**Real Analysis Exchange**
Vol. 20(1), 1994/95, pp. 204–211

Ján Borsík, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Slovakia
(e-mail: borsik@ccsun.tuke.sk)

# MAXIMAL ADDITIVE AND MAXIMAL MULTIPLICATIVE FAMILY FOR THE CLASS OF SIMPLY CONTINUOUS FUNCTIONS

**Abstract**

A function $f : X \to \mathbb{R}$ is simply continuous if for each open set $V$ in $\mathbb{R}$, the set $f^{-1}(V)$ is the union of an open and a nowhere dense set in $X$. The maximal additive and maximal multiplicative family for the class of all simply continuous functions is investigated.

## 1 Introduction

In what follows $X$ denotes a topological space. For a subset $A$ of a topological space Cl $A$ and Int $A$ denote the closure and the interior of $A$, respectively. The letters $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural and real numbers, respectively.

If $\mathcal{F}$ is a family of real functions on $X$, then a family $\mathfrak{A}(\mathcal{F})$ ($\mathfrak{M}(\mathcal{F})$ ) is called the maximal additive (maximal multiplicative) family for $\mathcal{F}$, if $\mathfrak{A}(\mathcal{F})$ ($\mathfrak{M}(\mathcal{F})$ ) is the set of all functions $f$ on $X$ such that $f + g \in \mathcal{F}$ ($f \cdot g \in \mathcal{F}$) for every $g \in \mathcal{F}$ (see [4]).

We recall that a function $f : X \to \mathbb{R}$ is *cliquish* at a point $x \in X$ (see [10]) if for each $\epsilon > 0$ and each neighborhood $U$ of $x$ there is a nonempty open set $G \subset U$ such that $|f(y) - f(z)| < \epsilon$ for each $y, z \in G$. A function $f : X \to \mathbb{R}$ is said to be cliquish if it is cliquish at each point $x \in X$.

A function $f : X \to \mathbb{R}$ is *simply continuous* (see [1]) if for each open set $V$ in $\mathbb{R}$, the set $f^{-1}(V)$ is the union of an open set and a nowhere dense set in $X$.

A function $f : X \to \mathbb{R}$ is *quasicontinuous* at a point $x \in X$ (see [10]) if for each neighborhood $U$ of $x$ and each neighborhood $V$ of $f(x)$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Denote by $Q_f$ the set of

Key Words: Maximal additive family, Maximal multiplicative family, Simply continuity
Mathematical Reviews subject classification: Primary 54C08
Received by the editors February 17, 1994

all points at which $f$ is quasicontinuous. If $Q_f = X$, then $f$ is said to be quasicontinuous.

It is easy to see that every quasicontinuous function is simply continuous and cliquish. In [11] it is shown that if $X$ is a Baire space, then every simply continuous function $f : X \to \mathbb{R}$ is cliquish.

In [3] the set $S_f$ of all simply continuity points of $f : X \to \mathbb{R}$ is defined as $S_f = \{x \in X :$ for each open neighborhood $V$ of $f(x)$ and for each neighborhood $U$ of $x$, the set $f^{-1}(V) \setminus \mathrm{Int}\, f^{-1}(V)$ is not dense in $U\}$ . It is shown that $f$ is simply continuous iff $S_f = X$. Further it is shown that $Q_f \subset S_f$ and the set $S_f \setminus C_f$ (where $C_f$ is the set of all continuity points of $f$) is of the first category.

The aim of this paper is to investigate the maximal additive and the maximal multiplicative family for the class of all real simply continuous functions. Denote by $\mathcal{S}$ the set of all simply continuous functions and let

$$\mathcal{T} = \{f : X \to \mathbb{R} : \cup\mathcal{G}(f) \text{ is dense in } X\},$$

where $\mathcal{G}(f) = \{G \subset X : G$ is open and $f$ is constant on $G\}$. (The class $\mathcal{T}$ contains nonmeasurable functions (for $X = \mathbb{R}$) ). We shall show that $\mathfrak{A}(\mathcal{S}) = \mathfrak{M}(\mathcal{S}) = \mathcal{T}$ for "nice" spaces $X$.

