_f(x)$ containing a_n. By (1), $M_n \cap Q \neq \emptyset$ and since $a_n \notin Q$, M_n must be an interval. Moreover, by (1), $f^i(M_n) \cap M_n = \emptyset$ whenever $i > 0$. Hence a_n must be an end-point of M_n since it is nonwandering. Assume first that $M_n = [a_n, q_n]$. Then $q_n \in Q$ and for some $\epsilon_n > 0$ we have $[a_n - \epsilon_n, a_n] \cap M_f(x) = \{a_n\}$ since otherwise, for any $\epsilon > 0$, $[a_n - \epsilon, a_n]$ contains infinitely many connected components of $M_f(x)$ and this would imply $a_n \in \bar{Q} = Q$ — a contradiction. Similarly find ϵ_n if $M_n = [q_n, a_n]$. Finally, set $U_n = [a_n - \epsilon_n/2, a_n]$, or $U_n = [a_n, a_n + \epsilon_n/2]$, respectively. Clearly, the intervals U_n are now pairwise disjoint.

(ii) First note that $f(y) = g(y)$ for any y in $M_f(x)$. Hence, keeping the notation from part (i), by (3) we have $g^i(V_n \cup M_n) = g^i(M_n) = f^i(M_n) \subset M_f(x)$ is a connected component of $M_f(x)$, disjoint from $V_n \cup M_n$, for any $i > 0$. Consequently, $a_n \notin \Omega(g)$, since $V_n \cup M_n$ is a neighborhood of a_n.

Now set $b_n = f(a_n) = g(a_n)$ and let V be an open interval containing b_n. Assume, to the contrary, that V can be taken so small that

$$g^i(V) \cap V = \emptyset \quad \text{for any} \quad i > 0. \tag{5}$$

Since $f(\Omega(f)) \subset \Omega(f)$, b_n is nonwandering for f. Hence there is an integer $k > 0$ such that $f^k(V) \neq g^k(V)$. Assume that k is a minimal such integer. By (2), $f^{k-1}(V)$ intersects some U_m with a_m as an endpoint. Consider the following two cases.

If $a_m \in f^{k-1}(V) (= g^{k-1}(V))$, then $m \neq n$ since otherwise b_n is in $g^k(V)$ and $g^k(V)$ intersects V, contrary to (5). But $m \neq n$ implies that a_m is not in the orbit of b_n, since the orbits of a_m and a_n are disjoint. Consequently, $g^k(V)$ intersects two different components of $M_f(x)$. By Lemma 5 we immediately get the result.

So assume that, for any m, $a_m \notin f^{k-1}(V)$. Then by (4), $g^k(V) \supset f^k(V)$ and since b_n is a nonwandering point of f, (5) cannot be true — a contradiction. $\square$

Lemma 7 *Let $Q \subset (0,1)$ be a Cantor set and let $\tilde{A}, \tilde{B}$ and $\tilde{D}$ be disjoint, countable subsets of Q. Let $\tilde{A}$ and $\tilde{D}$ be infinite and dense in Q and let $\tilde{B}$ be either finite or dense in Q. Then there exists a simple map $h \in C(Q,Q)$ such that $\tilde{A}$ and $\tilde{B} \cup \tilde{D}$ are full (i.e., backward and forward) orbits of h. Moreover, these orbits allow enumerations $\tilde{A} = \{\tilde{a}_n\}_{n=-\infty}^{\infty}$ and $\tilde{B} \cup \tilde{D} = \{\tilde{b}_n\}_{n=-\infty}^{\infty}$ such that $\tilde{B} = \{\tilde{b}_n\}_{n=0}^{k}$ where $0 \le k \le \infty$ and $h(\tilde{a}_n) = \tilde{a}_{n+1}$ and $h(\tilde{b}_n) = \tilde{b}_{n+1}$, for $-\infty < n < \infty$.*

PROOF. If $\tilde{B}$ is infinit,e then the proof is a slight modification of the proof of Theorem 3.7 in [3]. If $\tilde{B}$ has k elements where $0 < k < \infty$, choose m such that $2^m > k$ and define periodic portions $\{Q_n; 0 < n < 2^m\}$ of Q forming a simple orbit (cf. [1]) such that Q_n contains exactly one point of $\tilde{B}$, for $0 < n \le k$ and then proceed as in the preceding case. $\square$

Now we are able to give the main results. The following theorem gives a characterization of nonwandering sets of a map with zero topological entropy, attracted to a single infinite minimal ω-limit set.

