58. *Micro-local Properties of* $\prod_{j=1}^{n} f_{j+}^{s_j}$ *)

By Masaki KASHIWARA**) and Takahiro KAWAI***)

(Comm. by Kôsaku YOSIDA, M. J. A., April 12, 1975)

In connection with Sato's conjecture in S-matrix theory it has become important to investigate the micro-local properties of the function of the form $\prod_{j=1}^{n} f_{j+}^{s_j}$. See Kawai-Stapp [7] for example. The purpose of this note is to present some basic theorems on the micro-local structure of the function of the above form. The application of the results to the investigation of b-functions will be given somewhere else by the first author. See Kawai-Stapp [7] for the application of the results of this note to the micro-local study of the S-matrix and related functions.

The essential tool in our proof is the desingularization theorem of Hironaka (Hironaka-Lejeune-Teissier [5]). The usefulness of the desingularization theorem in investigating analytic properties of $\prod_{j=1}^{n} f_{j+}^{s_j}$ was first conjectured by Professor I. M. Gel'fand. See Bernstein-Gel'fand [3] and Atiyah [1]. See Björk [4] also. Note that Bernstein [2] proved Theorem 1 without making use of the desingularization theorem in the case when $n=1$ and f_1 is a polynomial.

In this note we use the same notations as in Sato-Kawai-Kashiwara [8] and Kashiwara [6] and do not repeat their definitions.

Theorem 1. *Let f_j ($j=1,\cdots,n$) be real valued real analytic functions defined on a real analytic manifold M. Let s_j ($j=1,\cdots,n$) be complex numbers with non-negative real part. Then there exists a maximally overdetermined system $\mathcal{M}$ of linear differential equations such that $u=\prod_{j=1}^{n} f_{j+}^{s_j}$ is a solution of system $\mathcal{M}$.*

Corollary 2. *Under the same assumptions as in Theorem 1 we can find a locally finite family of locally closed submanifolds N_i ($i=1, 2,\cdots$) of M such that*

(1) $$S.S.\ \prod_{j=1}^{n} f_{j+}^{s_j} \subset \bigcup_i \sqrt{-1}\, T_{N_i}^{*}M \cup \left(\bigcup_i \sqrt{-1}\, T_{N_i}^{*}N_i\right)$$

holds.

Relation (1) *implies further that*

(2) $$S.S.\ \prod_{j=1}^{n} f_{j+}^{s_j} \subset \bigcup_i \sqrt{-1}\, S_{N_i}^{*}M.$$

Theorem 3. *Let M be a real analytic manifold and X be its com-*

---

*) Supported by Miller Institute for Basic Research in Science.

**) Mathematical Institute, Nagoya University.

***) Department of Mathematics, University of California, Berkeley and Research Institute for Mathematical Sciences, Kyoto University.

plexification. Assume that there exists an analytic space (possibly with singularities) $\tilde{X}$ *such that* $\psi: \tilde{X} \to X$ *is a finite covering. Let* f_j $(j=1, \cdots, n)$ *be a multi-valued function on* X *such that* $f_j \circ \psi$ *is univalent and analytic on* $\tilde{X}$. *Assume further that there is an open set* D *in* $\psi^{-1}(M)$ *such that* $\psi|_D$ *is an imbedding such that*

(3) $$f_j \circ \psi|_D > 0, \qquad j=1, \cdots, n$$

and

(4) $$\prod_{j=1}^n (f_j \circ \psi)|_{\partial D} = 0$$

holds. Let s_j $(j=1, \cdots, n)$ *be complex numbers with non-negative real part. Then* $u = \prod_{j=1}^n f_{j+}^{s_j}$ *is a solution of a maximally overdetermined system* $\mathcal{M}$ *of linear differential equations.*

Corollary 4. *Under the same assumptions as in Theorem 3 we can find a locally finite family of locally closed submanifolds* N_i $(i=1, 2, \cdots)$ *of* M *such that*

(5) $$\widehat{S.S.}\ \prod_{j=1}^n f_{j+}^{s_j} \subset \bigcup_i \sqrt{-1}\, T_{N_i}^* M \cup \left(\bigcup_i \sqrt{-1}\, T_{N_i}^* N_i\right)$$

holds.

Relation (5) *implies further that*

(6) $$S.S.\ \prod_{j=1}^n f_{j+}^{s_j} \subset \bigcup_i \sqrt{-1}\, S_{N_i}^* M.$$

Remark. The same results holds also for the function of the form $\prod_{j=1}^n f_{j+}^{s_j} (\log f_{j+})^{m_j}$ for complex number s_j with non-negative real part and non-negative integer m_j.