In [3] it is shown that the set $S_f$ is pre-closed (i.e. Cl Int $S_f \subset S_f$). From this we obtain

**Lemma 1.1** *If $X \setminus S_f$ is nowhere dense, then $f$ is simply continuous.*

PROOF. We have $\emptyset = \mathrm{Int}\ \mathrm{Cl}\ (X \setminus S_f) = \mathrm{Int}\ (X \setminus \mathrm{Int}\ S_f) = X \setminus \mathrm{Cl}\ \mathrm{Int}\ S_f$. Therefore $X = \mathrm{Cl}\ \mathrm{Int}\ S_f \subset S_f$.□

The following lemma is proved in [2]. We recall that a $\pi$-base for $X$ is a family $\mathcal{A}$ of open subsets of $X$ such that every nonempty open subset of $X$ contains some nonempty $A \in \mathcal{A}$ (see [12]).

**Lemma 1.2** (See [2] .) *Let $X$ be a topological space such that the family of all open connected sets is a $\pi$-base for $X$. Let $h : X \to \mathbb{R}$ be a cliquish function such that $h^{-1}(0)$ is dense in $X$. Let $g : X \to \mathbb{R}$ be a continuous function which is constant on no nonempty open subset of $X$. Then $f = g + h$ is simply continuous.*

**Remark 1.1** *The assumption that $X$ has a $\pi$-base of open connected sets cannot be omitted. Let $X = C$ (the Cantor set) and let $[0,1] \setminus C = \cup_{n=1}^{\infty}(a_n, b_n)$ (contiguous intervals). Define $g(x) = x$ for all $x \in X$ and $h(x) = \frac{b_n - a_n}{2}$ for $x = a_n$, $h(x) = 0$ otherwise. Then $g$ is continuous and injective, $h$ is cliquish and $h^{-1}(0)$ is dense in $X$. However $f = g + h$ is not simply continuous.*

The following lemma is obvious.

**Lemma 1.3** *Let $X$, $Y$ and $Z$ be topological spaces.*

*1) If $f : Y \to Z$ is continuous and $g : X \to Y$ is simply continuous, then $f \circ g$ is simply continuous.*

*2) If $f : Y \to Z$ is a homeomorphism, then $g : X \to Y$ is simply continuous if and only if $f \circ g$ is simply continuous.*

## 2 Result

**Theorem 2.1** *Let $X$ be a Baire space such that the family of all connected open sets is a $\pi$-base for $X$ and there is a dense set in $X$ of the first category. Then $\mathfrak{A}(\mathcal{S}) = \mathfrak{M}(\mathcal{S}) = \mathcal{T}$.*

PROOF. $\mathcal{T} \subset \mathfrak{A}(\mathcal{S})$:
Let $f \in \mathcal{T}$ and $g \in S$. Let $x \in \mathcal{G}(f)$. Then there is an open $G$ such that $x \in G$ and $f$ is constant on $G$. Therefore $f(y) = a$ for each $y \in G$ and some $a \in \mathbb{R}$. Let $U$ be a neighborhood of $x$ and $V$ be an open neighborhood of $(f+g)(x) = a + g(x)$. Then $V - a = \{z \in \mathbb{R} : z + a \in V\}$ is an open neighborhood of $g(x)$. Since $g^{-1}(V-a) \setminus \text{Int } g^{-1}(V-a)$ is not dense in $U \cap G$ and $G \cap (f+g)^{-1}(V) = G \cap g^{-1}(V-a)$, we have $(f+g)^{-1}(V) \setminus \text{Int } (f+g)^{-1}(V)$ is not dense in $U$. Therefore $x \in S_{f+g}$. This yields $\cup\mathcal{G}(f) \subset S_{f+g}$. However $\cup\mathcal{G}(f)$ is open and dense. Hence $X \setminus S_{f+g}$ is nowhere dense and hence by Lemma 1.1 $f+g$ is simply continuous, i.e. $f \in \mathfrak{A}(\mathcal{S})$.

