144. *The Noetherian Properties of the Oblique Derivative Problems*

By Kazuhiro YAMAMOTO
Department of Mathematics, Hokkaido University

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§0. Introduction and results. In this note we establish the Noetherian properties of the oblique derivative problem, which is a degenerate elliptic boundary value problem studied in Ju. V. Egolov and V. K. Kondrat'ev [1]. They proved, though only partially, existence theorems by using the solution of the Dirichlet problem and elliptic regularization. Our method is different from theirs and our results are more complete and precise. Namely, we reduce the problem to the pseudo differential equation on the boundary, whose principal symbol is Lopatinskian of the considered boundary value problem. In virtue of this we can apply to our problem the regularizer, constructed as in G. I. Èskin [2].

Formulation of the problem. Let Ω be a bounded domain in R^{n+1} with the smooth boundary Γ and Γ_0 be an $(n-1)$-dimensional submanifold of Γ. We consider a second order elliptic differential operator with C^∞-coefficient;

$$L(y,D)=\sum_{k,j=1}^{n+1} a_{kj}(y)D_kD_j+\sum_{k=1}^{n+1} a_k(y)D_k+a(y),$$
$$D_j=\sqrt{-1}\frac{\partial}{\partial y_j}.$$

We assume the following two conditions;

i) $\sum_{k,j=1}^{n+1} a_{kj}(y)\eta_k\eta_j\neq 0$ if $(y,\eta)\in\bar{\Omega}\times(R^{n+1}\backslash 0)$.

ii) For each point $y\in\Omega$ and each pair of linearly independent vectors $\eta,\eta'\in R^{n+1}$, the polynomial $L^0(y,\eta+\tau\eta')$ in complex τ has only one root with negative imaginary part. Here $L^0(y,D)=\sum_{k,j=1}^{n+1} a_{kj}(y)D_kD_j$.

An operator ν is a non-zero C^∞-vector field given in a neighbourhood of Γ satisfying the following two conditions;

i) The coefficient of differentiation in the normal direction appearing in Γ does not vanish in $\Gamma-\Gamma_0$. However, it vanishes on Γ_0.

ii) The restriction of ν to Γ_0 is not a tangent vector of Γ_0.

We shall consider the following boundary value problem; $Lu=f$ in Ω, $\dfrac{\partial u}{\partial\nu}=g$ on Γ. From the assumption i), we remark that this boundary value problem is coercive in the outside of Γ_0 (cf. [3]). However, the problem is degenerate on Γ_0.

By § 2 of [1], for each point P of Γ_0, there exists a non-singular transformation Ψ from a neighbourhood of P to some neighbourhood of the origin of $(z_0, \cdots, z_n)$-space such that

i) The mapping Ψ transforms $\mathfrak{v}$ to $\dfrac{\partial}{\partial z_n}$.

ii) $\Psi(p)=0$ and the image of the manifold Γ_0 belongs to the hyperplane $z_0=z_n=0$.

iii) The image of $\bar{\Omega}$ is described by the inequality;

$$z_0 \geq \omega(z_1, \cdots, z_n),$$

where the equation $z_0=\omega(z_1, \cdots, z_n)$ satisfies the following conditions;

$$\left.\frac{\partial \omega}{\partial z_n}\right|_{z_n=0}=0, \quad \frac{\partial \omega}{\partial z_n} \neq 0 \qquad \text{for } 0<|z_n|<c.$$

By this transformation, the following three cases are possible;

a) $\omega(z_1, \cdots, z_n) \geq 0$ in some neighbourhood of the origin.

b) $\omega(z_1, \cdots, z_n) \leq 0$ in some neighbourhood of the origin.

c) ω is monotonic in z_n in some neighbourhood of the origin.

From the assumption of $\mathfrak{v}$ this classification does not depend on the choice of a point p on Γ_0. We shall suppose that there is a number $k_0>0$ such that in a neighbourhood of P

$$\left|\frac{\partial \omega}{\partial z_n}\right|(z) \geq c z_n^{k_0}, \qquad c>0.$$

Under this condition, we obtain the following three theorems. We assume $s>1$ in all the cases.

