156. *On the Difference between r Consecutive Ordinates of the Zeros of the Riemann Zeta Function*

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§ 1. **Introduction.** Let γ_n be the n-th ordinate of the zeros of the Riemann zeta function $\zeta(s)$ satisfying $0<\gamma_n\leq\gamma_{n+1}$. Here we are concerned with the following problems.

(i) To estimate $S_{r,k}(T)=\frac{1}{N(T)}\sum_{T<\gamma_n\leq 2T} d(\gamma_n, r)^k$ for integral $k\geqslant 1$ and $r\geqslant 1$, where $N(T)$ is the number of the zeros of $\zeta(s)$ in $0<\mathrm{Re}\, s<1$, $0<\mathrm{Im}\, s\leqslant T$ as usual and $d(\gamma_n, r)$ is $(\gamma_{n+r}-\gamma_n)/r$.

(ii) To estimate the number $N_r\left(\frac{C}{\log T}, T\right)$ of γ_n in $T<\gamma_n\leqslant 2T$ satisfying $d(\gamma_n, r)\geqslant C/\log T$.

As to (i) we have shown in [1], [3] that

$$S_{1,2}(T)\ll(\log T)^{-2}.$$

On the other hand the following result is announced in Zentralblatt [4];

$$S_{1,2k+1}(T)\ll\frac{(2k)!\,2^{2k}(2k+1)(\log\log T)^k}{k!\,(\log T)^{2k+1}}$$

for integral $k=o(\log T)$. Here we shall prove the following

Theorem 1. *Let $T>T_0$. Then for k in $1\leqslant k\ll(T\log T)^{2/3}$ and r in $1\leqslant r\ll k^{3/2}$, we have*

$$S_{r,k}(T)\ll\frac{(Ak)^{3k^2/(2k+1)}\,(\log(3+k))^k r^{-2k^2/(2k+1)}}{(\log T)^k},$$

where A is some positive absolute constant.

As to (ii) we have shown in [1], [3] that

$$N_r\left(\frac{2\pi(1+a)}{\log T}, T\right)\gg N(T)\exp\left(-(\log\log C)^{1-\varepsilon}\right)$$

for $C>C_0$, integral r less than $A\,(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}$ and

$$a=(A\,(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}-r)/(C+A\,(\log C)^{1/2}(\log\log C)^{1/2+\varepsilon}-r),$$

where A's above (and also in this paper) are some positive absolute constants and ε's are arbitrarily small positive numbers. Here we shall prove

Theorem 2. *For $T>T_0$, $C>C_0$ and r in $1\leqslant r\leqslant T\log T\,C^{-1}$, we have*

$$N_r\left(\frac{C}{\log T}, T\right)\ll N(T)\exp\left(-A(rC)^{2/3}(\log rC)^{-1/3}\right).$$

§ 2. **Proof of Theorem.**

2–1. To prove our theorem we use the following

Lemma 1.

$$\int_T^{2T} (S(t+h)-S(t))^{2k}dt=\frac{(2k)!}{(2\pi)^{2k}k!}T(\log(3+h\log T))^k$$
$$+0((Ak)^{4k}T(\log(3+h\log T))^{k-1/2})$$

uniformly for positive h, *integral* $k\geqslant 1$ *and* $T>T_0$, *where we put* $S(t)=\frac{1}{\pi}$ *arg* $\zeta(1/2+it)$ *as usual.*

(Cf. Main Theorem in [2].)

2–2. For convenience we put

$$S_{r,l}(h,T)=\frac{1}{N(T)}\sum_{\substack{T<\gamma_n\leq 2T\\ d(\gamma_n,r)\geqslant h}} d(\gamma_n,r)^l$$

for positive h and integral $l\geqslant 1$. Using Lemma 1 we shall prove first

Lemma 2. *For any integral* $l\geqslant 1$, *integral* $k\geqslant 1$, $T>T_0$, *positive* $h\gg(\log T)^{-1}$ *and integral* r *in* $1\leq r\leq T/h$, *we have*

$$S_{r,l}(h,T)\ll\frac{(Ak)^{4k}B(k,l)^{-1}(\log(3+rh\log T))^k}{r^{l-1}(\log T)^l(hr\log T)^{2k-(l-1)}},$$

where we put $B(k,1)=1$ *and* $B(k,l)=(k-1)(k-2)\cdots(k-(l-1))$ *for* $l\geqslant 2$.

Proof. We shall prove this by induction. Suppose that $d(\gamma_n,r)\geqslant h$. Then by the Riemann-von Mangoldt formula ((9.3.2) of [5]), we get

$$\int_{\gamma_n}^{\gamma_n+r-1/2hr}(S(t+rh/2)-S(t))^{2k}dt$$
$$\gg\left(r-\frac{rh\log T}{\pi}+0(1)\right)^{2k}(d(\gamma_n,r)-h/2)r$$
$$\gg(Arh\log T)^{2k}d(\gamma_n,r)r \qquad \text{for } h\log T\geqslant C_0.$$

We sum each side over γ_n satisfying $T<\gamma_n\leqslant 2T$ and $d(\gamma_n,r)\geqslant h$. Then using Lemma 1 we get

$$S_{r,1}(h,T)\ll\frac{(Ak)^{4k}(\log(3+hr\log T))^k}{(rh\log T)^{2k}(\log T)}.$$

Now suppose that our conclusion is true for $l-1$. Then for l,

$$S_{r,l}(h,T)\leqslant\frac{2}{N(T)}\sum_{\substack{T<\gamma_n\leq 2T\\ d(\gamma_n,r)\geqslant h}} d(\gamma_n,r)^{l-1}(d(\gamma_n,r)-h/2)$$
$$\leqslant\frac{2}{N(T)}\sum_{\substack{T<\gamma_n\leq 2T\\ d(\gamma_n,r)\geqslant h/2}} d(\gamma_n,r)^{l-1}\int_{h/2}^{d(\gamma_n,r)}dh$$
$$=2\int_{h/2}^{AT}S_{r,l-1}(h,T)dh$$
$$\ll\frac{(Ak)^{4k}B(k,l-1)^{-1}}{r^{l-2}(\log T)^{l-1}}\int_{h/2}^{AT}\frac{(\log(3+rh\log T))^k}{(rh\log T)^{2k-(1-2)}}dh$$
$$\ll\frac{(Ak)^{4k}B(k,l)^{-1}(\log(3+rh\log T))^k}{r^{l-1}(\log T)^l(rh\log T)^{2k-(l-1)}}.$$

2–3. Proof of Theorem 1. Now for $C>C_0$

$$S_{r,k}(T)\ll S_{r,k}(C/\log T,T)+C^k/(\log T)^k$$

$$\ll \frac{C^k}{(\log T)^k} + \frac{(Ak)^{3k}(\log(3+rC))^k}{r^{2k}C^{k+1}(\log T)^k}.$$

Here we choose $C=(Ak)^{3k/(2k+1)}r^{-2k/(2k+1)}$. Then we get our conclusion.

2–4. Proof of Theorem 2. By Lemma 2 we get

$$\frac{C^k}{(\log T)^k}N_r\left(\frac{C}{\log T}, T\right) \leqslant N(T)S_{r,k}\left(\frac{C}{\log T}, T\right)$$

$$\ll N(T)\frac{(Ak)^{3k}(\log(3+rC))^k}{r^{2k}C^{k+1}(\log T)^k}.$$

Choosing $k=[(r^2C^2/(e^3 \log Cr))^{1/3}A^{-1}]$ we get our conclusion.

References

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