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GROUPS WITH NORMALITY CONDITIONS FOR SUBGROUPS OF INFINITE RANK

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Abstract: A well-known theorem of B. H. Neumann states that a group has finite conjugacy classes of subgroups if and only if it is central-by-finite. It is proved here that if G is a generalized radical group of infinite rank in which the conjugacy classes of subgroups of infinite rank are finite, then every subgroup of G has finitely many conjugates, and so $G/Z(G)$ is finite. Corresponding results are proved for groups in which every subgroup of infinite rank has finite index in its normal closure, and for those in which every subgroup of infinite rank is finite over its core.

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1. Introduction

A subgroup X of a group G is called *almost normal* if it has only finitely many conjugates in G, or equivalently if the normalizer $N_G(X)$ of X has finite index in G. A famous theorem by B. H. Neumann [**12**] states that all subgroups of a group G are almost normal if and only if the centre $Z(G)$ has finite index. This result was later extended by I. I. Eremin [**10**], who proved that a group is finite over its centre if and only if all its abelian subgroups are almost normal. The imposition of the almost normality condition to the members of some other relevant system of subgroups gives rise to the consideration of interesting group classes (see for instance [**3, 4**]).

Recall that a group G is said to have *finite rank* $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. The investigation of the influence on a (generalized) soluble group of the behavior of its subgroups of infinite rank has been developed in a series of papers (see for instance [**5, 6, 8, 9**]). In many relevant cases, it turns out that the

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imposition of an embedding property χ to all subgroups of infinite rank of a group G implies that either G has finite rank or all its subgroups have the property χ. For instance, M. J. Evans and Y. Kim [**11**] have proved that if G is a locally soluble group of infinite rank in which all subgroups of infinite rank are normal, then G is a Dedekind group. More recently, N. N. Semko and S. N. Kuchmenko [**14**] have investigated the structure of groups in which many 'large' subgroups are almost normal, and among other results they have proved that if G is a generalized radical group whose subgroups of infinite rank are almost normal, then either G is central-by-finite or it has finite abelian section rank. Here a group G is called *generalized radical* if it has an ascending series whose factors are either locally finite or locally nilpotent, and G has *finite abelian section rank* if it has no infinite abelian sections of prime exponent. Since there exist soluble groups of infinite rank having finite abelian section rank, the quoted result by Semko and Kuchmenko has a gap, which is filled up by our main theorem.

Theorem A. *Let G be a generalized radical group in which every subgroup of infinite rank is almost normal. Then either G has finite rank or its centre $Z(G)$ has finite index.*

A subgroup X of a group G is said to be *nearly normal* if it has finite index in its normal closure X^G. In the same paper quoted above, Neumann also proved that in a group G all subgroups are nearly normal if and only if the commutator subgroup G' of G is finite. This theorem was extended to groups whose abelian subgroups are nearly normal by M. J. Tomkinson [**16**]. Here we will prove the following result.

Theorem B. *Let G be a generalized radical group in which every subgroup of infinite rank is nearly normal. Then either G has finite rank or its commutator subgroup G' is finite.*

Recall finally that a subgroup X of a group G is *normal-by-finite* if the core X_G of X in G has finite index in X. Although the structure of groups in which all subgroups are normal-by-finite is not completely clear, it was proved in [**1**] that locally finite groups with this property are abelian-by-finite, and this theorem was extended to more general situations in [**15**]. For groups with many normal-by-finite subgroups, we have the following result.

Theorem C. *Let G be a generalized radical group in which every subgroup of infinite rank is normal-by-finite. Then either G has finite rank or all its subgroups are normal-by-finite and G is abelian-by-finite.*

Most of our notation is standard and can be found in [**13**].

2. Subgroups with the Neumann property

In order to give a common approach to the various results of Neumann quoted in the introduction (and taking in mind the celebrated theorem of I. Schur on the finiteness of the commutator subgroup of a group whose centre has finite index), we introduce the following concept.

Let G be a group. We shall say that a subgroup X of G has the *Neumann property* if there exists a normal subgroup H of G such that the indices $|G : H|$ and $|H'X : X|$ are finite. In particular, a normal subgroup N of G has the Neumann property if and only if the factor group G/N is finite-by-abelian-by-finite. It is also clear that any subgroup containing a subgroup with the Neumann property likewise has the Neumann property.

