158. A Note on Character Sums

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§ 1. Introduction.

This is a continuation of the previous work (cf. [2]). We are concerned with the estimate of $\sum_{n\le X}\chi(n)$, where χ is a primitive character mod q. In [2] we showed

$$|\sum_{n\le X}\chi(n)|\ll_q\sqrt{X}\,q^{1/6},$$

where $\ll_q$ depends on prime factors of q. Here we will improve the dependence of $\ll_q$ on the prime factors of q. Hereafter implicit constants in $\ll$ are absolute. We will prove

Theorem. *Let χ be a primitive character* mod q. *Then for $X\le q^{2/3}$.*

$$|\sum_{n\le X}\chi(n)|\ll\sqrt{X}\,q^{1/6}B(q)$$

with

$$B(q)=\mathop{\mathrm{Min}}_{q=q_1q_2}\left\{q_1^{1/3}(\log q_1)^{\delta_2}\left[\left(\frac{1}{3}\log(q_2q_1^2)\right)^{R_2/2}A_2^{1/3}q_1^{1/3}\Big/\left(\prod_{p_i|q_2}\log p_i\right)\right.\right.$$
$$\left.\left.+(\log q_2)^{1/2}A_2^{1/6}\left(\prod_{p_i|q_2}p_i\Big/(p_i-1)\right)^{1/2}\right]^{1-\delta_2}\right\}$$

where

1) $\mathop{\mathrm{Min}}_{q=q_1q_2}$ *is taken over all decomposition of q into q_1q_2 such that if $q_2=\prod_{i=1}^{R_2}p_i^{r_i}$, then $p_i^{r_i}\|q$ and $r_i>r_0$, where r_0 is* 32, *say.*

2) $A_2=\prod_{p_i|q_2}p_i^{k_i}$, *where $K_i=0,2,1$ according as $r_i\equiv0,1,2$* (mod 3) *and $p_i^{r_i}\|q_2$.*

3) $\delta_2=\begin{cases}1 & \text{if } q_2=1\\ 0 & \text{if } q_2\ne1.\end{cases}$

§ 2. Proof of Theorem.

Let $q=q_1q_2$ and $q_2=\prod_{i=1}^{R_2}p_i^{r_i}$ with $p_i^{r_i}\|q$ and $r_i>r_0$. Let s_i be the least natural number larger than or equal to $r_i/3$. Write $d=\prod_{i=1}^{R_2}p_i^{s_i}$ and $k_i=0,2,1$ according as $r_i\equiv0,1,2$ (mod 3). We have by definition $r_i+k_i=3s_i$. Let $A_2=\prod_{i=1}^{R_2}p_i^{k_i}$. If $X\le d$, the theorem comes from a trivial estimate. (For $|\sum_{n\le X}\chi(n)|\le\sqrt{X}=\sqrt{X}\sqrt{X}\le\sqrt{X}d^{1/2}=\sqrt{X}\,q_2^{1/6}A_2^{1/6}$.) Hence we assume $d\le X\le q^{2/3}$. We see that

$$\left|\sum_{n\le X}\chi(n)\right|\le\sum_{\mu=0}^{\mu_0}\left|\sum_{N\le n\le N'}\chi(n)\right|,$$

where $N=2^{\mu}d$, $N'\le2N$, $\mu=0,1,\cdots,\mu_0$ and $2^{\mu_0}d\le X\le2^{\mu_0+1}d$. Hence the problem is reduced to the estimate of the sums of the type $\sum_{N\le n\le N'}\chi(n)$ under $d\le N\le q^{2/3}$, $N'\le2N$.

Now

$$(1)\qquad \Big|\sum_{N\le n\le N'}\chi(n)\Big|\le \sum_{a=1}^{d}{}'\Big|\sum_{n=N_1}^{N_2}\chi(a+ud)\Big|\le d^{1/2}\Big(\sum_{a=1}^{d}{}'\Big|\sum_{u=N_1}^{N_2}\chi(a+ud)\Big|^2\Big)^{1/2}$$
$$\le d^{1/2}\Big(\sum_{a=1}^{d}{}'\Big|\sum_{n=N_1}^{N_2}\chi_1(a+ud)e(\alpha' a^{*2}u^2+\beta' ua^*)\Big|^2\Big)^{1/2},$$

where the dash indicates that we sum only over a's relatively prime to d, $N_1=d^{-1}(N-a)$, $N_2=d^{-1}(N'-a)$, $\chi=\chi_1\chi_2$ with primitive characters $\chi_i \bmod q_i$ for $i=1$ and 2, a^* is determined by $aa^*\equiv 1 \pmod{q_2}$, $\alpha'=A_2\alpha''/d$ with $(\alpha'',d)=1$ and $e(Y)=\exp(2\pi iY)$. The last inequality comes from the following lemma. (For convenience we will add the proof which we can see in [2].)

