11. *The Mean Convergence for Ergodic Theorems*

By Kazuo ANZAI
Keio University

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1. **Introduction.** In this paper we deal with the equivalence of various ways of convergence in the ergodic theorems and establish an ergodic theorem for the family of operators. More precisely, it is shown that conditions considered by L. W. Cohen and W. F. Eberlein are equivalent in some sense. This result is applied to get a mean ergodic theorem for families of commuting operators. The details will be published elsewhere.

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2. **Preliminaries.** We call a matrix (a_{ni}) satisfying the condition (E) if the matrix satisfies the following conditions

(i) $\lim_{n\to\infty} a_{ni}=0$ $(i=0,1,2,\cdots)$,

(ii) $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_{ni}=1$,

(iii) $\sum_{i=0}^{\infty} |a_{ni}|\leqq K$ $(n=1,2,3,\cdots)$,

(iv) $\lim_{k\to\infty} \sum_{i=k}^{\infty} |a_{ni+1}-a_{ni}|=0$ uniformly in n.

Let T be a bounded linear mapping on a Banach space B such that $\|T^n\|\leqq A$. Then W. Cohen showed that this condition (E) is sufficient condition for the following to have. If $V_n x=\sum_{i=0}^{\infty} a_{ni}T^i x$ is sequentially compact, then the sequence converges strongly to an element $x_0\in B$ and $Tx_0=x_0$.

Let X be a locally convex space and T a continuous linear mapping of X to X. Let $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$, where (a_{ni}) is a matrix that $V_n(T)$ is well defined as a continuous linear mapping of X to X. Then we call $V_n(T)$ satisfying the condition (E_1) if

(i) $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_{ni}=1$,

(ii) $\lim_{n\to\infty} (I-T)V_n(T)x=0$ for $x\in X$,

(iii) $\{V_n(T): n=1,2,3,\cdots\}$ is equi-continuous.

Throughout this paper we denote by F_T and I_T the set of fixed points of mappings T and T^* respectively, where T^* is an adjoint mapping of T. $\frac{1}{n}\sum_{i=1}^{n} T^i$ is denoted by $M_n(T)$.

Remark 1. If T is a linear mapping on a complete locally convex space X such that the family of mappings $\{T^n: n=1,2,3,\cdots\}$ is equi-continuous, and let a matrix (a_{ni}) satisfy the condition (E), then $V_n(T)=\sum_{i=0}^{\infty} a_{ni}T^i$ satisfies the condition (E_1).

3. Equivalence. We shall show that for the summing process in the mean ergodic theorems, various matrices satisfying the condition (E) can be used indifferently.

Theorem 1. *Let X be a complete locally convex space and T a continuous linear mapping of X to X such that the family of mappings $\{T^n: n=1,2,3,\cdots\}$ is equi-continuous. If, for a given $x\in X$, there exists a matrix (a_{ni}) satisfying the condition* (E) *and a subsequence of $V_n(T)x=\sum_{i=0}^{\infty}a_{ni}T^ix$ which converges weakly to an element $x_0\in X$, then $x_0\in F_T$, and for every matrix (b_{ni}) satisfying the condition* (E),

$$x_0=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}T^ix=\lim_{n\to\infty}\sum_{i=0}^{\infty}b_{ni}T^ix.$$

Remark 2. Theorem 1 still remains to hold, even if we replace the condition (iv) on the matrix (a_{ni}) by the weaker condition

$$\text{(iv}'\text{)}\ \lim_{n\to\infty}(I-T)\sum_{i=0}^{\infty}a_{ni}T^i=0.$$

Remark 3. If there exists a constant $K(n)$ depending on n such that, for any $i(\geqq K(n))$, $a_{ni}=0$, $b_{n_i}=0$, the completeness assumption of Theorem 1 is not necessary.

Next, we shall show that Theorem 1 is still valid in the normed linear space if we replace the condition (E) by the condition (E_1).

Theorem 2. *Let X be a normed linear space and T a bounded linear operator on X such that $\|T\|\leqq 1$. If there exists a sequence $V_n(T)$ satisfying the condition* (E_1) *such that, for every $x\in X$, a subsequence of $V_n(T)x$ converges weakly to an element $x_0\in X$, then, for every $U_n(T)$ satisfying the condition* (E_1), *$x_0\in F_T$ and*

$$x_0=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}T^ix=\lim_{n\to\infty}U_n(T)x.$$

Corollary. *Let X and T satisfy the hypotheses of Theorem 2. Then the following conditions are equivalent.*

(i) *For any $x\in X$, there exists an $x_0\in X$ such that*

$$x_0=w-\lim_{n'\to\infty}M_{n'}(T)x.$$

(ii) *There exists a sequence $V_n(T)$ satisfying the condition* (E_1) *such that, for every $x\in X$, there exists an $x_0\in X$ which is the weak limit of the subsequence $V_{n'}(T)x$.*

(iii) *For any $x\in X$ and $U_n(T)$ satisfying the condition* (E_1), *there exists an $x_0\in X$ such that $x_0\in F_T$ and*

$$x_0=\lim_{n\to\infty}M_n(T)x=\lim_{n\to\infty}U_n(T)x.$$

(iv) *F_T separates the points of I_T.*

Let S be a compact Hausdorff space. Then a linear space M of continuous functions on S is called a function space if M separates the points of S and contains the constants. We consider M a normed linear space with sup-norm and denote by $\partial_M S$ the Choquet boundary.

