21. *On Linear Operators with Closed Range*

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Several conditions for a linear operator to have closed range are known (*e.g.* Browder [2], Baker [1], *etc.*). In this short report we will find another condition which is different from those due to Banach, Browder, Baker, *etc.* and will prove to be useful from practical point of view if we apply this theorem to the theory of boundary problems for linear differential equations (*cf.* [4], [5]). A special case was studied by Tréves [7], page 51, where spaces are Fréchet and operators become epimorphisms. Our proof depends on Pták's proof of the open mapping theorem.

Let E and F be (Hausdorff) locally convex spaces, and T a densely defined closed linear operator of E into F. Let E' denote the dual space of E. Let ${}^tT$ be the dual operator of T and $D({}^tT)$ its domain. We call $\mathfrak{B}$ a basis of continuous seminorms on E if its elements are continuous seminorms on E and for every continuous seminorm p on E there exist $q \in \mathfrak{B}$ and a positive constant C such that $p(x) \leqq C \cdot q(x)$, $x \in E$. For $x' \in E'$ and a continuous seminorm p on E, we write

$$\|x'\|_p = \inf \{C > 0\,;\, |x'(x)| \leqq C \cdot p(x),\, x \in E\}.$$

If there exists no such positive constant C, we set $\|x'\|_p = \infty$.

Recall that T is called almost open if, for each neighborhood U of $0 \in E$, the closure of $T(U)$ in F is a neighborhood of $0 \in F$. A locally convex space E is said to be B-complete if a linear continuous and almost open mapping of E onto any locally convex space F is open. Fréchet spaces and strong duals of Fréchet spaces are B-complete (*cf.* [6]).

Theorem. *Let E be a B-complete space and F a fully barrelled space, that is, every closed subspace of F is barrelled. Let $\mathfrak{B}_E$ and $\mathfrak{B}_F$ be bases of continuous seminorms on E and F respectively. Then the range of T is closed if and only if the following two conditions are satisfied.*

(1) *For every seminorm $p \in \mathfrak{B}_E$ there exists another seminorm $q \in \mathfrak{B}_F$ such that $y' \in D({}^tT)$ and $\|{}^tT(y')\|_p < \infty$ implies the existence of $z' \in D({}^tT)$, which satisfies ${}^tT(y') = {}^tT(z')$ and $z' = 0$ on the null space of q.*

(2) *For seminorms $p \in \mathfrak{B}_E$ and $q \in \mathfrak{B}_F$ there exist another seminorm $r \in \mathfrak{B}_E$ and a positive constant C such that the following holds. For every $y' \in D({}^tT)$, which is equal to zero on the null space of q, there*

exists another $z' \in D({}^tT)$, *which is also equal to zero on the null space of* q, *such that* ${}^tT(y') = {}^tT(z')$ *and*

$$\|z'\|_r \leqq C \cdot \|{}^tT(z')\|_p.$$

Outline of the proof. Assume that the range of T is closed. Let $p \in \mathfrak{P}_E$. Write $B = \{y \in R(T); |y'(y)| \leqq \|{}^tT(y')\|_p, y' \in D({}^tT)\}$. Since the range $R(T)$ of T is barrelled, B is a neighborhood of zero in $R(T)$. For some continuous seminorm q on F its restriction to $R(T)$ is the Minkowski functional of B. Now let $y' \in D({}^tT)$ and $\|{}^tT(y')\|_p < \infty$. Then y' is equal to zero on $R(T) \cap \operatorname{Ker} q$, where $\operatorname{Ker} q = \{y \in F; q(y) = 0\}$. Define a continuous linear form z'_0 on $R(T) + \operatorname{Ker} q$ by $z'_0 = y'$ on $R(T)$ and $z'_0 = 0$ on $\operatorname{Ker} q$. From the Hahn-Banach theorem there exists $z' \in F'$ whose restriction to $R(T) + \operatorname{Ker} q$ is equal to z'_0. Then it follows that $z' \in D({}^tT)$, ${}^tT(y') = {}^tT(z')$, and $z' = 0$ on $\operatorname{Ker} q$, and hence (1) holds.