Theorem A. *If the function $\omega(z)$ satisfies the condition* a), *then the operator $\mathfrak{A}_1(u)=\left(Lu, \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma}, u|_{\Gamma_0}\right)$ is a Noetherian operator from the Sobolev space $H_s(\Omega)$ to*

$$H_{s-2+k_0/(k_0+1)}(\Omega) \times H_{s-3/2+k_0/(k_0+1)}(\Gamma) \times H_{s-1+k_0/[2(k_0+1)]}(\Gamma_0).$$

Theorem B. *In the second case* b), *the operator $\mathfrak{A}_2(u, \rho) =\left(Lu, \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma}+G_s(\rho \times \delta_{\Gamma_0})\right)$ has a Noetherian property form $H_s(\Omega) \times H_{s+\alpha+(k_0+2)/[2(k_0+1)]}(\Gamma_0)$ to $H_{s-2+k_0/(k_0+1)}(\Omega) \times H_{s-3/2+k_0/(k_0+1)}(\Gamma)$. Where G_s is an elliptic classical pseudo differential operator on Γ of order α, $\left(s+\alpha-\dfrac{1}{k_0+1}<0\right)$. The symbol δ_{Γ_0} is the surface measure of Γ_0.*

Theorem C. *In the last case* c), *the operator $\mathfrak{A}_3(u)=\left(Lu, \left.\dfrac{\partial u}{\partial \nu}\right|_{\Gamma}\right)$ is a Noetherian operator from $H_s(\Omega)$ to $H_{s-2+k_0/(k_0+1)}(\Omega) \times H_{s-3/2+k_0/(k_0+1)}(\Gamma)$.*

Here the Noetherian property of an operator $\mathfrak{A}: \mathfrak{H}_1 \to \mathfrak{H}_2$ means that dim ker $\mathfrak{A}<\infty$ and dim $\mathfrak{H}_2/\mathrm{Im}\, \mathfrak{A}<\infty$.

§1. **Reduction to the boundary.** To reduce the problem to a pseudo differential equation on the boundary Γ, we need two propositions.

Proposition 1.1. *The following two assertions are equivalent;*

a) *The operator* $\mathfrak{A}(u,\rho)=\left(Lu, \left.\frac{\partial u}{\partial\nu}\right|_{\Gamma}+G_s(\rho\times\delta_{\Gamma_0}), u|_{\Gamma_0}\right)$ *is a Noetherian operator from* $H_s(\Omega)\times H_{s+\alpha-(k_0+2)/[2(k_0+1)]}(\Gamma_0)$ *to* $H_{s-2+k_0/(k_0+1)}(\Omega)$ $\times H_{s-3/2+k_0/(k_0+1)}(\Gamma)\times H_{s-1+k_0/[2(k_0+1)]}(\Gamma_0)$.

b) *The operator* $\mathfrak{D}(u,\rho)=\left(\left.\frac{\partial u}{\partial\nu}\right|_{\Gamma}+G_s(\rho\times\delta_{\Gamma_0}), u|_{\Gamma_0}\right)$ *has a Noetherian property from* $\mathfrak{H}_s\times H_{s+\alpha-(k_0+2)/[2(k_0+1)]}(\Gamma_0)$ *to* $H_{s-3/2+k_0/(k_0+1)}(\Gamma)$ $H_{s-1+k_0/[2(k_0+1)]}(\Gamma_0)$. *Where* $\mathfrak{H}_s=H_s(\Omega)\cap\{Lu=0 \text{ in } \Omega\}$ *and* $s>\frac{1}{k_0+1}$.

When there is no terms $G_s(\rho\times\delta_{\Gamma_0})$ and $u|_{\Gamma_0}$, the proposition is verified in [5]. In our case it is easily proved by an analogous method.

