101. *On QF-Extensions in an H-Separable Extension*

By Taichi NAKAMOTO
Okayama College of Science

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Throughout the present note, A/B will represent a ring extension with common identity 1, V the centralizer $V_A(B)$ of B in A, and C the center of A. Following K. Hirata [2], A/B is called an H-separable extension if $A\otimes_B A$ is A-A-isomorphic to an A-A-direct summand of a finite direct sum of copies of A. To be easily seen, A/B is H-separable if and only if there exist some $v_i \in V$ $(i=1, \cdots, m)$ and casimir elements $\sum_j x_{ij}\otimes y_{ij}$ of $A\otimes_B A$ (which means $(\sum_j x_{ij}\otimes y_{ij})x = x(\sum_j x_{ij}\otimes y_{ij})$ for all $x\in A$) such that $\sum_{i,j} x_{ij}\otimes y_{ij}v_i = 1\otimes 1$ (cf. [4; Proposition 1]). Such a system $\{v_i;\sum_j x_{ij}\otimes y_{ij}\}_i$ will be called an H-system for A/B. On the other hand, A/B is called a left QF-extension if ${}_BA$ is finitely generated (abbr. f.g.) projective and there exist some $f_r \in \mathrm{Hom}\,({}_BA_B, {}_BB_B)$ $(r=1, \cdots, n)$ and casimir elements $\sum_s c_{rs}\otimes d_{rs}$ of $A\otimes_B A$ such that $\sum_{r,s} c_{rs}f_r(d_{rs}) = 1$. Such a system $\{f_r;\sum_s c_{rs}\otimes d_{rs}\}_r$ will be called a left QF-system for A/B. Quite symmetrically, a right QF-extension and a right QF-system can be defined, and A/B is called a QF-extension if A/B is left QF and right QF. One will easily see that A/B is QF if and only if there exist a left QF-system and a right QF-system for A/B.

The notion of an H-system will provide a new technique to reconstruct the commutor theory in H-separable extensions developed in [2], [3] and [5]. In this note, we use the technique to prove the following which are motivated by [4; Theorems 4 and 5]:

Theorem 1. *Assume that A/B is an H-separable extension. Let B' be an intermediate ring of A/B with $V'=V_A(B')$ such that $V_A(V') = B'$ and ${}_{V'}V'_{V'} < \oplus_{V'}V_{V'}$ (V' is a V'-V'-direct summand of V).*

(1) *If there exists a left (resp. right) QF-system for A/B' then V'/C is right (resp. left) QF.*

(2) *If there exists a right (resp. left) QF-system for V'/C then $A_{B'}$ (resp. ${}_{B'}A$) is f.g. projective and there exists a left (resp. right) QF-system for A/B'.*

(3) *A/B' is QF if and only if so is V'/C.*

Theorem 2. *Assume that A/B is an H-separable extension. Let B' be an intermediate ring of A/B with $V'=V_A(B')$ such that ${}_{B'}B'_{B'} < \oplus_{B'}A_{B'}$.*

(1) *If there exists a left (resp. right) QF-system for B'/B, then ${}_{V'}V$ (resp. $V_{V'}$) is f.g. projective and there exists a right (resp. left)*

QF-system for V/V'.

(2) *If* $B=V_A(V)$ *and there exists a right* (*resp. left*) *QF-system for* V/V', *then* B'_B (*resp.* ${}_BB'$) *is f.g. projective and there exists a left* (*resp. right*) *QF-system for* B'/B.

(3) *In case* $B=V_A(V)$, B'/B *is QF if and only if so is* V/V'.

In order to prove those above, several results obtained previously in [1] and [3] will be required. However, for the sake of completeness, we shall give self-contained proofs to such preliminary results. In what follows, we assume always A/B is an H-separable extension with an H-system $\{v_i\,;\,\sum_j x_{ij}\otimes y_{ij}\}_i$.