Next let $p \in \mathfrak{P}_E$ and $q \in \mathfrak{P}_F$. Since T is an open mapping onto its range, there exist a seminorm $r^* \in \mathfrak{P}_F$ and a positive constant C such that $y \in R(T)$ and $r^*(y) \leqq 1$ implies the existence of some $x \in E$ which satisfies $p(x) \leqq C$ and $y = T(x)$. Let $y' \in D({}^tT)$ and $y' = 0$ on $\operatorname{Ker} q$. Then for every $y \in R(T)$ we have $|y'(y)| \leqq C \cdot r(y) \cdot \|{}^tT(y')\|_p$, where r is a continuous seminorm on F such that $q(y) = 0$ implies $r(y) = 0$, and $y \in R(T)$ implies $r(y) = \inf\{r^*(z); z \in R(T) \text{ and } q(y-z) = 0\}$. From the Hahn-Banach theorem there exists $z' \in F'$ such that $z' = y'$ on $R(T)$ and $|z'(y)| \leqq C \cdot r(y) \cdot \|{}^tT(y')\|_p$, $y \in F$. Hence we have $z' \in D({}^tT)$, ${}^tT(y') = {}^tT(z')$, and $\|z'\|_r \leqq C \cdot \|{}^tT(z')\|_p$, and thus (2) is proved.

Now we assume that the properties (1) and (2) are true. Let U be a neighborhood of 0 in E. There exist a seminorm $p \in \mathfrak{P}_E$ and a positive constant C such that the set $\{x \in E; p(x) \leqq C\}$ is contained in U. From (1) there exists a seminorm $q \in \mathfrak{P}_F$ which satisfies the requirement of (1). Then from (2) there exist a seminorm $r \in \mathfrak{P}_F$ and a positive constant C which satisfy the requirement of (2). Let N denote the orthogonal space of the null space of ${}^tT$, that is, $N = \{y \in F; {}^tT(y') = 0 \text{ implies } y'(y) = 0\}$.

Let $y \in N$. Take $y' \in D({}^tT)$ such that $\|{}^tT(y')\|_p < \infty$. From (1) there exists $z' \in D({}^tT)$ such that ${}^tT(y') = {}^tT(z')$ and $z' = 0$ on $\operatorname{Ker} q$. Then from (2) there exists $w' \in D({}^tT)$ such that ${}^tT(z') = {}^tT(w')$ and $\|w'\|_r \leqq C \cdot \|{}^tT(w')\|_p$. Since ${}^tT(y' - w') = 0$, we have $y'(y) = w'(y)$. Hence we obtain $|y'(y)| \leqq C \cdot r(y) \cdot \|{}^tT(y')\|_p$, and then $\lambda y \in B' = \{z \in N; |y'(z)| \leqq \|{}^tT(y')\|_p, y' \in D({}^tT)\}$ with some positive constant λ. We have thus proved that B' is absorbing in N. Since N is barrelled, B' is a neighborhood of 0 in N.

It is not difficult to prove that B' is contained in the closure of $T(U)$ in N. Then T is almost open as an operator of E into N. We can then conclude that the range of T is equal to the closed subspace

N, using the results due to Pták [6] and Mochizuki [3].

Remark. *If E and F are both Fréchet spaces, then the hypothesis of the theorem is satisfied.*

References

[1] Baker, J. W.: Operators with closed range. Math. Ann., **174**, 278–284 (1967).

[2] Browder, F. E.: Functional analysis and partial differential equations. I. Math. Ann., **138**, 55–79 (1959).

[3] Mochizuki, N.: On fully complete spaces. Tôhoku Math. J., **13**, 485–490 (1961).

[4] Ohwaki, S.: General boundary problems for linear differential equations. (to appear in Tech. Report R. I. M. S., **201** (1973)).

[5] ——: Elliptic boundary problems in non-compact manifolds. I (to appear).

[6] Pták, V.: Completeness and the open mapping theorem. Bull. Soc. Math. France, **86**, 41–74 (1958).

[7] Tréves, F.: Locally Convex Spaces and Linear Partial Differential Equations. Springer (1967).