2. On Spaces with a Map $CP^n \to M$ of Degree One

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§ 1. Introduction. Let M be a connected oriented closed topological m manifold. It is known in [2] that if $f\ S^m \to M$ is a map of degree one, then f is a homotopy equivalence. And moreover L.E. Spence has proved in [3] that if a map $f\ S^p \times S^q \to M$ is of degree one then M has the homotopy type of S^{p+q} or f is a homotopy equivalence. In this note we shall consider the case of complex projective space. Then we shall prove

Theorem. *If M admits a map $f\ CP^n \to M$ of degree one, then M has the homotopy type of S^{2n}, CP^n, or cohomological quartenion projective space. Especially if n is odd M has the homotopy type of S^{2n} or CP^n.*

Corollary. *Let QP^n be the n dimensional quaternion projective space. If M admits a map $f\ QP^n \to M$ of degree one, then M has the homotopy type of S^{4n} or f is a homotopy equivalence.*

§ 2. Some cohomological conditions. At first we note the following lemma in [2]

Lemma 1. *Let M, N be connected oriented closed topological n manifold. If $f: M \to N$ is a degree one map, then*

(1) *$f_{\#}\pi_1(M) \to \pi_1(N)$ is an epimorphism.*

(2) *$f_* H_i(M) \to H_i(N)$ is a split epimorphism.*

(3) *$f^* H^i(M) \to H^i(N)$ is a monomorphism.*

Now let $f: CP^n \to M$ be a map of degree one. Then we obtain from Lemma 1 that M is simply connected and $H^i(M) \cong 0$ ($i=1$ mod. 2). Thus we may assume that $H^{2k}(M) \cong Z$, and $H^i(M) \cong Z$ $(0<i<2k)$.

Lemma 2. $n \equiv 0$ (mod. 2) *and* $H^*(M) = \dfrac{Z[\alpha]}{(\alpha^{n/k}+1)}$

Proof. Let α be a generator of $H^{2k}(M)=Z$, and μ_M be the fundamental class of $H^{2n}(M)$. By (3) of Lemma 1 we have $f\ (\alpha) = mx^k$ $(\mathrm{m} \neq 0)$ where x denotes the generator of $H^2(CP^n)$. Therefore, from $f\ (\alpha^s) = m^s x^{ks}$, we obtain that

$$H^{2i}(M) \cong Z,\ i=0 \text{ (mod. } k) \text{ and } i \leqq n.$$

Suppose that $n=ks+r$ $(0<r<k)$. Then by the duality of $H^*(M)$, we have $H^{2r}(M)=0$. But this contradicts the assumption. Thus we have $k=0$ (mod. n). Next we suppose that $H^{2a}(M)=Z$ $(jk<a<(j+1)k \leqq n$, for some $j)$ and let β be a generator of $H^{2a}(M)$. Then we have

$f(\beta)=px^a (p\neq 0)$. Since $n-k<(n/k-1)k+a-jk<n$ and $f(\alpha^{(n/k)-j}\beta)=m^{(n/k-1)-j}\, px^{m-(j+1)k+a}$. We have $H^*(M)=Z(*=n-(j+1)k+a)$.

This again contradicts the assumption by duality. Thus the additive structure of $H_*(M)$ is as follows,

$$H^{2i}(M)=Z \qquad (i=0 \text{ mod. } k)$$
$$H^{2i}(M)=0 \qquad (i\neq 0 \text{ (mod. } k))$$

Now, since $f(\alpha^{n/k})=f(\alpha^{n/k}=mx^n$ and $f(\mu_M)=x$.
Obviously this means that $H^*(M)$ is isomorphic to the subring of $H^*(CP^n)$ generated by x^k. Thus we have Lemma 2.

§ 3. Proof of the main theorem. If $k=n$ M is obviously S^{2n} up to homotopy. So we assume $k<n$ then $4k$ skelton of M is the form $S^{2k}\cup e^{4k}$ up to homotopy. Hence by Adams' theorem k must be one of $\{1,2,4\}$. If $k=1$, f is a homotopy equivalence. If $k=2$, M is a simply connected cohomological quaternion projective space.

Lemma 3. *Let K be a Poincare complex of the form $S^8\cup e^{16}$ up to homotopy. Then there is no map f $CP^8\to K$ of degree one.*

Proof. Let $\alpha\in H^2(CP^8)$ and $\beta\in H^8(K)$ be generators. Suppose there exists a map f $CP^8\to K$ of degree one. Then by (3) of Lemma 1 $f^*(\beta)=\pm\alpha^4$ *and*

$$o=f^*\mathcal{P}_3^1(\beta)=\mathcal{P}_3^1 f^*(\beta)=\pm\mathcal{P}_3^1(\alpha)=\pm\alpha^4\neq o$$

where $\mathcal{P}_3$ is the 3rd reduced power operation.
This is a contradiction.
Thus we can eliminate the case $k=4$ and the proof is completed. Next let ρ $CP^{2n}\to QP^n$ be a restriction of the natural map $CP^{2n+1}\to QP^n$, and f $QP^n\to M$ be a map of degree one. Since the composition map $f\circ\rho$ $CP^{2n}\to M$ is of degree one, we can apply the theorem to this case. Then obviously M has the homotopy type of sphere or f is a homotopy equivalence. Thus we have the corollary.

Remark. In general we can not get more details about the case of $k=2$. However in the case of $M=S^4\cup e^8$ we can prove the following result.

If M admits a smooth structure up to homotopy, then M has the same homotopy type as a quaternion projective space. And moreover if M is a Poincare complex, then there exists M of two distinct kind. Of course one of them is a quarternion projective space and another one admits no smooth structure up to homotopy (see [1]).

References

[1] S. Sasao: An example of theorem of W. Browder. J. Math. Soc. Japan, **17**(2), 187–193 (1965).

[2] L. C. Siebenmann: On detecting open collars. Trans. Amer. Math. Soc., **142**, 201–227 (1969).
[3] L. E. Spence: On the image of $S^p \times S^q$ under mappings of degree one. Illinois J. of Math., **17**, 111–114 (1973).