97. *On Kronecker Limit Formula for Real Quadratic Fields*

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1. Let F be the real quadratic field with discriminant d embedded in the real field $\boldsymbol{R}$. Let χ be a primitive character of the group of the ideal class group modulo $\mathfrak{f}$ of F. *Assume that for a principal integral ideal* (μ) *of* F, $\chi((\mu))$ *is given by the following formula* (1).

(1) $$\chi((\mu))=\operatorname{sgn}(\mu)\chi_0(\mu),$$

where χ_0 is a character of the group of residue classes modulo $\mathfrak{f}$. Let $L_F(s,\chi)$ be the Hecke L-function of F associated with the character χ. In this note, we present a formula for the value $L_F(1,\chi)$ which seems to be new and suggestive. For previously known relevant results, we refer to E. Hecke [1], [2], G. Herglotz [3], C. Meyer [4], C. L. Siegel [6] and D. Zagier [7].

2. For a pair of positive numbers $a=(a_1,a_2)$, set

$$\begin{aligned} c_1(a)=&\frac{1}{a_1}\sum_{n=1}^{\infty}\left\{\psi\left(\frac{na_2}{a_1}\right)-\log\left(\frac{na_2}{a_1}\right)+\frac{a_1}{2na_2}\right\}\\ &+\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\log a_1-\frac{1}{2a_1}(\gamma-\log 2\pi)\\ &+\frac{a_1-a_2}{2a_1a_2}\log\frac{a_2}{a_1}-\frac{\gamma}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)\end{aligned}$$

and set

$$c_2(a)=\frac{1}{2a_1^2}\sum_{n=1}^{\infty}\left\{\psi'\left(\frac{na_2}{a_1}\right)-\frac{a_1}{na_2}\right\}+\frac{\pi^2}{12a_1^2}-\frac{1}{2a_1a_2}\log a_2+\frac{\gamma}{2a_1a_2},$$

where γ is the Euler constant and ψ is the logarithmic derivative of the gamma function.

Denote by $F(a,z)$ an entire function of z given by the following:

$$\begin{aligned} F(a,z)=&z\exp\{-c_1(a)z-c_2(a)z^2\}\Pi'\left(1+\frac{z}{na_1+ma_2}\right)\\ &\times\exp\left\{-\frac{z}{na_1+ma_2}+\frac{z^2}{2(na_1+ma_2)}\right\}\end{aligned}$$

where the product is over all pairs (n,m) of *non-negative integers* which are not simultaneously equal to zero.

We note that the function $F(a,z)^{-1}$ is the *double gamma function* introduced and studied by Barnes in [8].

Let $\varepsilon>1$ be the generator of the group of totally positive units of F. Choose a complete set of representatives $\mathfrak{a}_1,\mathfrak{a}_2,\cdots,\mathfrak{a}_{h_0}$ of the group of *narrow* ideal classes of F. For each k $(1\leq k\leq h_0)$ set

$$R_k(\mathfrak{f})=\{z=x+\varepsilon y \in (\mathfrak{a}_k\mathfrak{f})^{-1};\ x, y \in \boldsymbol{Q}, 0<x\leq 1, 0\leq y<1\}.$$

It is easy to see that $R_k(\mathfrak{f})$ is a finite subset of $(\mathfrak{a}_k\mathfrak{f})^{-1}$. For each $z \in R_k(\mathfrak{f})$, set

$$\chi_k(z)=\chi(\mathfrak{a}_k\mathfrak{f}(z)).$$

The L-function $L_F(s,\chi)$ is an entire function of s which satisfies the following functional equation (2).

$$(2)\qquad A^{1-s}\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{2-s}{2}\Big)L_F(1-s,\chi) = w(\chi)A^s\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big)L_F(s,\chi^{-1}),$$

where $A=\sqrt{dN(\mathfrak{f})}/\pi$ ($N(\mathfrak{f})$ is the norm of $\mathfrak{f}$), and $w(\chi)$ is a certain complex number of modulus 1.

Theorem 1. *Notations being as above, let χ be a primitive character of the group of ideal classes modulo $\mathfrak{f}$ of F which is of the form* (1), *then*

$$w(\chi)^{-1}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)=-\sum_{k=1}^{h_0}\sum_{z\in R_k(\mathfrak{f})}\chi_k^{-1}(z)\log\{F((1,\varepsilon),z)F((1,\varepsilon'),z')\}$$

(ε' *and* z' *are conjugates of* ε *and* z *repectively*).

Let $K\subset\boldsymbol{R}$ be a quadratic extension of F in which *exactly one of two archimedean primes of* F ramifies. Let $\mathfrak{d}$ be the relative discriminant of K with respect to F and let χ be the character of the group of ideal classes modulo $\mathfrak{d}$ of F which corresponds to K in class field theory. Assume that the fundamental unit η_0 of F is the m-th power of a primitive unit η_1 of K ($m\geq 1$). Further take a unit η of K so that $\pm\eta_1$ and η generate the group of units of K. Denote by η^σ the conjugate of η with respect to F and denote by h_F (resp. h_K) the class number of F (resp. K). We may assume that $\eta>|\eta^\sigma|>0$.