$\mathfrak{A}(\mathcal{S}) \subset \mathcal{T}$:
Let $f \notin \mathcal{T}$. We shall show that there is $g \in \mathcal{S}$ such that $f + g \notin \mathcal{S}$. Evidently, we can assume that $f \in \mathcal{S}$ (otherwise we choose $g = 0$). Since $f \notin \mathcal{T}$, the set $\cup\mathcal{G}(f)$ is not dense in $X$ and hence there is a nonempty open subset $B$ of $X$ such that $B \cap (\cup\mathcal{G}(f)) = \emptyset$. Then $f$ is constant on no nonempty open subset of $B$. We have two possibilities:

a) The set $B \setminus C_f$ is not dense in $B$.
Then there is a nonempty open $H \subset B$ such that $H \subset C_f$. Therefore $f$ is continuous on $H$ and it is constant on no nonempty open subset of $H$. Evidently $H$ satisfies the assumptions of Theorem 2.1.

Let $T \subset H$ be a dense set in $H$ of the first category. Then $T = \cup_{n=1}^{\infty} T_n$, where each $T_n \subset H$ is a nowhere dense set in $H$. We can assume that the sets $T_n$ are pairwise disjoint. Define $h : X \to \mathbb{R}$ as

$$h(x) = \begin{cases} \frac{1}{n} & \text{for } x \in T_n, \\ 0 & \text{otherwise.} \end{cases}$$

Since $h^{-1}((\varepsilon,\infty))$ is a nowhere dense set for each $\varepsilon > 0$, the function $h$ is cliquish. Further $h^{-1}(0)$ is dense in $X$. However, $h^{-1}((0,\infty)) = T$ is a dense set in $H$ with the empty interior and hence $h$ is not simply continuous.

We put $g = h - f$. Then $g_{|H} = h_{|H} - f_{|H}$ is a cliquish function on $H$, $(h_{|H})^{-1}(0)$ is dense in $H$ and $f_{|H}$ is continuous and constant on no nonempty open subset of $H$. Hence by Lemma 1.2 $g_{|H}$ is simply continuous on $H$. Since $h = 0$ on $X \setminus H$, we have $g$ is simply continuous on Int $(X \setminus H)$. Therefore $S_g \subset H \cup (X \setminus \mathrm{Cl}\, H)$. However $X \setminus (H \cup (X \setminus \mathrm{Cl}\, H)) = \mathrm{Cl}\, H \setminus H$ is nowhere dense and thus $X \setminus S_g$ is nowhere dense. According to Lemma 1.1 $g \in \mathcal{S}$. However $f + g = h \notin \mathcal{S}$.

b) The set $B \setminus C_f$ is dense in $B$.
Since $X$ is Baire and $f$ is simply continuous, $X \setminus C_f$ is of the first category ([1]) and therefore $C_f$ is dense in $X$. Denote by $\mathcal{U}(x)$ the family of all neighborhoods of $x$ and $C(f,x) = \bigcap_{U \in \mathcal{U}(x)} \mathrm{Cl}\, f(C_f \cap U)$.