Lemma. *Let χ be a primitive character* mod q. *Let $q=\prod_{i=1}^{R}p_i^{r_i}$ and $d=\prod_{i=1}^{R}p_i^{s_i}$, where s_i is the least natural number larger than or equal to $r_i/3$ and let $k_i=3s_i-r_i$ for $i=1,2,\cdots,R$. Let $A=\prod_{i=1}^{R}p_i^{k_i}$.*

Then for any u,

$$\chi(1+ud)=e(\alpha' u^2+\beta' u),$$

where $\alpha'=A\alpha''/d$ with $(\alpha'',d)=1$.

Proof. Since χ is decomposed uniquely into primitive characters $\chi_i \bmod p_i^{r_i}$

$$\chi(1+ud)=\prod_{i=1}^{R}\chi_i(1+ud)=\prod_{i=1}^{R}\chi_i\Big(1+p_i^{s_i}\Big(\prod_{j\ne i}p_j^{s_j}u\Big)\Big)$$
$$=\prod_{i=1}^{R}e\Big(\alpha_i\Big(u\prod_{j\ne i}p_j^{s_j}\Big)^2+\beta_i\Big(u\prod_{j\ne i}p_j^{s_j}\Big)\Big),$$

where

$$\alpha_i=\frac{a_i}{p_i^{r_i-2s_i}}=\frac{a_i}{p_i^{s_i-k_i}}$$

$$\beta_i=-\frac{2a_i}{p_i^{r_i-s_i}}\quad\text{with}\quad a_i=a_{\chi_i}\not\equiv 0 \pmod{p_i^{r_i}}$$

by Postnikov-Gallagher expression for prime power modulus case (cf. [2]). Hence

$$\chi(1+ud)=e\Big(\sum_{i=1}^{R}a_i\Big(u\prod_{u\ne i}p_j^{s_j}\Big)^2+\sum_{i=1}^{R}\beta_i\Big(u\prod_{j\ne i}p_j^{s_j}\Big)\Big).$$

Take

$$\alpha'=\sum_{i=1}^{R}\alpha_i\Big(\prod_{j\ne i}p_j^{s_j}\Big)^2=\sum_{i=1}^{R}-\frac{a_i}{p_i^{s_i-k_i}}\prod_{j\ne i}p_j^{2s_j}=\sum_{i=1}^{R}\frac{a_i}{p_i^{3s_i-k_i}}\Big(\prod_{j=1}^{R}p_j^{s_j}\Big)^2$$
$$=\frac{A}{d}\Big(\sum_{i=1}^{R}a_i\prod_{j\ne i}p_j^{r_j}\Big)=\frac{A}{d}\alpha''\quad\text{and}\quad(\alpha'',d)=1.$$

$$\beta'=\sum_{i=1}^{R}\beta_i\prod_{j\ne i}p_j^{s_j}.$$

Q.E.D. of Lemma.

Since $\tau(\chi_1)\chi_1(a+ud)=\sum_{c=1}^{q_1}\overline{\chi_1(c)}e(c(a+ud)/_{q_1})$, where $\tau(X)$ is the Gaussian sum of χ_1, the last inequality (1) becomes

$$\leq d^{1/2}\Big(\sum_{a=1}^{d}{}'\Big|\sum_{u=N_1}^{N_2}\sum_{c=1}^{q_1}\frac{\chi_1(c)}{\tau(\chi_1)}e(c(a+ud)/q_1)e(\alpha' a^{*2}u^2+\beta' ua^*)\Big|^2\Big)^{1/2}$$

$$(2)\qquad \leq d^{1/2}\Big(\sum_{a=1}^{d}{}'\frac{q_1}{|\tau(\chi_1)|^2}\sum_{c=1}^{q_1}\Big|\sum_{u=N_1}^{N_2}e\Big(\alpha' a^{*2}u^2+u\Big(\beta' a^*+\frac{cd}{q_1}\Big)+\frac{ca}{q_1}\Big)\Big|^2\Big)^{1/2}$$

$$\leq d^{1/2}\Big(\sum_{c=1}^{q_1}\sum_{a=1}^{d}{}'\Big|\sum_{u=N_1}^{N_2}e\Big(\alpha' a^{*2}u^2+u\Big(\beta' a^*+\frac{cd}{q_1}\Big)+\frac{ca}{q_1}\Big)\Big|^2\Big)^{1/2}.$$

So in the extremal case, i.e., if $q_2=1$, the conclusion comes from Weyl sum estimate as in the proof of Polya-Vinogradov's theorem. Hence hereafter we assume $q_2\neq 1$. By van der Copput's method (cf. [3]), (2) becomes

$$(3)\qquad \leq d^{1/2}q_1^{1/2}(\log q_1)^{\delta_2}S^{1/2},$$

where $S=\sum_{a=1}^{\prime d}\sum_{|u|\leq N_2-N_1}\mathrm{Min}\,(N_2-N_1, 1/\|\alpha' a^{*2}u\|)$ and $\|\lambda\|=\mathrm{Min}\,(\lambda-[\lambda], 1-\lambda+[\lambda])$. Now let us express u as $u=u'\sum_{i=1}^{R_2}p_i^{\tau_i}$ with $(u',d)=1$ and