Theorem 8 *Let $Q \subset (0,1)$ be a Cantor set and A, B disjoint countable sets of points in $I \setminus Q$ such that A is either empty or infinite. Then the following two statements* **P1** *and* **P2** *are equivalent.*

P1. *There exists a map $f \in E_0(I,I)$ such that $Q \cup A \cup B$ is the set of nonwandering points of f attracted to Q and such that $Q \cup A$ is a (maximal) ω-limit set for f and $B = \Omega(f) \setminus \omega(f)$.*

P2. *(i) Every interval contiguous to Q contains at most two points of $A \cup B$,*

(ii) Each of the intervals $[0, \min Q], [\max Q, 1]$ contains at most one point of $A \cup B$,

(iii) If $A \ne \emptyset$, then A is infinite and the intervals contiguous to Q that intersect A are dense in the system of intervals contiguous to Q (with respect to the natural ordering in I),

(iv) If $B \ne \emptyset$, then the system of intervals contiguous to Q that contain at most one point of $A \cup B$ is dense in the system of intervals contiguous to Q (with respect to the natural ordering in I).

PROOF. **P1** $\Rightarrow$ **P2**: This implication is true when $B = \emptyset$, cf., e.g., Theorem 6.5 in [3]. So let B be nonempty. By (1) and Lemma 4, the points of $A \cup B$ must be end-points of nondegenerate connected components of $M = M_f(x)$, for any

x in Q. To see this note that any interior point of M is wandering. Since any component of M contains at least one point of Q, it follows that any such component contains at most one point of $A \cup B$. This implies (i) and (ii). Property (iii) follows from Theorem 6.5 in [3]. To prove (iv) note that by Proposition 2, there is a point b in B that has no preimage in $Q \cup A \cup B$. But as above, b is an endpoint of a nondegenerate connected component of an invariant set M. Hence there is a sequence $\{b_n\}_{n=0}^{\infty}$ of points in M such that $b_0 = b$ and $f(b_{n+1}) = b_n$, for any n. By the continuity of f, each b_n must be in a non-degenerate component of M. Hence by (1), the intervals contiguous to Q and containing points b_n must be dense in the set of all intervals contiguous to Q. To finish the proof denote by M_n the component of M containing b_n. An induction argument shows that $M_n \cap (A \cup B) = \emptyset$, for $n > 0$.