Since the problem $LU=0$ in Ω, $U|_\Gamma=g$ on Γ is coercive, there exists a regularizer R which takes function $\varphi\in H_{s-1/2}(\Gamma)$ into a element of $H_s(\Omega)$ such that

$$LR\varphi=0,\quad (R\varphi)|_\Gamma=\varphi+S_1\varphi,\quad R(U|_\Gamma)=U+S_2U \quad \text{if } LU=0 \quad \text{when } U\in H_s(\Omega).$$

Where S_1, S_2 are smoothing operators. Let T be the operator T: $\varphi\to\frac{\partial}{\partial\nu}(R\varphi)|_\Gamma$. It was known (cf. [6]) that the operator T is a classical pseudo-differential operator on Γ. Its principal symbol is Lopatinskian of the boundary value problem $Lu=f$, $\left.\frac{\partial u}{\partial\nu}\right|_\Gamma=g$. Our considered Lopatinskian $A(x,\xi)$ is defined by the following way (cf. [3] (4.7)). By the local transformation in a neighbourhood of a point P on Γ, in the half space $\{(x,t)\in R^{n+1}, t\geq 0\}$ we consider the problem $L(x,t,D)u$ $=f$ in $t>0$, $\frac{\partial u}{\partial\nu}=g$ on $t=0$. From our assumptions, $L^0(x,0,\xi,\tau)$ $=C(\tau-\tau^+(x,\xi))(\tau-\tau^-(x,\xi))$, where $\operatorname{Im}\tau^+>0$, $\operatorname{Im}\tau^-<0$, and C is a non-vanishing factor. Denote by $\mathfrak{d}=\sum_{j=1}^n a_j(x,t)\frac{\partial}{\partial x_j}+b(x,t)\frac{\partial}{\partial t}$. Then we define $A(x,\xi)=\frac{1}{i}\left(\sum_{j=1}^n a_j(x,0)\xi_j+b(x,0)\tau^-(x,\xi)\right)$. Throughout this paper, we adopt a Russian Fourier transform;

$$f(\eta)=\frac{1}{(2\pi)^{n+1}}\int e^{i\langle y,\eta\rangle}f(y)dy \qquad \text{for } f\in\mathcal{S}(R^{n+1}).$$

Proposition 1.2. *The following two properties are equivalent;*

a) *The operator* $\mathfrak{D}(U,\rho)=\left(\frac{\partial U}{\partial\nu}+G_s(\rho\times\delta_{\Gamma_0}), U|_{\Gamma_0}\right)$ *is a Noetherian* *operator from* $\mathfrak{H}_s \times H_{s+\alpha-(k_0+2)/[2(k_0+1)]}(\Gamma_0)$ *to* $H_{s-3/2+k_0/(k_0+1)}(\Gamma) \times H_{s-1+k_0/[2(k_0+1)]}(\Gamma_0)$.

b) *The operator* $\tilde{T}(u,\rho)=(Tu+G_s(\rho\times\delta_{\Gamma_0}), u|_{\Gamma_0})$ *is a Noetherian operator from* $H_{s-1/2}(\Gamma)\times H_{s+\alpha-(k_0+2)/[2(k_0+1)]}(\Gamma_0)$ *to* $H_{s-3/2+k_0/(k_0+1)}(\Gamma) \times H_{s-1+k_0/[2(k_0+1)]}(\Gamma_0)$.

Since the Dirichlet problem is coercive, the proof is trivial.

§2. Sketch of proofs of theorems. First, we shall compute Lopatinskian in a neighbourhood of a degenerate point.