Further set

$$\begin{aligned} E &= \{x \in B : C(f,x) = \emptyset\}, \\ D &= \{x \in B : C(f,x) = \{f(x)\}\}. \end{aligned}$$

Let $x \in C_f$ and $W = [f(x)+1, f(x)-1]$. Then there is an open neighborhood $U_x$ of $x$ such that $\mathrm{Cl}\, f(U_x) \subset W$. Let $u \in U_x$. Then $(\mathrm{Cl}\, f(U_x \cap U \cap C_f))_{U \in \mathcal{U}(u)}$ is a family of closed subsets of $W$ with the finite intersection property. Hence $\bigcap_{U \in \mathcal{U}(u)} \mathrm{Cl}\ f(U \cap U_x \cap C_f)) \neq \emptyset$ and therefore $C(f,u) \neq \emptyset$. This yields $U_x \cap E = \emptyset$. From this we obtain

$$C_f \cap \mathrm{Cl}\, E = \emptyset \tag{1}$$

and therefore $E$ is a nowhere dense set. Evidently $C_f \subset D$. Now put

$$\begin{aligned} A_1 &= \{x \in B : \forall U \in \mathcal{U}(x)\, \forall n \in \mathbb{N}\, \exists t \in U \cap C_f : f(t) > n\}, \\ A_2 &= \{x \in B : \forall U \in \mathcal{U}(x)\, \forall n \in \mathbb{N}\, \exists t \in U \cap C_f : f(t) < -n\}, \\ A_3 &= D \setminus (A_1 \cap A_2). \end{aligned}$$

Let $J \subset B$ be a nonempty open set. Then there is $v \in J \cap C_f$. Hence there is an open neighborhood $P \subset J$ of $v$ and $k \in \mathbb{N}$ such that $f(P) \subset (-k,k)$. Therefore $P \cap A_1 = \emptyset = P \cap A_2$ and $A_1, A_2$ are nowhere dense sets. Hence there is a nonempty open set $L \subset B$ such that $L \cap (A_1 \cup A_2 \cup E) = \emptyset$. Since $C_f \subset D$, we have $D$ is dense in $L$.

Now we shall show that the set $B \setminus (D \cup E)$ is not nowhere dense in $B$. Suppose to the contrary that $B \setminus (D \cup E)$ is nowhere dense in $B$. Then there is a nonempty open $M \subset L$ such that $M \cap (B \setminus (D \cup E)) = \emptyset$. Therefore $M \subset D \cup E$ and since $L \cap (A_1 \cup A_2 \cup E) = \emptyset$, $M \subset A_3$.

Let $x \in M$. Then there is a neighborhood $U \subset M$ of $x$ and $n \in \mathbb{N}$ such that $f(t) \in (-n, n)$ for each $t \in U \cap C_f$. Then for every $t \in U \cap C_f$ there is an open neighborhood $U_t \subset U$ such that $f(U_t) \subset (-n, n)$. Let $K = \bigcup_{t \in U \cap C_f} U_t$. Then $K$ is an open dense set in $U$ and $f(K) \subset (-n, n)$.

Since $B \setminus C_f$ is dense in $B$, there is $u \in K \setminus C_f$. Since $u \notin C_f$, there is $\varepsilon > 0$ such that for each neighborhood $S \subset K$ of $u$ there is $w_S \in S$ such that

$$|f(u) - f(w_S)| > 2\varepsilon. \tag{2}$$

Since $K \subset U$, we have $u \in M \subset D$.

Suppose that for each neighborhood $P \subset K$ of $u$ there is $y_P \in P \cap C_f$ such that $|f(y_P) - f(u)| \geq \varepsilon$. Then $f(y_P) \in [-n, f(u) - \varepsilon] \cup [f(u) + \varepsilon, n]$. Therefore $(\mathrm{Cl}\, f(P \cap C_f) \setminus (f(u) - \varepsilon, f(u) + \varepsilon))_{P \in \mathcal{U}(u), P \subset K}$ is a family of closed subsets of $[-n, n]$ with the finite intersection property. Therefore there is

$$s \in \bigcap_{P \in \mathcal{U}(u), P \subset K} \mathrm{Cl}\, f(P \cap C_f) \setminus (f(u) - \varepsilon, f(u) + \varepsilon).$$

Thus $s \in C(f, u)$ and since $|s - f(u)| \geq \varepsilon$, we obtain $s \neq f(u)$. However then $u \notin D$, a contradiction.