First, we consider the A-A-homomorphism $\eta: A\otimes_B A\to \mathrm{Hom}_C(V, A)$ $(a_1\otimes a_2\mapsto(v\mapsto a_1va_2))$. Since $\sum_{i,j} x_{ij}\otimes y_{ij}a_1v_ia_2=\sum_{i,j} a_1x_{ij}\otimes y_{ij}v_ia_2=a_1\otimes a_2$, we see that η is a monomorphism. Moreover, $\sum_{i,j} x_{ij}\otimes y_{ij}av_i=a\otimes 1$ implies $\sum_{i,j} g(x_{ij})\otimes y_{ij}av_i=g(a)\otimes 1$ ($g\in \mathrm{Hom}\, A_B, A_B), a\in A$). Applying η, we obtain $\sum_{i,j} g(x_{ij})vy_{ij}av_i=g(a)v$ $(v\in V)$. In particular, if there exists a right B-epimorphism $p: A\to B$ which induces the identity map on B then for $a\in V_A(V)$ we have $p(a)=\sum_{i,j} p(x_{ij})y_{ij}av_i=\sum_{i,j} p(x_{ij})y_{ij}v_ia$ $=a$, which means that if $B_B<\oplus A_B$ then $V_A(V)=B$. Finally, given $h\in \mathrm{Hom}_C(V, A)$, there holds $\sum_{i,j} x_{ij}vy_{ij}h(v_i)=h(\sum_{i,j} x_{ij}vy_{ij}v_i)=h(v)$, namely, η is an epimorphism. Summarizing the facts mentioned above, we obtain the following:

Lemma 1. (1) $\sum_{i,j} g(x_{ij})vy_{ij}av_i=g(a)v$ ($g\in \mathrm{Hom}\,(A_B, A_B), a\in A$, $v\in V$).

(2) $\sum_{i,j} v_iax_{ij}vg(y_{ij})=vg(a)$ ($g\in \mathrm{Hom}\,({}_BA, {}_BA), a\in A, v\in V$).

(3) V_C *is f.g. projective and* η *is an isomorphism whose inverse is given by* $h\mapsto\sum_{i,j} x_{ij}\otimes y_{ij}h(v_i)$ (cf. [1; p. 112]).

(4) *If* $B_B<\oplus A_B$ (*resp.* ${}_BB<\oplus {}_BA$) *then* $V_A(V)=B$ ([3; Proposition 1.2]).

Next, the map $\xi: V\otimes_C V\to \mathrm{Hom}\,({}_BA_B, {}_BA_B)$ $(u_1\otimes u_2\mapsto(a\mapsto u_1au_2))$ is a V-V-isomorphism, whose inverse is given by $h\mapsto\sum_i\sum_j h(x_{ij})y_{ij}\otimes v_i$ $=\sum_i v_i\otimes\sum_j x_{ij}h(y_{ij})$ (Lemma 1). Now, let V' be a subring of V with $B'=V_A(V')$ such that $V_A(B')=V'$ and ${}_{V'}V'_{V'}<\oplus_{V'}V_{V'}$. If $h\in \mathrm{Hom}\,({}_{B'}A_{B'}$, ${}_{B'}A_{B'})$ then $\xi^{-1}(h)=\sum_i\sum_j h(x_{ij})y_{ij}\otimes v_i=\sum_i v_i\otimes\sum_j x_{ij}h(y_{ij})\in(V'\otimes_C V)$ $\cap(V\otimes_C V')=V'\otimes_C V'$ $(\subset V\otimes_C V)$. This proves the following:

Lemma 2. *Let* V' *be a subring of* V *with* $B'=V_A(V')$ *such that* $V_A(B')=V'$ *and* ${}_{V'}V'_{V'}<\oplus_{V'}V_{V'}$. *Then*, ξ *induces a* V'-V'-*isomorphism* $V'\otimes_C V'\cong \mathrm{Hom}\,({}_{B'}A_{B'}, {}_{B'}A_{B'})$, *and so an element* h *of* $\mathrm{Hom}\,({}_{B'}A_{B'}, {}_{B'}A_{B'})$ *is in* $\mathrm{Hom}\,({}_{B'}A_{B'}, {}_{B'}B'_{B'})$ *if and only if* $\xi^{-1}(h)$ *is a casimir element of* $V'\otimes_C V'$.

