146. *Average Powers of Gaussian White Noise*

By Yoshihei HASEGAWA
Mathematics Department, Nagoya University

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§1. Introduction and Theorem.

The purpose of this note is to prove a limit theorem of the average powers of the gaussian white noise, which is a generalization of the Brownian oscillation due to P. Lévy [3; §41].

P. Lévy developed extensively Gâteaux's constructive study of the gaussian white noise in his book [2]. In his original idea the choice of the complete orthonormal system $\{\xi_n\}$ in the real Hilbert space $L^2[0,1]$ plays important roles, however there seems to be no other result depending on the choice of the system $\{\xi_n\}$. Our theorem stated below does depend on the choice of the system $\{\xi_n\}$.

Now, we shall introduce the measure of gaussian white noise. Let E be a nuclear subspace of the real Hilbert space $L^2[0,1]$ which is dense in the space $L^2[0,1]$ and satisfies the relation

$$E\subset L^2[0,1]\subset E^*,$$

where E^* stands for the dual space of E. For the characteristic functional

$$C(\xi)=\exp\left(-\frac{1}{2}\|\xi\|^2\right),$$

$\|\xi\|$ being the $L^2[0,1]$-norm of $\xi\in E$, there corresponds a probability measure μ on E^* such that,

$$C(\xi)=\int_{E^*}e^{i(x,\xi)}\mu(dx),$$

where (x,ξ), $x\in E^*$, $\xi\in E$ is the canonical bilinear form which links the spaces E and E^* We call μ the measure of gaussian white noise (see T. Hida [1]).

Next we define *the average powers* $\{\rho_N(x)\,;\,x\in E^*\}_{N=1}^{\infty}$ for a complete orthonormal system $\{\xi_n\}_{n=1}^{\infty}$ in $L^2[0,1]$ as follows;

$$\rho_N(x)=\frac{1}{N}\sum_{n=1}^{N}(x,\,\xi\cdot\xi_n{}^2),$$

where ξ is a bounded measurable function on the interval $[0,1]$.

The system $\{\xi_n\}_{n=1}^{\infty}$ is called *weakly equally dense*, if it satisfies

$$\lim_{N\to\infty}\int_0^1\eta(u)\left(\frac{1}{N}\sum_{n=1}^{N}\xi_n^2(u)-1\right)du=0,$$

for any bounded measurable function η on $[0,1]$.

Theorem. *Let* $\{\xi_n\}_{n=1}^{\infty}$ *be a complete orthonormal system. Then*

$$\mu\left(x \in E^* ; \lim_{N\to\infty} \rho_N(x) = \int_0^1 \xi^2(u)du\right) = 1$$

holds for any bounded measurable function ξ, *if and only if the system* $\{\xi_n\}_{n=1}^{\infty}$ *is weakly equally dense.*

§ 2. Proof of the theorem.

Noting that a random variable (x, η) is subject to the normal distribution $N(0, \|\eta\|^2)$ by the definition of the measure μ, we can prove the following three formulas (1), (2) and (3).

$$(1) \qquad E[\rho_N] = \int_0^1 \xi^2(u) \frac{1}{N} \sum_{n=1}^{N} \xi_n^2(u)du.$$

$$(2) \qquad E[|\rho_N(x) - E[\rho_N]|^4] = 12\left[\int_0^1\int_0^1 \xi^2(u)\xi^2(v)\Phi_N^2(u,v)dudv\right]^2$$
$$+48\int_0^1\int_0^1\int_0^1\int_0^1 \xi^2(u_1)\xi^2(u_2)\xi^2(u_3)\xi^2(u_4)\Phi_N(u_1,u_2)\Phi_N(u_1,u_3)$$
$$\times \Phi_N(u_2,u_4)\Phi_N(u_3,u_4)du_1du_2du_3du_4,$$

where

$$\Phi_N(u,v) = \frac{1}{N}\sum_{n=1}^{N} \xi_n(u)\xi_n(v).$$

$$(3) \qquad \int_0^1\int_0^1 \Phi_N^2(u,v)dudv = \frac{1}{N}, \qquad N = 1, 2, 3, \cdots.$$

Although the proof of this formula is immediate, it is essential in our theory.

Now we shall prove our theorem. First we estimate each term in the right side of (2). Since ξ is bounded,

$$\text{, say} \qquad |\xi(u)| \leqq M,$$

the formula (3) implies

$$12\left[\int_0^1\int_0^1 \xi^2(u)\xi^2(v)\Phi_N^2(u,v)dudv\right]^2 \leqq \frac{12M^8}{N^2},$$

and

$$\left|48\int_0^1\int_0^1\int_0^1\int_0^1 \xi^2(u_1)\xi^2(u_2)\xi^2(u_3)\xi^2(u_4)\Phi_N(u_1,u_2)\Phi_N(u_1,u_3)\right.$$
$$\left.\times \Phi_N(u_2,u_4)\Phi_N(u_3,u_4)du_1du_2du_3du_4\right|$$
$$\leqq 48M^8\left(\int_0^1\int_0^1\int_0^1\int_0^1 \Phi_N^2(u_1,u_2)\Phi_N^2(u_3,u_4)du_1du_2du_3du_4\right)^{1/2}$$
$$\left(\int_0^1\int_0^1\int_0^1\int_0^1 \Phi_N^2(u_1,u_3)\Phi_N^2(u_2,u_4)du_1du_2du_3du_4\right)^{1/2}$$
$$\leqq \frac{48M^8}{N^2}.$$

We therefore have the following basic estimates:

$$E[|\rho_N(x) - E[\rho_N]|^4] \leqq \frac{60M^8}{N^2}, \qquad N = 1, 2, 3, \cdots.$$

Hence by the Tychebycheff's inequality we have for any positive number ε,

$$\mu(x \in E^* ; |\rho_N(x) - E[\rho_N]| \geq \varepsilon) \leqq \frac{60M^8}{\varepsilon^4 N^2}.$$

Since

$$\sum_{N=1}^{\infty} \frac{60M^8}{\varepsilon^4 N^2} < +\infty,$$

the Borel-Cantelli's lemma can be applied to show that

$$\lim_{N\to\infty} (\rho_N(x) - E[\rho_N]) = 0, \qquad \text{for a.e. } x \in E^*.$$

If, in particular, the system $\{\xi_n\}$ is weakly equally dense, we have

$$\lim_{N\to\infty} E[\rho_N] = \lim_{N\to\infty} \int_0^1 \xi^2(u) \frac{1}{N} \sum_{n=1}^{N} \xi_n^2(u) du = \int_0^1 \xi^2(u) du, \tag{4}$$

which implies

$$\lim_{N\to\infty} \rho_N(x) = \int_0^1 \xi^2(u) du \qquad \text{for a.e. } x \in E^*.$$

The converse statement is obvious from the equality (4). This completes the proof of our theorem.

References

[1] T. Hida: Stationary Stochastic Processes. Princeton Lecture Note (1968).
[2] P. Lévy: Problèmes concrets d'analyse fonctionnelle. Gauthier-Villars, Paris (1951).
[3] ——: Processus stochastiques et mouvement brownien. Gauthier-Villars, Paris (1965).