Therefore there is an open neighborhood $Z \subset K$ of $u$ such that $f(y) \in (f(u) - \varepsilon, f(u) + \varepsilon)$ for each $y \in Z \cap C_f$. Since $w_Z \in Z \subset D$, there is an open neighborhood $J \subset K$ of $w_Z$ such that $|f(w_Z) - f(t)| < \varepsilon$ for each $t \in J \cap C_f$. Since $J \cap Z$ is a nonempty open set, there is $z \in J \cap Z \cap C_f$. Then we have $|f(w_Z) - f(z)| < \varepsilon$ and $|f(z) - f(u)| < \varepsilon$. Therefore

$$|f(u) - f(w_Z)| \leq |f(u) - f(z)| + |f(z) - f(w_Z)| < 2\varepsilon,$$

contrary to (2). Therefore the set $B \setminus (D \cup E)$ is not nowhere dense in $B$.

Then there is a nonempty open $H \subset B$ such that $B \setminus (D \cup E)$ is dense in $H$. If $x \in B \setminus (D \cup E)$, then there is $x^* \in C(f, x)$ such that $f(x) \neq x^*$. Define a function $g : X \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^*, & \text{if } x \in H \setminus (D \cup E), \\ f(x), & \text{if } x \in H \cap (D \cup E), \\ 0, & \text{if } x \in X \setminus H. \end{cases}$$

We shall show that $g$ is simply continuous. Let $x \in C_f \cap H$ and $\varepsilon > 0$. Then there is an open neighborhood $F \subset H$ of $x$ such that $f(F) \subset (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. According to (1) $U = F \setminus \mathrm{Cl}\, E$ is an open neighborhood of $x$. Let $u \in U$. Then $u \notin E$. If $u \in H \cap D$, then $g(u) = f(u) \in (f(x) - \frac{\varepsilon}{2}, f(x) + \frac{\varepsilon}{2})$. If $u \notin D$, then $u \in H \setminus (D \cup E)$ and hence $g(u) \in C(f, u) \subset \mathrm{Cl}\, f(F) \subset$

$[f(x)-\frac{\varepsilon}{2}, f(x)+\frac{\varepsilon}{2}] \subset (f(x)-\varepsilon, f(x)+\varepsilon)$. Therefore $x \in C_g$. If $x \in X \setminus \mathrm{Cl}\, H$, then evidently $x \in C_g$. Therefore

$$(C_f \cap H) \cup (X \setminus \mathrm{Cl}\, H) \subset C_g \tag{3}$$

and $C_g$ is dense set in $X$.

Let $x \in H \setminus E$. Let $U \subset H$ be a neighborhood of $x$ and $\varepsilon > 0$. Since $g(x) \in C(f,x)$, we have $f(U \cap C_f) \cap (g(x)-\frac{\varepsilon}{2}, g(x)+\frac{\varepsilon}{2}) \neq \emptyset$. Let $t \in U \cap C_f$ be such that $|g(x) - f(t)| < \varepsilon$. By (3) we have $t \in C_g$. Since $C_f \subset D$, we have $g(t) = f(t)$. Then there is a nonempty open $G \subset U$ such that $t \in G$ and $|g(u) - g(t)| < \frac{\varepsilon}{2}$ for each $u \in G$. Then for each $u \in G$ we have

$$|g(u) - g(x)| \leq |g(u) - g(t)| + |g(t) - g(x)| < \varepsilon.$$

Therefore $x \in Q_g$.