Corollary to Theorem 1. *Notation being as above,*

$$(|\eta^\sigma|/\eta)^{h_K}=\prod_{k=1}^{h_0}\prod_{z\in R_k(\mathfrak{d})}\{F((1,\varepsilon),z)F((1,\varepsilon'),z')\}^{mh_F\chi_k(z)}.$$

3. The next two propositions are proved by straightforwards arguments involving only elementary theory of functions.

Proposition 1. *The function $F(a,z)$ is an entire function of z of order 2 which is symmetric with respect to a_1 and a_2 and satisfies the following difference equations*:

$$F(a,z+a_1)=\frac{1}{\sqrt{2\pi}}F(a,z)\Gamma\Big(\frac{z}{a_2}\Big)\exp\Big\{\Big(\frac{z}{a_2}-\frac{1}{2}\Big)\log a_2\Big\},$$

$$F(a,z+a_2)=\frac{1}{\sqrt{2\pi}}F(a,z)\Gamma\Big(\frac{z}{a_1}\Big)\exp\Big\{\Big(\frac{z}{a_2}-\frac{1}{2}\Big)\log a_1\Big\}.$$

For a positive number $\lambda<1$, we denote by $I_\lambda(+\infty)$ (resp. $I_\lambda(1)$) the integral path in the complex plane consisting of the linear segment $(+\infty,\lambda)$ (resp. $(1,\lambda)$), the counterclockwise circle of radius λ around the origin and of the linear segment $(\lambda, +\infty)$ (resp. $(\lambda, 1)$).

Proposition 2. *Notations being as above, if a_1 and a_2 are linearly independent over the rational number field $\boldsymbol{Q}$, there exists a constant $C(a)$ which does not depend on z such that*

$$-\frac{1}{2\pi i}\int_{I_\lambda(+\infty)} \frac{e^{(a_2+a_2-z)t}}{(e^{a_1t}-1)(e^{a_2t}-1)}\frac{\log t}{t}dt$$

$$=\log F(a, z)+\frac{(\gamma-\pi i)}{2a_1a_2}\{z^2-(a_1+a_2)z\}+C(a)$$

$$(\mathrm{Re}\, z>0,\quad 0<\lambda<1,\quad \lambda<(2\pi)/a_1,\quad \lambda<(2\pi)/a_2).$$

For a pair of *positive numbers* $a=(a_1, a_2)$ and a pair of *non-negative numbers* $x=(x_1, x_2)\neq 0$, set

$$\zeta(s, a, x)=\sum_{n,m=0}^{\infty}\prod_{k=1}^{2}\{x_1+m+(x_2+n)a_k\}^{-s},$$

then the Dirichlet series $\zeta(s, a, x)$ is absolutely convergent if $\mathrm{Re}\, s>1$ and is extended to a meromorphic function in the whole complex plane.

Proposition 3. *Notations being as above, we have*

$$\left\{\frac{d}{ds}\zeta(s, a, x)\right\}_{s=0}=-\log\{F(1, a_1), x_1+x_2a_1)F((1, a_2), x_1+x_2a_2)\}$$

$$+(2\gamma-3\pi i)\zeta(0, a, x)-\frac{\gamma-\pi i}{2}\sum_{k=1}^{2}\left\{\frac{(x_1+x_2a_k)^2}{a_k}-\left(1+\frac{1}{a_k}\right)(x_1+x_2a_k)\right\}$$

$$+\left\{\frac{1}{4}i\pi\left(\frac{1}{a_1}+\frac{1}{a_2}\right)+\frac{a_1-a_2}{4a_1a_2}\log\left(\frac{a_2}{a_1}\right)\right\}B_2(x_1)+\pi iB_1(x_1)B_1(x_2)$$

$$+\frac{\pi i}{4}(a_1+a_2)B_2(x_2)+C((1, a_1))+C((1, a_2)),$$

where B_1 and B_2 are, respectively, the first and the second Bernoulli polynomial and the constant $C((1, a_k))$ $(k=1, 2)$ is as in Proposition 2.

Proof. If $\mathrm{Re}\, s>1$, we have the following integral representation for $\zeta(s, a, x)$.

$$\Gamma(s)^2\zeta(s, a, x)=\int_0^\infty\int_0^\infty (t_1t_2)^{s-1}g(t_1, t_2)dt_1dt_2,$$

where $g(t_1, t_2)=\dfrac{\exp\{(1-x_1)(t_1+t_2)+(1-x_2)(a_1t_1+a_2t_2)\}}{\{1-\exp(t_1+t_2)\}\{1-\exp(a_1t_1+a_2t_2)\}}$.

The integral in the right side of the above equality is equal to

$$\int_0^\infty t^{2s-1}\int_0^1 u^{s-1}g(t, tu)dtdu+\int_0^\infty t^{2s-1}\int_0^1 u^{s-1}g(tu, t)dtdu.$$

Hence, for a sufficiently small positive number λ, we have

$$(1-\exp 4\pi is)(1-\exp 2\pi is)\Gamma(s)^2\zeta(s, a, x)$$

$$(3)\qquad =\int_{I_\lambda(+\infty)} t^{2s-1}dt\int_{I_\lambda(1)} u^{s-1}g(t, tu)du+\int_{I_\lambda(+\infty)} t^{2s-1}dt$$

$$\times\int_{I_\lambda(1)} u^{s-1}g(tu, t)du.$$

Proposition 3 now follows easily from (3) and Proposition 2.

4. It follows from the functional equation (2) that

$$(4)\qquad w(\chi)^{-1}\frac{\sqrt{dN(\mathfrak{f})}}{2\pi}L_F(1,\chi)=\left\{\frac{d}{ds}L_F(s,\chi^{-1})\right\}_{s=0}.$$

On the other hand, it is easy to see that

$$(5)\qquad L_F(s,\chi)=\sum_{k=1}^{h_0}\sum_{z\in R_k(\mathfrak{f})}\chi_k(z)N(\mathfrak{a}_k\mathfrak{f})^{-s}\zeta(s,(\varepsilon,\varepsilon'),(x,y))\qquad (z=x+\varepsilon y).$$

Theorem 1 is now an immediate consequence of (4), (5) and Proposition 3. Details will appear elesewhere.

Remark. The method in the proof of Proposition 3 has been applied in [5] for the evaluation of zeta-functions of totally real algebraic number fields at non positive integers.

References

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