Lemma 1. *Let G be a finitely generated soluble group. If all subgroups of infinite rank of G have the Neumann property, then G has finite rank.*

Proof: Assume for a contradiction that G has infinite rank. As G is soluble, it contains an abelian subgroup A of infinite rank, and A can be chosen to be either free abelian or a direct product of subgroups of prime order. Then

$$A = A_1 \times A_2,$$

where both factors A_1 and A_2 have infinite rank. By hypothesis, A_1 and A_2 have the Neumann property, and hence there exist normal subgroups H_1 and H_2 of G such that the indices $|G : H_i|$ and $|H_i'A_i : A_i|$ are finite, for $i = 1, 2$. Consider the normal subgroup $H = H_1 \cap H_2$ of G. Then the indices $|H'A_1 : A_1|$ and $|H'A_2 : A_2|$ are finite, and so the commutator subgroup H' of H is finite, because $A_1 \cap A_2 = \{1\}$. Moreover, the factor group G/H is finite, so that G is finite-by-abelian-by-finite and hence polycyclic. This contradiction shows that G has finite rank. □

Lemma 2. *Let G be a generalized radical group whose subgroups of infinite rank have the Neumann property. Then G is locally (soluble-by-finite).*

Proof: Assume for a contradiction that the statement is false. Since the hypotheses are inherited by subgroups, we may choose a finitely generated counterexample G. It follows from a result of N. S. Černikov [**2**] that G has infinite rank. Let $\mathfrak{L}$ be a chain of normal subgroups of G such that for every $L \in \mathfrak{L}$ the factor group G/L is not soluble-by-finite, and put

$$K = \bigcup_{L \in \mathfrak{L}} L.$$

Assume that G/K is soluble-by-finite. Then G/K has finite rank by Lemma 1, so that K has infinite rank and hence it has the Neumann property. It follows that G/K is finite-by-abelian-by-finite, and so K is the normal closure of a finite subset (see for instance [**13**, Part 1, Lemma 1.43]). This contradiction shows that G/K is not soluble-by-finite, and hence an application of Zorn's lemma yields that $\mathfrak{L}$ has a maximal element M. As G/M is a counterexample, replacing G by G/M we may suppose that all proper homomorphic images of G are soluble-by-finite. Clearly, G contains a non-trivial normal subgroup N, which is either locally finite or locally nilpotent. The factor group G/N is soluble-by-finite, and hence it has finite rank by Lemma 1. Thus N has infinite rank, and so it contains an abelian subgroup A of infinite rank. Since A has the Neumann property, there exists a normal subgroup H of G such that the indices $|G : H|$ and $|H'A : A|$ are finite. Therefore G is soluble-by-finite, and this contradiction completes the proof of the lemma. □

The following lemma is a direct consequence of a result by D. J. S. Robinson [**13**, Part 2, Lemma 10.39].

Lemma 3. *Let G be a locally soluble group of finite rank. Then G has a characteristic ascending series with abelian factors.*

It is known that a group of infinite rank contains an abelian subgroup of infinite rank, provided that it is either locally finite or locally nilpotent. Our next lemma shows that a similar conclusion also holds for generalized radical groups which are also locally (soluble-by-finite).

Lemma 4. *Let G be a generalized radical group which is also locally (soluble-by-finite). If G has infinite rank, then it contains an abelian subgroup of infinite rank.*

Proof: Assume for a contradiction that G has no abelian subgroups of infinite rank. It is known that G contains a locally soluble subgroup U of infinite rank (see [**7**]). Let M be the subgroup generated by all normal subgroups of finite rank of U. It follows from Lemma 3 that M has an U-invariant ascending series with abelian factors. In particular, M is hyperabelian, so that it must have finite rank, because there are no abelian subgroups of infinite rank. Therefore all non-trivial normal subgroups of U/M have infinite rank. As G is a generalized radical group, U/M has a non-trivial normal subgroup L/M which is either locally finite or locally nilpotent. Moreover, L/M has infinite rank, and so it contains an abelian subgroup of infinite rank V/M. Then V is a hyperabelian group of infinite rank, and hence it has an abelian subgroup of infinite rank. This contradiction proves the statement. □

A result similar to our main theorems can be proved also for the Neumann property.

Theorem 5. *Let G be a generalized radical group in which every subgroup of infinite rank has the Neumann property. Then either G has finite rank or it is finite-by-abelian-by-finite.*

Proof: It follows from Lemma 2 that the group G is locally (soluble-by-finite), so that by Lemma 4 it contains an abelian subgroup A of infinite rank. Clearly, A contains a subgroup of the form $B = B_1 \times B_2$, where both factors B_1 and B_2 are subgroups of infinite rank. Then B_1 and B_2 have the Neumann property, and hence there exist normal subgroups of finite index H_1 and H_2 of G such that the indices $|H_1'B_1 : B_1|$ and $|H_2'B_2 : B_2|$ are finite. It follows that the intersection $H = H_1 \cap H_2$ is a normal subgroup of finite index of G, and the indices $|H'B_1 : B_1|$ and $|H'B_2 : B_2|$ are finite. Therefore the commutator subgroup H' of H is finite, and G is finite-by-abelian-by-finite. □

3. Proofs of the theorems

The proofs of our main theorems essentially depend on the fact that in all three cases the subgroups have the Neumann property. In order to prove Theorem A, we need also the following result, which shows that certain groups are rich of normal subgroups.