$$|u'|\leq Nd^{-1}\prod_{i=1}^{R_2}p_i^{-\tau_i}.$$

Then

$$S\leq\sum_{i=1}^{R_2}\sum_{\tau_i=0}^{\bar\tau_i}\sum_{u'}{}''\sum_{a=1}^{d}{}'\mathrm{Min}\,(\mathrm{Nd}^{-1}, 1/\eta(u')),$$

where $\bar\tau_i\leq(\log \mathrm{Nd}^{-1})/\log p_i$ for each $\mathrm{i}=1,2,\cdots,R_2$, $\eta(u')=\|\alpha'' a^{*2}u'/D\|$ with $D=dA_2^{-1}\prod_{i=1}^{R_2}p_i^{-\tau_i}$ and the double dash indicates that we sum over all u' satisfying $|u'|\leq \mathrm{Nd}^{-1}\prod_{i=1}^{R_2}p_i^{-\tau_i}$.

Now the variable a runs over all residue classes mod D with the multiplicity $\prod_{i=1}^{R_2}p_i^{\tau_i}A_2$. a^{*2} runs over all residue classes mod D with the multiplicity less than $2\sum_{i=1}^{R_2}p_i^{\tau_i}A_2$. And $\alpha'' a^{*2}u'$ runs over all residue classes mod D with the same multiplicity as a^{*2}. Hence

$$S\leq\sum_{i=1}^{R_2}\sum_{\tau_i=0}^{\bar\tau_i}\sum_{u'}{}''2A_2\prod_{i=1}^{R_2}p_i^{\tau_i}\sum_{a=1}^{D}\mathrm{Min}\,(\mathrm{Nd}^{-1}, 1/\|a/D\|)$$

$$\leq\sum_{i=1}^{R_2}\sum_{\tau_i=0}^{\bar\tau_i}\mathrm{Nd}^{-1}\prod_{i=1}^{R_2}p_i^{-\tau_i}(\mathrm{ND}^{-1}+d\,\mathrm{Log}\,D)$$

$$\leq N^2d^{-2}A_2\prod_{i=1}^{R_2}\bar\tau_i+\mathrm{Nd}^{-1}d\log q_2\cdot\sum_{i=1}^{R_2}\sum_{\tau_i=0}^{\bar\tau_i}\prod_{i=1}^{R_2}p_i^{-\tau_i}$$

Since

$$N\leq q^{2/3}, N^2d^{-2}\leq N\sum_{i=1}^{R_2}p_i^{-2/3k_i}q_1^{2/3}=NA_2^{-2/3}q_1^{2/3}.$$

Hence

$$S\leq NA_2^{1/3}q_1^{2/3}\Big(\frac{1}{3}\log(q_2q_1^2)\Big)^{R_2}\Big/\Big(\prod_{p_i|q_2}\log p_i\Big)+N\log q_2\cdot\prod_{p_i|q_2}p_i/(p_i-1).$$

Hence (3) becomes

$$\leq N^{1/2}d^{1/2}q_1^{1/2}(\log q_1)^{\delta_2}\Big\{A_2^{1/6}q_1^{1/3}\Big(\frac{1}{3}\log(q_2q_1^2)\Big)^{R_2/2}\Big/\Big(\prod_{p_i|q_2}\log p_i\Big)^{1/2}$$
$$+(\log q_2)^{1/2}\Big(\prod_{p_i|q_2}p_i/(p_i-1)\Big)^{1/2}\Big\}$$

$$\leq N^{1/2}q_2^{1/6}A_2^{1/6}q_1^{1/2}(\log q_1)^{\delta_2}\Big\{A_2^{1/6}q_1^{1/3}\Big(\frac{1}{3}\log(q_2q_1^2)\Big)^{R_2/2}\Big/\Big(\prod_{p_i|q_2}\log p_i\Big)^{1/2}$$
$$+(\log q_2)^{1/2}\Big(\prod_{p_i|q_2}p_i/(p_i-1)\Big)^{1/2}\Big\}$$
$$\leq N^{1/2}q^{1/6}A_2^{1/6}q_1^{1/3}(\log q_1)^{\delta_2}\Big\{A_2^{1/6}q_1^{1/3}\Big(\frac{1}{3}\log(q_2q_1^2)\Big)^{R_2/2}\Big/\Big(\prod_{p_i|q_2}\log p_i\Big)^{1/2}$$
$$+(\log q_2)^{1/2}\Big(\prod_{p_i|q_2}p_i/(p_i-1)\Big)^{1/2}\Big\}.$$

Hence

$$\Big|\sum_{n\leq X}\chi(n)\Big|\leq\sqrt{X}\,q^{1/6}q_1^{1/3}(\log q_1)^{\delta_2}\Big\{A_2^{1/3}q_1^{1/3}\Big(\frac{1}{3}\log(q_2q_1^2)\Big)^{R_2/2}\Big/\Big(\prod_{p_i|q_2}\log p_i\Big)^{1/2}$$
$$+A_2^{1/6}(\log q_2)^{1/2}\Big(\prod_{p_i|q_2}p_i/(p_i-1)\Big)^{1/2}\Big\}.$$

Since this is true for any decomposition of q into $q=q_1q_2$, we get our conclusion.

Q.E.D.

References

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