Proof of Theorem 1. (3) is only a combination of (1) and (2). Let $q: V\to V'$ be an arbitrary V'-V'-epimorphism which induces the identity map on V'.

(1) Let $\{f_r; \sum_s c_{rs}\otimes d_{rs}\}_r$ be a left QF-system for A/B', and $g_r: V' \to C$ the maps given by $v' \mapsto \sum_s c_{rs}v'd_{rs}$. Then, by Lemma 2 and Lemma 1 (2), $\sum_i q(v_i)\otimes\sum_j x_{ij}f_r(y_{ij}) = \sum_i v_i \otimes \sum_j x_{ij}f_r(y_{ij})$ are casimir elements of $V'\otimes_C V'$ and $\{g_r; \sum_i q(v_i)\otimes\sum_j x_{ij}f_r(y_{ij})\}_r$ is a right QF-system for V'/C. Furthermore, V'_C is f.g. projective as a C-direct summand of f.g. projective V_C.

(2) Let $\{g_r; \sum_s u'_{rs}\otimes v'_{rs}\}_r$ be a right QF-system for V'/C, and $f_{ij}: A \to B'$ the right B'-homomorphism given by $a \mapsto \sum_{r,s} u'_{rs}y_{ij}ag_rq(v_i)v'_{rs}$. Then, $\{f_{ij}; x_{ij}\}_{i,j}$ is an f.g. projective coordinate system (abbr. FGP-system) for $A_{B'}$. In fact, by Lemma 1 we have $\sum_{i,j} x_{ij}f_{ij}(a) = \sum_{i,j,r,s} x_{ij}u'_{rs}y_{ij}ag_rq(v_i)v'_{rs} = a\sum_{r,s} g_rq(\sum_{i,j} x_{ij}u'_{rs}y_{ij}v_i)v'_{rs} = a\sum_{r,s} g_r(u'_{rs})v'_{rs} = a$. Finally, if $f_i: {}_{B'}A_{B'} \to {}_{B'}B'_{B'}$ are given by $a \mapsto \sum_{r,s} u'_{rs}ag_rq(v_i)v'_{rs}$ then $\sum_{i,j} x_{ij}f_i(y_{ij}) = \sum_{i,j} x_{ij}f_{ij}(1) = 1$, which means that $\{f_i; \sum_j x_{ij}\otimes y_{ij}\}_i$ is a left QF-system for A/B'.

Lemma 3 ([3; Proposition 1.3]). *Let B' be an intermediate ring of A/B with $V' = V_A(B')$ such that ${}_{B'}B'_{B'} < \oplus_{B'}A_{B'}$. Then, η induces a B'-B'-isomorphism $B'\otimes_B B' \cong \mathrm{Hom}({}_{V'}V_{V'}, {}_{V'}A_{V'})$. Moreover, $V_A(V') = B'$, and so an element h of $\mathrm{Hom}({}_{V'}V_{V'}, {}_{V'}A_{V'})$ is in $\mathrm{Hom}({}_{V'}V_{V'}, {}_{V'}V'_{V'})$ if and only if $\eta^{-1}(h)$ is a casimir element of $B'\otimes_B B'$.*