Lemma 6. *Let G be a group, and let A be an abelian normal subgroup of G such that each subgroup of A has finitely many conjugates in G. If A has infinite rank, then there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of finitely generated G-invariant subgroups of A such that*

$$\langle X_n \mid n \in \mathbb{N}\rangle = \operatorname*{Dr}_{n\in\mathbb{N}} X_n$$

and $r(X_n) < r(X_{n+1})$ for every positive integer n. Moreover, the subgroups X_n can be chosen torsion-free if A has infinite torsion-free rank, and of prime exponent otherwise.

Proof: Assume first that the abelian subgroup A has infinite torsion-free rank. Put $X_1 = \{1\}$, and suppose that $X_2, \dots, X_k$ are finitely generated torsion-free G-invariant subgroups of A such that

$$\langle X_1, \dots, X_k\rangle = X_1 \times \cdots \times X_k$$

and $r(X_i) < r(X_{i+1})$ for all $i < k$. Consider the finitely generated G-invariant subgroup

$$X = \langle X_1, \dots, X_k\rangle,$$

and let M be a subgroup of A which is maximal with respect to the condition $M \cap X = \{1\}$. Then A/MX is periodic, and so A/M has finite torsion-free rank. Since M has finitely many conjugates in G, the factor group A/M_G has likewise finite torsion-free rank. It follows that the core M_G has infinite torsion-free rank, and hence it contains a finitely generated torsion-free subgroup Y_{k+1} such that $r(X_k) < r(Y_{k+1})$. The normal closure Y_{k+1}^G is a finitely generated subgroup of M, and contains a characteristic torsion-free subgroup X_{k+1} of finite index. Therefore

$$\langle X_1, \dots, X_k, X_{k+1} \rangle = X_1 \times \cdots \times X_k \times X_{k+1}$$

and the statement is proved in this case.

Assume now that A has finite torsion-free rank, and let T be the subgroup consisting of all elements of finite order of A. As A/T has finite rank, T must have infinite rank, so that also its socle S has infinite rank. Put $X_1 = \{1\}$, and suppose that $X_2, \dots, X_k$ are finite G-invariant subgroups of S with prime-power order such that

$$\langle X_1, \dots, X_k \rangle = X_1 \times \cdots \times X_k$$

and $r(X_i) < r(X_{i+1})$ for all $i < k$. Then

$$E = \langle X_1, \dots, X_k \rangle$$

is a finite subgroup of S, and $S = E \times V$ for a suitable subgroup V. Since V has finitely many conjugates in G, the factor group S/V_G is finite, and hence the core V_G contains a finite subgroup Y_{k+1} of prime-power order such that $r(X_k) < r(Y_{k+1})$. The normal closure $X_{k+1} = Y_{k+1}^G$ is likewise finite of prime-power order, and it is contained in V, so that

$$\langle X_1, \dots, X_k, X_{k+1} \rangle = X_1 \times \cdots \times X_k \times X_{k+1}$$

and the proof is complete. □

Lemma 7. *Let G be a group in which every subgroup of infinite rank is almost normal. Then all subgroups of infinite rank of G have the Neumann property.*

Proof: Let X be any subgroup of infinite rank of G. Then the normalizer $N_G(X)$ has finite index in G and all subgroups of the factor group $N_G(X)/X$ are almost normal, so that the first Neumann's theorem yields that $N_G(X)/X$ is finite over its centre. Therefore $N_G(X)/X$ has finite commutator subgroup by Schur's theorem. If H is the core of $N_G(X)$ in G, it follows that the indices $|G : H|$ and $|H'X : X|$ are finite, so that the subgroup X has the Neumann property. □