Then we have the following.

Proposition 3. *Let M be a function space on compact Hausdorff space and T a bounded linear operator on M such that $\|T\| \leqq 1$. If there exists a sequence $V_n(T)$ satisfying the condition* (E_1) *such that, for every $f \in M$, there exist an element $f_0 \in M$ and a subsequence $V_{n'}(T)$ of $V_n(T)$ such that*

$$f_0(x) = \lim_{n' \to \infty} V_{n'}(T) f(x) \qquad \text{for } x \in \partial_M S,$$

then, for every $U_n(T)$ satisfying the condition (E_1) *and an $f \in M$, there exists an $f_0 \in M$ such that $Tf_0 = f_0$ and*

$$f_0 = \lim_{n \to \infty} M_n(T) f = \lim_{n \to \infty} U_n(T) f.$$

Theorem 2 and Proposition 3 yield the following.

Corollary. *Let (S, Ω, μ) be a positive σ-finite measure space, and T a contraction linear operator on L^1. If there exists a sequence $V_n(T)$ satisfying the condition* (E_1) *such that, for every $f \in L^1$, there exist an $f_0 \in L^1$ and a subsequence $V_{n'}(T)f$ of $V_n(T)f$ such that*

$$\int f_0 g \, d\mu = \lim_{n' \to \infty} \int g V_{n'}(T) f \, d\mu$$

whenever g is an extreme point of the unit ball of L^∞. Then, for every $f \in L^1$, there exists an $f_0 \in L^1$ such that

$$f_0 = T f_0 \text{ and } f_0 = \lim_{n \to \infty} M_n(T) f = \lim_{n \to \infty} V_n(T) f.$$

4. Several operators. Theorem 4. *Let X be a locally convex space. Consider a finite number of commuting continuous linear mappings from X to X such that*

(a) *for every $x \in X$ and T_j $(1 \leqq j \leqq J)$, there exists an $x_{0_j} \in X$ depending on x and T_j such that $x_{0_j} = w - \lim_{n' \to \infty} M_{n'}(T_j) x$,*

(b) *for each T_j $(1 \leqq j \leqq J)$, the family $\{T_j^n : n = 1, 2, 3, \cdots\}$ is equicontinuous.*

Let T be a convex combination $\sum_{j=1}^J \alpha_j T_j$ of linear mappings T_j $(1 \leqq j \leqq J)$, where $0 < \alpha_j < 1$ and $\sum_{j=1}^J \alpha_j = 1$. Then, for every $x \in X$, there exists an $x_0 \in X$ depending on x such that

(1) $x_0 = \lim_{n \to \infty} M_n(T) x = \lim_{n \to \infty} M_n(T_1) M_n(T_2) \cdots M_n(T_J) x$,

(2) $x_0 \in F_T = \bigcap_{j=1}^J F_{T_j}$.

Let S be a compact Hausdorff space and $C_R(S)$ a Banach space of real valued continuous functions on S. A Markov operator T is a positive ($Tf \geqq 0$ whenever $f \geqq 0$) linear mapping with $T1 = 1$. We call a Markov operator T uniformly mean stable (*u.m.s.*) if $M_n(T)f$ converges uniformly for every f in $C_R(S)$. We have the following corollary which states a more precise result than Sine's [7, Theorem].

Corollary. *Let $T = \sum_{j=1}^J \alpha_j T_j$ be a convex combination of commuting u.m.s. Markov operator T_j $(1 \leqq j \leqq J)$, where $\sum_{j=1}^J \alpha_j = 1$ and $0 < \alpha_j < 1$ $(1 \leqq j \leqq J)$. Then, for any $f \in C_R(S)$, there exists an $f_0 \in C_R(S)$ which satisfies the following*

(1) $f_0=\lim_{n\to\infty} M_n(T)f=\lim_{n\to\infty} M_n(T_1)M_n(T_2)\cdots M_n(T_J)f$,

(2) $f_0 \in \bigcap_{j=1}^{J} F_{T_j}$.

References

[1] K. Anzai: On a mean ergodic theorem for several operators (to appear in some journal).

[2] ——: On the weak conditions for ergodic theorems (to appear in some journal).

[3] L. W. Cohen: On the mean ergodic theorem. Ann. of Math., **41**, 505–509 (1940).

[4] W. F. Eberlein: Abstract ergodic theorems and weak almost periodic functions. Trans. Amer. Math., **67**, 217–240 (1949).

[5] R. Sine: A mean ergodic theorem. Proc. Amer. Math. Soc., **24**, 438–439 (1970).

[6] ——: Convex combination of uniformly mean stable Markov operators. Proc. Amer. Math. Soc., **51**, 123–126 (1975).

[7] K. Yosida: Mean ergodic theorem in Banach space. Proc. Imp. Acad. Sci., **14**, 292–294 (1938).