This and (3) give $X \setminus Q_g \subset E \cup (\mathrm{Cl}\, H \setminus H)$ and thus $X \setminus Q_g$ is a nowhere dense set. Since $Q_g \subset S_g$, according to Lemma 1.1 we get that $g$ is a simply continuous function and also $-g \in \mathcal{S}$. However the function $h = f - g$ is not simply continuous, since $h^{-1}(\mathbb{R} \setminus \{0\}) \cap H = H \setminus (D \cup E)$ is a dense set with the empty interior in $H$

$\mathcal{T} \subset \mathfrak{M}(\mathcal{S})$:

If $c \in \mathbb{R}$ and $f : X \to \mathbb{R}$ is simply continuous, then similar to the proof that $c + f$ we can prove that $c \cdot f$ is simply continuous. Let $f \in \mathcal{T}$ and $g \in \mathcal{S}$. Then $\cup \mathcal{G}(f) \subset S_{f \cdot g}$ and by Lemma 1.1 $f \cdot g \in \mathcal{S}$.

$\mathfrak{M}(\mathcal{S}) \subset \mathcal{T}$:

Let $f \notin \mathcal{T}$. We can assume that $f \in \mathcal{S}$. (Otherwise we choose $g = 1$.)

$\alpha$) Let $f$ be positive. Then by Lemma 1.3 $\ln f \notin \mathcal{T}$ and since $\mathfrak{A}(\mathcal{S}) = \mathcal{T}$, there is a simply continuous function $h : X \to \mathbb{R}$ such that $\ln f + h$ is not simply continuous. Then by Lemma 1.3 $e^h \in \mathcal{S}$ and $f \cdot e^h = e^{\ln f + h} \notin \mathcal{S}$. Similarly for negative $f$.

$\beta$) Let $f$ be positive (negative) on some nonempty open set $G$. Then by $\alpha$) there is a simply continuous function $h : G \to \mathbb{R}$ such that $f \cdot h$ is not simply continuous (on $G$). Let $g : X \to \mathbb{R}$, $g(x) = h(x)$ for $x \in G$ and $g(x) = 0$ otherwise. Then by Lemma 1.1 $g \in \mathcal{S}$ and $f \cdot g \notin \mathcal{S}$.

$\gamma$) Let $f^{-1}((0,\infty))$ be dense on some nonempty open set $G$. Then simply continuity of $f$ gives $\mathrm{Int}\, f^{-1}((0,\infty)) \neq \emptyset$ and by $\beta$) there is $g \in \mathcal{S}$ such that $f \cdot g \notin \mathcal{S}$. Similarly if $f^{-1}((-\infty,0))$ is dense on some nonempty open set $G$.

$\delta$) Let $f^{-1}((0,\infty))$ and $f^{-1}((-\infty,0))$ be nowhere dense sets. Then there is a nonempty open dense set $G$ such that $f(y) = 0$ for each $y \in G$. However then $f \in \mathcal{T}$, a contradiction.□

## 3 Remarks

**Remark 3.1** *By* [8, Proposition 1.9] *every $T_1$-space with no isolated points having a $\sigma$-locally finite base has a dense subspace of the first category.*

**Remark 3.2** *Theorem 2.1 does not hold for an arbitrary topological space. Let $X$ be as in* [5], *i.e.* $X = \mathbb{N}$, $\mathcal{D}$ *an ultrafilter on $X$, which contains no finite sets and* $\mathcal{E} = \mathcal{D} \cup \{\emptyset\}$ *be a topology on $X$. Then each function on $X$ is simply continuous and each nonempty open subset of $X$ is infinite. Hence* $\mathcal{S} = \mathbb{R}^X = \mathfrak{A}(\mathcal{S}) = \mathfrak{M}(\mathcal{S}) \neq \mathcal{T}$.