Proof. Obviously, $(B'\otimes_B A)\cap(A\otimes_B B') = B'\otimes_B B' (\subset A\otimes_B A)$ and η induces a B'-B'-monomorphism $\eta_{B'}: B'\otimes_B B' \to \mathrm{Hom}({}_{V'}V_{V'}, {}_{V'}A_{V'})$. Now, let $p: A \to B'$ be an arbitrary B'-B'-epimorphism which induces the identity map on B'. Since $\sum_j x_{ij}\otimes y_{ij}$ is a casimir element, $\sum_j p(x_{ij})vy_{ij}$ and $\sum_j x_{ij}vp(y_{ij})$ are in V' ($v \in V$). Accordingly, if $h \in \mathrm{Hom}({}_{V'}V_{V'}, {}_{V'}A_{V'})$ then by Lemma 1 (1) we have $\sum_{i,j} p(x_{ij})vy_{ij}h(v_i) = h(\sum_{i,j} p(x_{ij})vy_{ij}v_i) = h(p(1)v) = h(v)$, and similarly by Lemma 1 (2) $\sum_{i,j} h(v_i)x_{ij}vp(y_{ij}) = h(v)$. Hence, $\sum_{i,j} p(x_{ij})\otimes y_{ij}h(v_i) = \eta^{-1}(h) = \sum_{i,j} h(v_i)x_{ij}\otimes p(y_{ij}) \in (B'\otimes_B A)\cap(A\otimes_B B') = B'\otimes_B B'$, which means that $\eta_{B'}$ is an epimorphism. In particular, considering h as the map defined by $v \mapsto xv$ with $x \in V_A(V')$, we have $p(x) = \sum_{i,j} p(x_{ij})y_{ij}xv_i = x\sum_{i,j} p(x_{ij})y_{ij}v_i = x$ (Lemma 1 (1)), which proves $V_A(V') = B'$.

Lemma 4. *Let V' be a subring of V with $B' = V_A(V')$ such that $V_A(B') = V'$. Let B^* be an intermediate ring of B'/B with $V^* = V_A(B^*)$.*

(1) *If ${}_{B'}B'_B < \oplus_{B'}A_B$ then the map ${}_{B^*}\xi': V^*\otimes_{V'} V \to \mathrm{Hom}({}_{B^*}B'_B, {}_{B^*}A_B)$ $(v^*\otimes v \mapsto (b' \mapsto v^*b'v))$ is a V^*-V-isomorphism, whose inverse is given by $h \mapsto \sum_i\sum_j hp(x_{ij})y_{ij}\otimes v_i$, where $p: A \to B'$ is an arbitrary B'-B-epimorphism which induces the identity map on B'. In particular, ${}_B\xi': V\otimes_{V'} V \to \mathrm{Hom}({}_BB'_B, {}_BA_B)$ is a V-V-isomorphism. Moreover, if $h \in \mathrm{Hom}({}_BB'_B, {}_BB_B)$ and $b' \in B'$ then $\sum_i\sum_j hp(x_{ij})y_{ij}\otimes v_i$ is a casimir element of $V\otimes_{V'} V$ and $\sum_{i,j} hp(x_{ij})y_{ij}b'v_i = h(b')$.*

(2) *If ${}_BB'_{B'} < \oplus_BA_{B'}$ then the map $\xi'_{B^*}: V\otimes_{V'} V^* \to \mathrm{Hom}({}_BB'_{B^*}, {}_BA_{B^*})$ $(v\otimes v^* \mapsto (b' \mapsto vb'v^*))$ is a V-V^*-isomorphism, whose inverse is given by $h \mapsto \sum_i v_i\otimes\sum_j x_{ij}hp(y_{ij})$, where $p: A \to B'$ is an arbitrary B-B'-epimor*

phism which induces the identity map on B'.

Proof. It is enough to prove (1). If $h \in \mathrm{Hom}\,({}_{B*}B'_B, {}_{B*}A_B)$ then $\sum_j hp(x_{ij})y_{ij} \in V^*$ and $\sum_{i,j} hp(x_{ij})y_{ij}b'v_i = hp(b') = h(b')$ (Lemma 1 (1)), which means that the map defined by $h \mapsto \sum_i \sum_j hp(x_{ij})y_{ij} \otimes v_i$ is a right inverse of ${}_{B*}\xi'$. Moreover, noting that $\sum_j p(x_{ij})vy_{ij} \in V'$, we have $\sum_i \sum_j v^* p(x_{ij})vy_{ij} \otimes v_i = v^* \otimes v$ (Lemma 1 (1)), which means that the above map is the inverse of ${}_{B*}\xi'$.