Proof of Theorem A: Of course, we may suppose that the group G has infinite rank. It follows from Lemma 7 that all subgroups of infinite rank of G have the Neumann property, so that G is locally (soluble-by-finite) by Lemma 2. Application of Lemma 4 yields that there exists in G an abelian subgroup A of infinite rank. Clearly, A contains a subgroup of the form

$$B = B_1 \times B_2,$$

where both factors B_1 and B_2 are subgroups of infinite rank. Then B_1 and B_2 are almost normal in G, so that their normalizers $N_G(B_1)$ and $N_G(B_2)$ have finite index in G. Moreover, the factor groups $N_G(B_1)/B_1$ and $N_G(B_2)/B_2$ are central-by-finite, since all their subgroups are almost normal. Put

$$C_i/B_i = Z\big(N_G(B_i)/B_i\big)$$

for $i = 1, 2$. Then $C_1 \cap C_2$ is a subgroup of finite index of G, so that also its core C has finite index. As

$$C' \leq C_1' \cap C_2' \leq B_1 \cap B_2 = \{1\},$$

the normal subgroup C is abelian. Since C has infinite rank, it follows from Lemma 6 that there exists a sequence $(X_n)_{n\in\mathbb{N}}$ of finitely generated G-invariant subgroups of C (which are either all torsion-free or all of prime exponent) such that

$$X = \langle X_n \mid n \in \mathbb{N}\rangle = \operatorname*{Dr}_{n\in\mathbb{N}} X_n$$

and $r(X_n) < r(X_{n+1})$ for each positive integer n.

Let H be any subgroup of finite rank of G. As C is abelian, the normalizer $N_G(H\cap C)$ is a subgroup of finite index of G, and of course it contains H. Replacing G by the group of infinite rank $N_G(H\cap C)/H\cap C$, in order to prove that H is almost normal in G, it can be assumed without loss of generality that $H \cap C = \{1\}$. Consider now two G-invariant subgroups of infinite rank Y_1 and Y_2 of X such that $Y_1 \cap Y_2 = \{1\}$. Then the products HY_1 and HY_2 are almost normal subgroups of G, and hence also

$$H = HY_1 \cap HY_2$$

is almost normal. Therefore all subgroups of G are almost normal, and so the factor group $G/Z(G)$ is finite. □

Lemma 8. *Let G be a group in which every subgroup of infinite rank is nearly normal. Then all subgroups of infinite rank of G have the Neumann property.*

Proof: Let X be any subgroup of infinite rank of G. Then X has finite index in its normal closure X^G. Moreover, all subgroups of the factor group G/X^G are almost normal, so that it has finite commutator subgroup by the second theorem of Neumann. It follows that the index $|G'X : X|$ is finite, and the subgroup X has the Neumann property. □

Proof of Theorem B: Of course, we may suppose that the group G has infinite rank. It follows from Lemma 8 that all subgroups of infinite rank of G have the Neumann property, so that G is locally (soluble-by-finite) by Lemma 2, and hence it contains an abelian subgroup A of infinite rank by Lemma 4. Clearly, A contains a subgroup of the form $B = B_1 \times B_2$, where both factors B_1 and B_2 are subgroups of infinite rank. Then B_1 and B_2 are nearly normal in G, so that the indices $|B_1^G : B_1|$ and $|B_2^G : B_2|$ are finite, and hence the intersection

$$N = B_1^G \cap B_2^G$$

is a finite normal subgroup of G. On the other hand, the factor groups G/B_1^G and G/B_2^G have finite commutator subgroups, since all their subgroups are nearly normal. Therefore also the commutator subgroup G' of G is finite. □

Lemma 9. *Let G be a generalized radical group in which every subgroup of infinite rank is normal-by-finite. Then all subgroups of infinite rank of G have the Neumann property.*

Proof: Let X be any subgroup of infinite rank of G. Then the core X_G has finite index in X, and in particular X_G has infinite rank. It follows that all subgroups of the factor group G/X_G are normal-by-finite. As every periodic homomorphic image of G is locally finite, we have that G/X_G contains an abelian normal subgroup A/X_G such that G/A is finite (see [**15**, Theorem 2]). Clearly, the index $|A'X : X|$ is finite, and hence the subgroup X has the Neumann property. □

Proof of Theorem C: Of course, we may suppose that the group G has infinite rank. It follows from Lemma 9 that all subgroups of infinite rank of G have the Neumann property, so that G is locally (soluble-by-finite) by Lemma 2, and hence there exists in G an abelian subgroup A of infinite rank by Lemma 4. Clearly, A contains a subgroup B of infinite rank which is either free abelian or a direct product of subgroups of prime order. Let H be any subgroup of finite rank of G. Clearly, B contains a subgroup C such that $H \cap C = \{1\}$ and $C = C_1 \times C_2$, where C_1 and C_2 have infinite rank. Since C_1 and C_2 are normal-by-finite in G, they can be even chosen to be G-invariant. Then the products HC_1 and HC_2 are normal-by-finite subgroups of G, and so also

$$H = HC_1 \cap HC_2$$

is normal-by-finite. Therefore all subgroups of G are normal-by-finite. As all periodic homomorphic images of G are locally finite, it follows that G contains an abelian subgroup of finite index (see [**15**, Theorem 2]). □

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