49. *Automorphic Forms and Algebraic Extensions of Number Fields**)

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§0. The purpose of this paper is to present a result on an arithmetical relation between Hilbert cusp forms over a totally real algebraic number field, which is a cyclic extension of the rational number field $\boldsymbol{Q}$ with a prime degree l, and cusp forms of one variable. The details of this result will appear in [7].

Let F be a totally real algebraic number field, and $\mathfrak{o}$ be its maximal order. For an even positive integer κ, let $S_\kappa(\Gamma)$ denote the space of Hilbert cusp forms of weight κ with respect to the subgroup $\Gamma = GL_2(\mathfrak{o})^+$ consisting of all elements with totally positive determinants in $GL_2(\mathfrak{o})$. For a place (archimedean or non-archimedean) v of F, let F_v be the completion of F at v. For a non-archimedean place v $(=\mathfrak{p})$, let $\mathfrak{o}_\mathfrak{p}$ be the ring of $\mathfrak{p}$-adic integers of F_v. Let F_A be the adele ring of F, and consider the adele group $GL_2(F_A)$. Let $\mathfrak{U}_F$ be the open subgroup $\prod_{\mathfrak{p}:\text{ non-archimedean}} GL_2(\mathfrak{o}_\mathfrak{p}) \times \prod_{v:\text{ archimedean}} GL_2(F_v)$ of $GL_2(F_A)$. Then we can consider the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$ and its action $\mathfrak{T}$ on $S_\kappa(\Gamma)$ as in G. Shimura [8].

For the ordinary modular group $SL_2(\boldsymbol{Z})$ $(=GL_2(\boldsymbol{Z})^+)$, we also consider its adelization $\mathfrak{U}_\boldsymbol{Q} = \prod_p GL_2(\boldsymbol{Z}_p) \times GL_2(\boldsymbol{R})$ and the Hecke ring $R(\mathfrak{U}_\boldsymbol{Q}, GL_2(\boldsymbol{Q}_A))$. The latter is acting on the space $S_\kappa(SL_2(\boldsymbol{Z}))$ of cusp forms of weight κ with respect to $SL_2(\boldsymbol{Z})$.

§1. The space $\mathcal{S}_\kappa(\Gamma)$. Suppose F is a cyclic extension of $\boldsymbol{Q}$ of degree l. We fix an embedding of F into the real number field $\boldsymbol{R}$ and a generator σ of the Galois group Gal $(F/\boldsymbol{Q})$ of the extension $F/\boldsymbol{Q}$, then all the distinct embeddings of F into $\boldsymbol{R}$ are given by σ^i, $0 \le i \le l-1$. We consider the group $GL_2(F)$ as a subgroup of $GL_2(\boldsymbol{R})^l$ by $g \to (g, {}^\sigma g, \cdots, {}^{\sigma^{l-1}}g)$ for $g \in GL_2(F)$. For this fixed generator σ, we define an operator T_σ on $S_\kappa(\Gamma)$ by the permutation of variables, namely $T_\sigma f(z_1, \cdots, z_l) = f(z_2, \cdots, z_l, z_1)$ for $f \in S_\kappa(\Gamma)$. Using this T_σ, we define a new subspace $\mathcal{S}_\kappa(\Gamma)$ of $S_\kappa(\Gamma)$, to be called "the space of symmetric Hilbert cusp forms", as follows;

$$\mathcal{S}_\kappa(\Gamma) = \{f \in S_\kappa(\Gamma) \mid \mathfrak{T}(e)T_\sigma f = T_\sigma \mathfrak{T}(e) f \quad \text{for any } e \in R(\mathfrak{U}_F, GL_2(F_A))\}.$$

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Obviously $\mathcal{S}_\kappa(\Gamma)$ is stable under the action of $R(\mathfrak{U}_F, GL_2(F_A))$, and we get a new representation $\mathfrak{T}_\mathcal{S}$ of the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$ on the space $\mathcal{S}_\kappa(\Gamma)$.

Now we assume

0) The weight $\kappa \geq 4$.
1) The degree $l = [F : \boldsymbol{Q}]$ is a prime.
2) The class number of F is one.
3) The maximal order $\mathfrak{o}$ has a unit of any signature distribution.
4) F is tamely ramified over $\boldsymbol{Q}$.

As a consequence of 2) and 4), the conductor of $F/\boldsymbol{Q}$ is a prime q.

Our result claims that the representation $\mathfrak{T}_\mathcal{S}$ of $R(\mathfrak{U}_F, GL_2(F_A))$ on $\mathcal{S}_\kappa(\Gamma)$ can be obtained from those on the spaces of cusp forms $S_\kappa(SL_2(\boldsymbol{Z}))$ and $S_\kappa(\Gamma_0(q), \chi)$ for various characters χ of $(\boldsymbol{Z}/q\boldsymbol{Z})^\times$ of order l, where

$$\Gamma_0(q) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c \equiv 0 \text{ mod. } q\right\},$$

and $S_\kappa(\Gamma_0(q), \chi)$ is the space of cusp forms g which satisfy $g(\gamma z) = \chi(d)(cz+d)^\kappa g(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$.