Proof of Theorem 2. (3) is a combination of (1) and (2), and $V_A(V') = B'$ by Lemma 3. Let $p: A \to B'$ be an arbitrary B'-B'-epimorphism which induces the identity map on B'.

(1) Let $\{f_r; \sum_s c'_{rs} \otimes d'_{rs}\}_r$ be a left QF-system for B'/B. If $g_i: {}_{V'}V \to {}_{V'}V'$ and $g'_r: {}_{V'}V_{V'} \to {}_{V'}V'_{V'}$ are defined by $v \mapsto \sum_{r,s} c'_{rs} v (\sum_j f_r p(x_{ij})y_{ij}) d'_{rs}$ and $v \mapsto \sum_s c'_{rs} v d'_{rs}$ respectively, then, by Lemma 1 (1) and Lemma 4 (1), one will easily see that $\{g_i; v_i\}_i$ is an FGP-system for ${}_{V'}V$ and $\{g'_r; \sum_i \sum_j f_r p(x_{ij})y_{ij} \otimes v_i\}_r$ is a right QF-system for V/V'.

(2) Let $\{g_r; \sum_s u_{rs} \otimes v_{rs}\}_r$ be a right QF-system for V/V', and $f_{ij}: B' \to B$ the right B-homomorphism given by $b' \mapsto \sum_{r,s} u_{rs} p(y_{ij} g_r(v_i)) b' v_{rs}$. By Lemma 3, $\sum_{i,j} x_{ij} \otimes y_{ij} g_r(v_i) = \sum_{i,j} p(x_{ij}) \otimes p(y_{ij} g_r(v_i))$ is a casimir element of $B' \otimes_B B'$. By Lemma 1, we have then $\sum_{i,j} p(x_{ij}) f_{ij} (b') = \sum_{r,s} \sum_{i,j} p(x_{ij}) u_{rs} p(y_{ij} g_r(v_i)) b' v_{rs} = \sum_{r,s} \sum_{i,j} x_{ij} u_{rs} y_{ij} g_r(v_i) b' v_{rs} = \sum_{r,s} g_r(\sum_{i,j} x_{ij} u_{rs} y_{ij} v_i) b' v_{rs} = b' \sum_{r,s} g_r(u_{rs}) v_{rs} = b'$, which means that $\{f_{ij}; p(x_{ij})\}_{i,j}$ is an FGP-system for B'_B. Finally, if $f_r: {}_BB'_B \to {}_BB_B$ are given by $b' \mapsto \sum_s u_{rs} b' v_{rs}$ then $\sum_{i,j,r} p(x_{ij}) f_r(p(y_{ij} g_r(v_i))) = \sum_{i,j} p(x_{ij}) f_{ij}(1) = 1$. This proves that $\{f_r; \sum_{i,j} p(x_{ij}) \otimes p(y_{ij} g_r(v_i))\}_r$ is a left QF-system for B'/B.

References

[1] K. Hirata: Some types of separable extensions of rings. Nagoya Math. J., **33**, 107–115 (1968).

[2] ——: Separable extensions and centralizers of rings. Nagoya Math. J., **35**, 31–45 (1969).

[3] K. Sugano: Note on semisimple extensions and separable extensions. Osaka J. Math., **4**, 265–270 (1967).

[4] ——: Separable extensions and Frobenius extensions. Osaka J. Math., **7**, 291–299 (1970).

[5] ——: On some commutor theorems of rings. Hokkaido Math. J., **1**, 242–249 (1972).