66. *A Remark on Picard Principle. II*

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The purpose of this note is to announce two results on the Picard principle in the unpublished papers [10] and [11] which will be published later elsewhere.

1. A nonnegative locally Hölder continuous function $P(z)$ on $0<|z|\leq 1$ will be referred to as a *density* on $\Omega: 0<|z|<1$. The *elliptic dimension* of a density P on Ω at $\delta: z=0$, $\dim P$ in notation, is the dimension of the half module $\mathcal{P}$ of nonnegative solutions of $\Delta u = Pu$ on Ω with vanishing boundary values on $\partial\Omega: |z|=1$. More precisely, let $\mathcal{P}_1$ be the convex set of $u \in \mathcal{P}$ with the normalization $\int_0^{2\pi} [u_r(re^{i\theta})]_{r=1} d\theta = -1$. Then we define

$$\dim P = \sharp(ex[\mathcal{P}_1]) \tag{1}$$

where $ex[\mathcal{P}_1]$ is the set of extreme points of $\mathcal{P}_1$ and $\sharp$ denotes the cardinal number. We say that the *Picard principle* is valid for P at δ if $\dim P=1$. The study of Picard principle is initiated by Picard, Stozek, and Bouligand. The present formulation as well as the first step to a systematic study is taken by Brelot [1]. For further developments and related works we refer to Heins [3], Ozawa [12], [13], Hayashi [2], Nakai [6]-[9], Kawamura-Nakai [5], among others. The first of our announcements is the following practical test of the Picard principle [10]:

Theorem. *The Picard principle is valid at δ for any finite density P on Ω, i.e. for any density P with the following property*

$$\int_\Omega P(z)dxdy < \infty \qquad (z=x+iy). \tag{2}$$

We shall give an outline of the proof of the above in no. 4. The proof is based on a general theory on the Picard principle originally obtained by Heins [3] and Hayashi [2]. We state this in the next no.

2. Let Ω be an *end* of an m dimensional $(m\geq 2)$ C^∞ Riemannian manifold, i.e. Ω is a manifold with a compact smooth relative boundary $\partial\Omega$ and a single ideal boundary compact δ. A typical example is the one in no. 1: $\Omega: 0<|z|<1$, $\partial\Omega: |z|=1$, $\delta: z=0$. Consider an elliptic differential operator L on $\bar{\Omega}$ given by

$$Lu(x) = \Delta u(x) + b(x)\cdot\nabla u(x) + c(x)u(x) \tag{3}$$

for $u\in C^2(\Omega)$, where Δ is the Laplace-Beltrami operator on the

Riemannian manifold, ∇ the gradient, $b(x)$ a covariant vector of class C^2 on Ω and of class C^1 on $\bar{\Omega}$, and $c(x)$ a locally Hölder continuous function on $\bar{\Omega}$. The *elliptic dimension* of L at δ, $\dim L$ in notation, is given, as in (1), by the following:

$$\dim L = \sharp(ex[\mathscr{P}_1]) \tag{4}$$

where $\mathscr{P}_1$ is the convex set of nonnegative solutions u of $Lu=0$ on Ω with vanishing boundary values on $\partial\Omega$ and with the normalization $\int_{\partial\Omega} (\partial u/\partial n)dS = -1$ where $\partial/\partial n$ is the inner normal derivative and dS the surface element of $\partial\Omega$. We say that the *Picard principle* is valid for L at δ if $\lim_{x\to\delta} u(x)$ exists for every bounded solution u of $Lu=0$ on a neighborhood of δ in Ω. We know that $\dim L > 0$ if and only if $\dim L^* > 0$ where L^* is the adjoint operator to L. In this case there exists a positive solution v of $L^*u=0$ on Ω with boundary values 1 on $\partial\Omega$. We denote by e_L the smallest of such functions v. The *associated operator* $\hat{L}$ with L in (3) is then given by

$$\hat{L}u(x) = \Delta u(x) + (\nabla \log e_L^2(x) - b(x)) \cdot \nabla u(x) \tag{5}$$

for $u \in C^2(\Omega)$. Concerning an operator L and its associated operator $\hat{L}$ we have the following duality relation [11],*) to announce which is our second purpose of this note:

Theorem. *The Picard principle is valid for an operator L at δ if and only if the Riemann theorem is valid for the associated operator $\hat{L}$ at δ.*

3. We state an outline of proof of the above. Let Ω^* be the Martin compactification of Ω with respect to L (cf. e.g. Itô [4], Šur [14]) and $\hat{\mathscr{B}}$ be the Banach space of bounded solutions of $\hat{L}u=0$ with continuous boundary values on $\partial\Omega$. We can see that $\hat{\mathscr{B}} \subset C(\Omega^*)$ and $\hat{\mathscr{B}} | (\Omega^* - \Omega)$ separates points in $\Omega^* - \Omega$. From this the assertion follows.

As an application of the above theorem we state the following rather pathological example. Assume that the *harmonic dimension* of δ is 1, i.e. $\dim \Delta = 1$. The Ω in no. 1 is an example of such. Consider an operator L_c on Ω given by

$$L_c u(x) = \Delta u(x) + \nabla \log e_c^2(x) \cdot \nabla u(x) + c(x)u(x) \tag{6}$$

for $u \in C^2(\Omega)$ where $c(x)$ is a locally Hölder continuous function on $\bar{\Omega}$ such that the equation $\Delta u(x) = c(x)u(x)$ posseses a solution $u > 0$ on Ω with boundary values 1 on $\partial\Omega$, and $e_c(x)$ is the smallest of such functions u. This is the case, for example, when $c(x) \geq 0$ on Ω. By a direct computation we see that $\hat{L}_c = \Delta$ and therefore

$$\dim L_c = 1. \tag{7}$$

Observe that the coefficients of L_c can have arbitrarily high order

*) This is based on an invited hour lecture at the Central Section Meeting of the Mathematical Society of Japan in December, 1974.

singularities at δ and yet (7) is valid. This also adds an example to [9] to show the complexity of the elliptic dimension.

4. Sketch of proof of Theorem in no. 1. Let $e=e_P$ be the P-unit, i.e. e is the unique bounded solution of $\Delta u=Pu$ on Ω with boundary values 1 on $\partial\Omega$. Then the associated operator $\hat{L}$ with $L=\Delta-P$ is given by

$$\hat{L}u(z)=\Delta u(z)+2\nabla \log e(z)\cdot\nabla u(z) \tag{8}$$

for $u\in C^2(\Omega)$. Let u be a bounded solution of $\hat{L}u=0$ on Ω, or more generally a function with the following property:

$$m(r)\leq u(z)\leq M(r)$$

on $0<|z|\leq r$ for every $r\in(0,1]$ where $m(r)=\min_{|z|=r}u(z)$ and $m(r)=\max_{|z|=r}u(z)$. We can then show that $\lim_{z\to 0}u(z)$ exists if $u\in C^1(\Omega)$ and satisfies

$$\int_\Omega |\nabla u(z)|^2\,dxdy<\infty. \tag{9}$$

The condition (9) is satisfied for every bounded solution u of $\hat{L}u=0$ on Ω if the coefficient of (8) has the following property:

$$\int_\Omega |\nabla \log e(z)|^2\,dxdy<\infty. \tag{10}$$

In general we have the following inequality

$$\int_\Omega |\nabla \log e(z)|^2\,dxdy\leq\int_\Omega P(z)(1-e(z))dxdy. \tag{11}$$

Therefore the condition (2) implies (10) by (11) and a fortiori the Riemann theorem is valid at δ for $\hat{L}$ in (8). By the theorem in no. 2 we conclude that the Picard principle is valid for L, i.e. for the finite density P at δ.

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