To give a meaningful description for the above, we shall define a "natural" homomorphism $\lambda : R(\mathfrak{U}_F, GL_2(F_A)) \to R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$ in the next section § 2. On the other hand, $R(\mathfrak{U}_Q, GL_2(F_A))$ is acting not only on $S_\kappa(SL_2(\boldsymbol{Z}))$ but also on $S_\kappa(\Gamma_0(q), \chi)$. For a prime p, let $T(p)$ and $T(p,p)$ be the elements of $R(\mathfrak{U}_Q, GL_2(F_A))$ given in § 2. Then for $p \neq q$, $T(p)$ and $T(p,p)$ act on $S_\kappa(\Gamma_0(q), \chi)$ in the usual manner ([9]). For q, let $\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q) = \bigcup_{\nu=1}^{d} \alpha_\nu \Gamma_0(q)$ be a disjoint union, and put for $g \in S_\kappa(\Gamma_0(q), \chi)$

$$g \left| \left[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q)\right] = \sum_\nu \chi(d_\nu) \frac{(\det \alpha_\nu)^{\kappa/2}}{(-c_\nu z + a_\nu)^\kappa} g(\alpha_\nu^{-1} z)\right.$$

where $\alpha_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$. And we define the action of $T(q)$ and $T(q,q)$ on $S_\kappa(\Gamma_0(q), \chi)$ by

$$T(q)g = g \left| \left[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q)\right] + g \right| \left[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q)\right]^*$$

$$T(q,q)g = g.$$

Here $\left[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q)\right]^*$ denotes the adjoint operator of $\left[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\Gamma_0(q)\right]$ with respect to the Petersson inner product. These actions of $T(p)$ and $T(p,p)$ can be extended to that of $R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$, and we obtain the representations $\mathfrak{T}_1$ and $\mathfrak{T}_\chi$ of $R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$ on $S_\kappa(SL_2(\boldsymbol{Z}))$ and $S_\kappa(\Gamma_0(q), \chi)$, respectively. Thus $S_\kappa(SL_2(\boldsymbol{Z}))$ (resp. $S_\kappa(\Gamma_0(q), \chi)$) can be viewed as a $R(\mathfrak{U}_F, GL_2(F_A))$-module by the action $\mathfrak{T}_1 \circ \lambda$ (resp. $\mathfrak{T}_\chi \circ \lambda$). In these notations, we can prove

Theorem. *There exists a subspace S of $\bigoplus_{\chi} S_\kappa(\Gamma_0(q),\chi)$ such that*

$$\mathcal{S}_\kappa(\Gamma)\simeq S_\kappa(SL_2(\boldsymbol{Z}))\oplus S$$
$$(\text{and } \bigoplus_{\chi} S_\kappa(\Gamma_0(q),\chi)\simeq S\oplus S)$$

as $R(\mathfrak{U}_F, GL_2(F_A))$-modules, where in $\bigoplus_{\chi}$, χ runs through all characters of order l of $(\boldsymbol{Z}/q\boldsymbol{Z})^\times$.

This theorem can be derived by standard arguments from the following equality between the traces of the operators.

Theorem. $\operatorname{tr}\mathfrak{T}_{\mathcal{S}}(e)=\operatorname{tr}\mathfrak{T}_1(\lambda(e))+\frac{1}{2}\sum_{\chi}\operatorname{tr}\mathfrak{T}_\chi(\lambda(e))$

for any $e\in R(\mathfrak{U}_F, GL_2(F_A))$.

§ 2. The homomorphism $\lambda: R(\mathfrak{U}_F, GL_2(F_A))\to R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$. Let $\mathfrak{a}$ (resp. n) be an integral ideal of F (resp. a positive integer), and $T(\mathfrak{a})$ (resp. $T(n)$) be the sum of all integral elements in $R(\mathfrak{U}_F, GL_2(F_A))$ (resp. $R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$) of norm $\mathfrak{a}$ (resp. n). For a prime ideal $\mathfrak{p}$ of F (resp. a prime p), let $T(\mathfrak{p},\mathfrak{p})$ (resp. $T(p,p)$) denote the double coset $\mathfrak{U}_F\alpha\mathfrak{U}_F$ (resp. $\mathfrak{U}_Q\alpha\mathfrak{U}_Q$), where the $\mathfrak{p}$-component (resp. p-component) of α is $\begin{pmatrix}\pi & 0\\ 0 & \pi\end{pmatrix}$ $\left(\text{resp. }\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}\right)$ with a prime element π of $\mathfrak{o}_\mathfrak{p}$, and the other component is the identity. We define elements $U(\mathfrak{p}^m)$ (resp. $U(p^m)$) of $R(\mathfrak{U}_F, GL_2(F_A))$ (resp. $R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$) for a prime ideal $\mathfrak{p}$ of F (resp. a prime p) and a non-negative integer m by