P2 $\Rightarrow$ **P1**: Assume first that B is infinite. By (iv) and (i), there is a countably infinite set $D \subset [\min Q, \max Q]$ disjoint from $Q \cup A \cup B$ and such that any interval $J \subset \mathrm{conv}(Q)$ complementary to Q contains exactly two points of $A \cup B \cup D$ and $\bar{D} \supset Q$. Assign to every point p in $A \cup B \cup D$ a point $\phi(p)$ in Q such that there is no point from $A \cup B \cup D$ between p and $\phi(p)$. Set $\tilde{A} = \phi(A), \tilde{B} = \phi(B)$ and $\tilde{D} = \phi(D)$. Let $h, \{\tilde{a}_n\}_{n=-\infty}^{\infty}$ and $\{\tilde{b}_n\}_{n=-\infty}^{\infty}$ be as in Lemma 7. Using techniques similar to those employed in [3] (cf. Theorem 4.1 and the proof of Theorem 6.2) we can get a map $f \in E_0(I, I)$ such that $f|Q = h$ and the points $\{\, a_n\}_{n=-\infty}^{\infty}$, $\{\, b_n\}_{n=-\infty}^{\infty}$ are isolated ω-limit points of f satisfying $f(a_n) = a_{n+1}$ and $f(b_n) = b_{n+1}$, for any n. In fact, for each n, let M_n^a be the compact interval with a_n and $\tilde{a}_n$ as end-points and let M_n^b be defined similarly with b_n and $\tilde{b}_n$. Then put $M = \bigcup_{n=-\infty}^{\infty}\{M_n^a \cup M_n^b\} \cup Q$. Extend h to a continuous map $\tilde{h} : M \to M$ so that $\tilde{h}$ is linear on any $\{M_n^a\}$ and any $\{M_n^b\}$, $\tilde{h}(M_n^a) = M_{n+1}^a$ and $\tilde{h}(M_n^b) = M_{n+1}^b$. Clearly, we get a simple map $\tilde{h}$ and a suitable extension of $\tilde{h}$ yields f. Now applying Lemma 6 to f we get a map g such that A and B have the desired properties.

If B is finite, the construction of g is similar with the exception that we let $M_n^b = \{\tilde{b}_n\}$ for $n > k$ (i.e., we "blow up" only the points $\{\tilde{b}_n\}_{n=-\infty}^{k}$ for some $k > 0$; cf. also Remark 6.4 in [3]). □

Theorem 9 *There is a map $F \in C(I, I)$ with zero topological entropy, possessing a unique maximal (with respect to inclusions) infinite ω-limit set $\omega_F(y)$, of the form $Q \cup P$, where Q is a Cantor set and P is a countable set of isolated points. Moreover, F has a countably infinite set $W = \Omega(F) \setminus \omega(F)$ and also satisfies the following conditions.*

(i) There is an infinite sequence $\{p_{0n}\}_{n=1}^{\infty}$ of points in P with mutually disjoint orbits. More precisely, the orbit $Orb_F(p_{0n}) = O_n$ of any p_{0n} contains a chain $P_n = \{p_{in}\}_{i=-\infty}^{\infty}$ such that $F(p_{in}) = p_{i+1,n}$, for any i and $P_n =$

$O_n \cap \Omega(F) \subset P$ if n is even while $\{p_{in}\}_{i=-\infty}^{0} \subset P$ and $\{p_{in}\}_{i=1}^{\infty} \subset Q$ for n odd.

(ii) Every set of the form $Q \cup P_{n(1)} \cup P_{n(2)} \cup \dots$ where $\{n(i)\}$ is a finite or infinite set of positive integers, is an ω-limit set for F.

(iii) Consequently, the system $S_F(x)$ of ω-limit sets contained in $\omega_F(y) = Q \cup P$ has the cardinality of continuum and in fact, contains chains of arbitrary countable order type.

(iv) There is an infinite sequence $\{w_{0n}\}_{n=1}^{\infty}$ in W with mutually disjoint orbits. Moreover, for any n, $Orb\,(w_{0n}) \cap \Omega(F) = \{w_{in}\}_{i=0}^{\infty}$ and for any $k \in N \cup \{\infty\}$ there are infinitely many n such that $W_n = Orb(w_{0n}) \cap W = \{w_{in}\}_{i=0}^{k}$.

(v) $\Omega(F) = Q \cup P \cup W \cup Per(F)$.

Proof. There is a map F with the above described properties, but with $W = \emptyset$, cf. [3, Remark 6.4]. By applying Lemma 6 to a countable system of orbits in P we get the result. □

Remark 1 *By Proposition* 3 *and Theorem* 8 *we can describe the set $\Omega(f) \setminus \omega(f)$ for maps in $E_0(I, I)$. In fact, by Lemma* 4*, this set is contained in the union of a family of maximal simple sets and by Proposition* 1*, each such maximal set contains a subinterva. Hence the family is countable. It is easy to see that it can be infinite.*

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