Lemma 2.1. *There is a non-singular transformation from the neighbourhood of a point on* Γ_0 *to some neighbourhood of the origin of* R^{n+1} *with the following properties. If* $A(x^0,\xi^0)=0$ *for some point,* $(x^0,\xi^0)\in\Gamma\times(R^n\backslash 0)$, *then for any positive integer* N *there exists a neighbourhood* V_N of $(x^0,\xi^0)\in R^n\times(R^n\backslash 0)$ *such that*

$$A(x,\xi)=\left(\xi_n-\frac{\partial\omega}{\partial z_n}(x)\right)(\lambda_1(x,\xi')+i\lambda_2(x,\xi'))B(x,\xi) \qquad (x,\xi)\in V_N. \tag{2.1}$$

Here $\xi=(\xi',\xi_n)=(\xi_1,\cdots,\xi_{n-1},\xi_n)$, *the functions* $\lambda_i(x,\xi')$ $(i=1,2)$ *are real and homogeneous of degree* 1 *in* ξ' *and belong to* $C^N(V_N)$. *The sign of* $\lambda_2(x,\xi')$ *is negative in* V_N. *The function* $B(x,\xi)$ *is a non-zero* $C^N(V_N)$-*element and homogeneous of degree* 0 *in* ξ.

Proof. By the local transformation Ψ described in § 0, the equation of Γ is $z_0=\omega(z_1,\cdots,z_n)$. Let a map Φ be given by the equations

$$x_0=z_0-\omega(z_1,\cdots,z_n),\ x_i=z_i \qquad (i=1,\cdots,n).$$

In the space $(x_0,x)=(x_0,x_1,\cdots,x_n)$ the image of $\bar{\Omega}$ becomes $x_0\geq 0$ and the vector field $\mathfrak{v}$ becomes $\frac{1}{i}\left(\frac{i\partial}{\partial x_n}-\frac{\partial\omega}{\partial z_n}(x)\frac{i\partial}{\partial x_0}\right)$. Therefore we obtain the following equality

$$A(x,\xi)=\frac{1}{i}\xi_n-\frac{1}{i}\frac{\partial\omega}{\partial z_n}(x)\tau^-(x,\xi), \tag{2.2}$$

where $\tau^-(x,\xi)$ is the root of $L(0,x,\tau,\xi)=0$ with $\operatorname{Im}\tau^-(x,\xi)<0$. Since $A(x,\xi)=0$ if and only if $x_n=\xi_n=0$ and $\frac{\partial A}{\partial\xi_n}(x',0,\xi',0)\neq 0$, by Lemma 2.1 of [2] we obtain the following division in some neighbourhood V_N of $(x',0,\xi',0)$ where $x'=(x_1,\cdots,x_{n-1})$, $\xi'=(\xi_1,\cdots,\xi_{n-1})$;

$$A(x,\xi)=(\xi_n-\tilde{\lambda}_1(x,\xi')-i\tilde{\lambda}_2(x,\xi'))B(x,\xi). \tag{2.3}$$

Here $\tilde{\lambda}_i(x,\xi)$ $(i=1,2)$ are real and positively homogeneous of degree 1 in ξ'; $B(x,\xi)$ is a non-zero $C^N(V_N)$-element and positively homogeneous of degree 0 in ξ. When $\xi_n=0$, we have

$$(\tilde{\lambda}_1(x,\xi')+i\tilde{\lambda}_2(x,\xi'))B(x,\xi',0)=\frac{1}{i}\frac{\partial\omega}{\partial z_n}(x)\tau^-(x,\xi',0).$$

Therefore if we put $\lambda_1(x,\xi')+i\lambda_2(x,\xi')=-i\tau^-(x,\xi',0)B^{-1}(x,\xi',0)$, we obtain (2.1). In the equality (2.3) if we put $x_n=\xi_n=0$ then $\tilde{\lambda}_1(x',0,\xi')=\tilde{\lambda}_2(x',0,\xi')=0$ for $B(x',0,\xi',0)\neq 0$. Substitute $x_n=0$ into (2.2) and