Denote by $\mathcal{C}$ the class of all continuous functions and by $\mathcal{Q}$ the class of all quasicontinuous functions. Further set

$$\begin{aligned} \mathcal{C}^\star &= \{f : X \to \mathbb{R} : X \setminus C_f \text{ is nowhere dense}\}, \\ \mathcal{Q}^\star &= \{f : X \to \mathbb{R} : X \setminus Q_f \text{ is nowhere dense}\}. \end{aligned}$$

By [9] $\mathcal{Q}^\star$ is the lattice generated by $\mathcal{Q}$ (if $X$ is a separable metrizable space without isolated points). In [7] it is shown that $\mathfrak{A}(\mathcal{Q}) = \mathcal{C}$ and in [6] that $\mathfrak{M}(\mathcal{Q}) = \{f \in \mathcal{Q} :$ if $x \notin C_f$, then $f(x) = 0$ and $x \in \mathrm{Cl}\ (C_f \cap f^{-1}(0))\}$ ($X$ is an arbitrary topological space). Therefore $\mathfrak{A}(\mathcal{Q}) \neq \mathfrak{M}(\mathcal{Q})$. We shall show that for $\mathcal{Q}^\star$ we have $\mathfrak{A}(\mathcal{Q}^\star) = \mathfrak{M}(\mathcal{Q}^\star)$.

**Theorem 3.1** *Let $X$ be a Baire space. Then* $\mathfrak{A}(\mathcal{Q}^\star) = \mathfrak{M}(\mathcal{Q}^\star) = \mathcal{C}^\star$.

PROOF. It is easy to see that $Q_f \cap C_g \subset Q_{f+g} \cap Q_{f \cdot g}$. Hence $X \setminus Q_{f+g} \subset (X \setminus Q_f) \cup (X \setminus C_g)$. Therefore $\mathcal{C}^\star \subset \mathfrak{A}(\mathcal{Q}^\star) \cap \mathfrak{M}(\mathcal{Q}^\star)$.

Let $f \in \mathcal{Q}^\star \setminus \mathcal{C}^\star$. Then there is a nonempty open set $B$ such that $B \setminus C_f$ is dense in $B$. The function $g$ from part b) of the proof of Theorem 2.1 is such that $X \setminus Q_f$ is nowhere dense and $X \setminus Q_{f-g}$ is not nowhere dense.

Now if $f^{-1}(0)$ is dense, then there is an open dense set $G$ such that $f(y) = 0$ for every $y \in G$. However then $X \setminus C_f$ is nowhere dense, a contradiction. Hence $f^{-1}(0)$ is not dense and the proof is the same as for $\mathfrak{M}(\mathcal{S})$.□

## References


[1] N. Biswas, *On some mappings in topological spaces*, Bull. Calcutta Math. Soc. **61** (1969), 127–135.

[2] J. Borsík and J. Doboš, *A note on real cliquish functions*, Real Analysis Exch. **18** (1992-93), 139–145.

[3] J. Borsík and J. Doboš, *On simple continuity points*, Real Analysis Exch. **16** (1990-91), 552–558.

[4] A. M. Bruckner, *Differentiation of Real Functions*, Springer-Verlag, New York, 1978.

[5] J. Doboš, *Simple continuity and cliquishness*, Časopis Pěst. Mat. **112** (1987), 355–358.

[6] Z. Grande, *On the maximal multiplicative family for the class of quasi-continuous functions*, Real Analysis Exch. **15** (1989-90), 437–441.

[7] Z. Grande and L. Soltysik, *Some remarks on quasicontinuous real functions*, Problemy Mat. **10** (1990), 79–86.

[8] R. C. Haworth and R. A. McCoy, *Baire Spaces*, Dissertationes Math. **141**, Warszawa, 1977.

[9] T. Natkaniec, *On the maximum and the minimum of quasi-continuous functions*, Math. Slovaca **42** (1992), 103–110.

[10] T. Neubrunn, *Quasi-continuity,* Real Analysis Exch **14** (1988–89), 295–306.

[11] A. Neubrunnová, *On certain generalizations of the notion of continuity,* Mat. časopis **23** (1973), 374–380.

[12] J. C. Oxtoby, *Cartesian product of Baire spaces*, Fund. Math. **49** (1961), 156–170.