$$U(\mathfrak{o})=2T(\mathfrak{o})$$
$$(\text{resp. } U(1)=2T(1))$$
$$U(\mathfrak{p}^m)=\begin{cases}T(\mathfrak{p}), & m=1\\ T(\mathfrak{p}^m)-N\mathfrak{p}T(\mathfrak{p},\mathfrak{p})T(\mathfrak{p}^{m-2}), & m\geq 2\end{cases}$$
$$\left(\text{resp. } U(p^m)=\begin{cases}T(p), & m=1\\ T(p^m)-pT(p,p)T(p^{m-2}), & m\geq 2\end{cases}\right),$$

where $N\mathfrak{p}$ is the cardinality of $\mathfrak{o}/\mathfrak{p}$. Then the correspondence $U(\mathfrak{p}^m)\to U(N\mathfrak{p}^m)$ can be extended to a homomorphism λ from $R(\mathfrak{U}_F, GL_2(F_A))$ to $R(\mathfrak{U}_Q, GL_2(\boldsymbol{Q}_A))$.

§ 3. **Applications.** Our result is related to the recent works of the following authors.

(I) In their joint work [2], K. Doi and H. Naganuma studied a relation between cusp forms with respect to $SL_2(\boldsymbol{Z})$ and Hilbert cusp forms over real quadratic fields. More precisely, let $\varphi(s)=\sum_{n=1}^{\infty}a_n n^{-s}$, $a_1=1$, be the Dirichlet series associated with a cusp form of weight κ with respect to $SL_2(\boldsymbol{Z})$ which is a common eigen-function for all Hecke operators, and let χ be the real character corresponding to a real quadratic field $F=\boldsymbol{Q}(\sqrt{D})$ in the sense of class field theory. If we put $\varphi(s,\chi)=\sum_{n=1}^{\infty}\chi(n)a_n n^{-s}$, then $\varphi(s)\varphi(s,\chi)$ can be expressed in the following form with suitable coefficients $C_\mathfrak{a}$ which are defined for every integral ideal $\mathfrak{a}$ in F;

$$\varphi(s)\varphi(s,\chi)=\sum_{\mathfrak{a}} C_{\mathfrak{a}} N\mathfrak{a}^{-s}.$$

For a Grössen-character ξ of F, we set

$$D(s,\varphi,\chi,\xi)=\sum_{\mathfrak{a}} \xi(\mathfrak{a}) C_{\mathfrak{a}} N\mathfrak{a}^{-s}.$$

In [2], K. Doi and H. Naganuma tried to prove a functional equation of $D(s,\varphi,\chi,\xi)$ and proved it for the case where the conductor of ξ is one, and showed that if the maximal order of F is an Euclidean domain, the Dirichlet series $\varphi(s)\varphi(s,\chi)$ is actually associated with a Hilbert cusp form over F and the function

$$h(z_1,z_2)=\sum_{\substack{\mathfrak{a}=(\mu)\\ \mu/\sqrt{q}\gg 0}} C_{\mathfrak{a}} \sum_{\varepsilon\in E_+} \exp\left(2\pi\sqrt{-1}\left(\frac{\varepsilon\mu}{\sqrt{q}}z_1+{}^{\sigma}\left(\frac{\varepsilon\mu}{\sqrt{q}}\right)z_2\right)\right)$$

on the product $\mathfrak{H}\times\mathfrak{H}$ of the complex upper half planes is a Hilbert cusp form over F. Moreover in [6], H. Naganuma showed that a similar result holds also for cusp forms of "Neben" type (in Hecke's sense) with a prime level. Now from our present result for $l=2$, it can be proved that $\varphi(s)\varphi(s,\chi)$ is the Dirichlet series associated with a Hilbert cusp form over a real quadratic field F, and that Doi-Naganuma's construction is "injective" under the condition of this paper.

(II) In [5], H. Jacquet studied the similar theme as Doi-Naganuma's, in a more general (adelic and representation-theoretic) point of view, hence this result should have a close connection to ours.

(III) F. Hirzebruch [3] [4] and R. Busam [1] gave a dimension formula for the subspace $S_\kappa(\hat{\Gamma})$ of $S_\kappa(\Gamma)$ consisting of elements f such that $T_\sigma f=(-1)^{\kappa/2}f$. Since there is an obvious relation

$$\dim S_\kappa(\hat{\Gamma})=\frac{1}{2}(\dim S_\kappa(\Gamma)+(-1)^{\kappa/2}\dim \mathcal{S}_\kappa(\Gamma)),$$

our result can be viewed as a generalization of their formula.

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