(2.3), we show $B(x', 0, \xi) = -i$. Thus for small number ε, $|\mathrm{Re}\, B(x, x_n, \xi)| < \varepsilon$, $|\mathrm{Im}\, B(x', x_n, \xi) + 1| < \varepsilon$ in a small neighbourhood of $(x', 0, \xi', 0)$. By the definition, $\lambda_2(x, \xi') = -(\mathrm{Re}\, \tau^-(x, \xi', 0) \mathrm{Re}\, B^{-1}(x, \xi', 0) - \mathrm{Im}\, \tau^-(x, \xi', 0)$ $\mathrm{Im}\, B^{-1}(x, \xi', 0))$. Since $\mathrm{Im}\, \upsilon^-(x, \xi', 0) < 0$, $\lambda_2(x, \xi') < 0$ in the neighbourhood of $(x', 0, \xi', 0)$. The proof is complete.

We now return to the proofs of theorems. By Propositions 1.1 and 1.2, we shall verify the Noetherian property of the operator $\tilde{T}$. Let a symbol $G_s^0(x, \xi)$ be the principal symbol of $G_s(x, D)$. We shall consider the Noetherian property of an operator $\tilde{T}_0(u, \rho) = (A(x, D)u + G_s^0(x, D)(\rho \times \delta_{\Gamma_0}), u|_{\Gamma_0})$ instead of $\tilde{T}(u, \rho)$. If we can construct the right regularizer and the left regularizer of the operator $\tilde{T}_0$, they are also the right and left regularizer of the operator $\tilde{T}$. For the lower order terms does not disturb to make up a regularizer of $\tilde{T}$. By this reduction, we can adapt Èskin's discussion [2] to our case. Since $\lambda_2(x, \xi') < 0$, we see that our cases a), b) correspond to Eskin's singular cases while the case c) to his non-singular case. From § 3 in [2], for the singular cases we require boundary operators on Γ_0 or potential operators to have a Noetherian property. Since the null set of Lopatinskian $A(x, \xi)$ is a connected $(2n-2)$-dimensional submanifold of the cotangent bundle on Γ, in the case a) we consider the operator adjoining to $A(x, D)$ one boundary operator on Γ_0; in the case b) we consider the operator adding a potential operator to $A(x, D)$. We determine auxiliary operators in the following way. $A(x', 0, \xi', \lambda(x', 0, \xi')) \equiv 0$, where $\lambda(x, \xi') = \lambda_1(x, \xi') + i\lambda_2(x, \xi'))$. Thus, in particular the function $e(x', \xi') \equiv 1$ satisfies $A(x', 0, \xi', \lambda(x', 0, \xi')) e(x', \xi') \equiv 0$. Take the identity operator as the boundary operator on Γ, and as the potential operator an elliptic pseudo-differential operator on Γ. Then the conditions (3.20), (3.21) in [2] are satisfied. Summing up, we consider the following operators; in case a) $\tilde{T}_1(u) = (Au, u|_{\Gamma_0})$, in case b) $\tilde{T}_2(u, \rho) = (Au + G_s^0(\rho \times \delta_{\Gamma_0}))$, where $G_s^0(x, \xi)$ is a non-zero homogenous function of degree α in ξ, $\left(s + \alpha - \dfrac{1}{k_0 + 1} < 0\right)$ and in the last case c) $\tilde{T}_3(u) = Au$. As in § 6 of [2], these operator has a right and left regularizer. Thus we verified the theorems.

Remark 2.2. Since Eskin's theorem remains valid even for $k_0 = \infty$, our theorems are also valid for such a case. In this case, our results coincide with the results of [1].

Remark 2.3. If the function $\omega(z)$ is of class C^∞, the positive number k_0 must be a integer. However in our method it is not necessary that the boundary Γ is of class C^∞. For example let $\omega(z) = z_n^{N+1+m(z)} \omega'(z)$ in $(z_0, \cdots, z_n)$-space, where N is a sufficiently large positive integer depended on s, $m(z)$ is a non-negative C^∞-function and $\omega'(z) \neq 0$ in some neighbourhood of $z = 0$.

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