The proof of Theorems 1 and 3 are given in the following two steps:

We first apply Hironaka's theorem on desingularization of analytic spaces so that we find composite of monoidal transforms $\varphi: M' \to M$ with $\varphi^{-1}(\{x \in M; \prod_{j=1}^n f_j(x) = 0\})$ being normal crossing in the case of Theorem 1. In the case of Theorem 3, it is sufficient to find $\varphi: \tilde{X}' \to \tilde{X}$ so that $\tilde{X}'$ is a manifold (i.e. without singularities) and that $\varphi^{-1}(\{\tilde{x} \in \tilde{X}; \prod_{j=1}^n (f_j \circ \psi)(\tilde{x}) = 0\})$ is normal crossing. Then it is clear that $v = \prod_{j=1}^n (f_j \circ \varphi)_+^{s_j}$ $(\prod_{j=1}^n (f_j \circ \psi \circ \varphi)_+^{s_j}$ in the case Theorem 3) satisfies a maximally overdetermined system $\mathcal{N}$ of linear differential equations.

Nextly we apply the following Lemma 5 to $\int v(\varphi^* dx)/dx$, which is equal to $u(x)$ by the definition of the (generalized) integration procedure along fiber. Note that Lemma 5 is a natural and powerful generalization of Theorem 3.5.5 in Sato-Kawai-Kashiwara [8] Chapter II. There the natural map from ρ^{-1} Supp $\mathcal{N} \cap \tilde{\omega}^{-1}(U)$ to U is assumed to be finite for an open set $U \subset P^*X$. In this sense Lemma 5 globalizes the above quoted theorem. See also Sato-Kawai-Kashiwara [8] Chapter I Theorem 2.3.1′ for an analogous theorem on integration along fiber of microfunctions.

Corollary 2 and Corollary 4 follow from Theorem 1 and Theorem 3, respectively, because of the invertibility of elliptic (pseudo-) differential operators (Sato-Kawai-Kashiwara [8] Chapter II Theorem

2.1.1). In the course of the proof we use the stratification techniques to assert that $\widehat{S.S.}\ \mathcal{M}\cap\sqrt{-1}\,T^*M$ has the conormal structure shown in the left hand side of (1) and (5).

Lemma 5. *Let Y and X be complex manifolds and $\varphi\colon Y\to X$ be projective. That is, Y can be imbedded into $X\times \boldsymbol{P}^N$ for some N and φ is a restriction to Y of the natural projection from $X\times\boldsymbol{P}^N$ to X. Let $\mathcal{N}$ be a $\mathcal{D}_Y^f$-Module and assume that there exists a coherent $\mathcal{O}_Y$-Module $\mathcal{N}_0$ such that $\mathcal{N}=\mathcal{D}_Y^f\mathcal{N}_0$ holds. Then $\mathcal{M}^k=R^k\varphi_*(\mathcal{D}^f_{X\leftarrow Y}\otimes^L_{\mathcal{D}_Y^f}\mathcal{N})$ is a coherent $\mathcal{D}_X^f$-Module for every k and*

$$\widehat{S.S.}\ \mathcal{M}^k\subset\tilde{\omega}\rho^{-1}\widehat{S.S.}\ \mathcal{N} \tag{7}$$

holds. Here $\tilde{\omega}=\tilde{\omega}_\varphi$ ($\rho=\rho_\varphi$, resp.) denotes the natural projection from $Y\times_X T^*X$ to $T^*X(T^*Y$, resp.).

The details of this note will be published elsewhere.

References

[1] Atiyah, M. F.: Resolution of singularities and division of distributions. Comm. Pure Appl. Math., **23**, 145–150 (1970).

[2] Bernstein, I. N.: Modules over a ring of differential operators. Study of the fundamental solutions of equations with constant coefficients. Funkts. Analiz i Ego Prilozhen., **5** (2), 1–16 (1970) (in Russian).

[3] Bernstein, I. N., and S. I. Gel'fand: Meromorphic properties of the function P^λ. Funkts. Analiz i Ego Prilozhen., **3** (1), 84–85 (1969) (in Russian).

[4] Björk, J. E.: Dimensions over algebras of differential operators (to appear).

[5] Hironaka, H., M. Lejeune, and B. Teissier: Résolution des singularités des espaces analytiques complexes (to appear).

[6] Kashiwara, M.: Index theorem for a maximally overdetermined system of linear differential equations. Proc. Japan Acad., **49**, 803–804 (1973).

[7] Kawai, T., and H. P. Stapp: Micro-local study of the S-matrix singularity structure (to appear in Proc. Int. Symp. on Math. Problems in Theoretical Physics, Kyoto (1975)). The details of this report will appear elsewhere.

[8] Sato, M., T. Kawai, and M. Kashiwara: Microfunctions and Pseudo-Differential Equations. Lecture Note in Math. No. 287, Springer, Berlin-Heidelberg-New York, pp. 265–529 (1973).