81. *On Deformations of Quintic Surfaces*

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Let S_0 be a non-singular hypersurface of degree 5 in the projective 3-space $\boldsymbol{P}^3$ defined over $\boldsymbol{C}$. For brevity, we call S_0 a non-singular quintic surface.

By a surface, we shall mean a compact complex manifold of complex dimension 2, unless explicit indications are given. We say that a surface S is a deformation of S_0 if there exists a finite sequence of surfaces $S_0, S_1, \cdots, S_k, \cdots, S_n = S$ such that, for each k, S_k and S_{k-1} belong to one and the same complex analytic family of surfaces.

If S is a deformation of a non-singular quintic surface, S has the following numerical characters:

(*) $$p_g=4, \quad q=0, \quad c_1^2=5,$$

where p_g, q and c_1^2 denote the geometric genus, the irregularity and the Chern number of S, respectively. In particular S is an algebraic surface (see [5], Theorem 9). Moreover, since S_0 is minimal, Theorem 23 of Kodaira [5] asserts that

(**) S is minimal.

In this note, we shall give a statement of the results on the structures and deformations of surfaces which satisfy the conditions (*) and (**). Details will be published elsewhere.

1. Structures.

Theorem 1. *Let S be a minimal algebraic surface with $p_g=4$, $q=0$, and $c_1^2=5$. Then the canonical system $|K|$ on S has at most one base point. There are two cases:*

Case I. *$|K|$ has no base point. In this case, there exists a birational holomorphic map $f: S \to S'$ of S onto a (possibly singular) quintic surface S' in $\boldsymbol{P}^3$. S' has at most rational double points as its singularities.*

Case II. *$|K|$ has one base point b. Let $\pi: \tilde{S} \to S$ be the quadric transformation with center at b. Then this case is divided as follows:*

Case IIa. *There exists a surjective holomorphic map $f: \tilde{S} \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ of degree 2.*

Case IIb. *There exists a surjective holomorphic map $f: \tilde{S} \to \sum_2$ of degree 2, where $\sum_2$ denotes the Hirzebruch surface of degree 2, i.e., $\sum_2$ is a rational ruled surface with a section Δ_0 with $(\Delta_0)^2 = -2$.*

The proof is based on a detailed analysis of the rational map Φ_K:

$S \to \boldsymbol{P}^3$ defined by the canonical system $|K|$. The holomorphic maps f in the above statement are derived from Φ_K.

Corollary. *If $|K|$ has a base point, then there exists a surjective holomorphic map $g: S \to \boldsymbol{P}^1$ whose general fibre is an irreducible non-singular curve of genus 2. In particular, the rational map Φ_{2K} defined by the bicanonical system $|2K|$ is not birational.*

Conversely, we can construct every surface of type II as follows: First we construct a double covering S' of $W = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ or $\sum_2$ with appropriate branch locus on W. S' is a normal surface. Let $\tilde{S}$ be the minimal resolution of singularities of S' (see [1], p. 81). We construct S' so that $\tilde{S}$ contains one exceptional curve E. Contracting E to a point, we obtain a minimal algebraic surface with $p_g = 4$, $q = 0$, and $c_1^2 = 5$.

2. Deformations. First, we give some results on small deformations of a surface in consideration.

Proposition 1. i) *The classes of surfaces of type* I *and of type* IIa *are, respectively, closed under small deformations.*

ii) *A surface of type* I *and a surface of type* IIa *do not belong to one and the same family (with non-singular base space).*

Theorem 2. *Let S be a surface of type* IIb *of which the canonical bundle is ample. Let $p: \mathcal{S} \to M$ be the Kuranishi family of deformations of $S = p^{-1}(0)$ with $0 \in M$* (see [7]). *Then*

i) *$M = M_0 \cup M_1$ where each M_i $(i = 0, 1)$ is a 40-dimensional manifold,*

ii) *$N = M_0 \cap M_1$ is a 39-dimensional manifold,*

iii) *$S_t = p^{-1}(t)$ is a non-singular quintic surface, a surface of type* IIa, *or a surface of type* IIb *according as $t \in M_0 - N$, $t \in M_1 - N$, or $t \in N$.*

We now indicate an outline of the proof of Theorem 2. Let S denote a surface as in Theorem 2 and let Θ_S denote the sheaf of germs of holomorphic vector fields on S. We have $\dim H^1(S, \Theta_S) = 41$ and $\dim H^2(S, \Theta_S) = 1$. Let $D = \{t \in \boldsymbol{C}^{41} : |t| < \varepsilon\}$ with $\varepsilon > 0$ sufficiently small. Then there exists a $(0,1)$-form $\varphi(t)$ with coefficients in Θ_S depending holomorphically on $t \in D$ such that

$$M = \{t \in D : \boldsymbol{H}[\varphi(t), \varphi(t)] = 0\},$$

where $\boldsymbol{H}$ denotes the projection onto the space of harmonic forms with respect to a Hermitian metric on S and $[\ ,\ ]$ denotes the Poisson bracket. Since $\dim H^2(S, \Theta_S) = 1$, we may regard $\boldsymbol{H}[\varphi(t), \varphi(t)]$ as a holomorphic function on D. We can prove that

$$\boldsymbol{H}[\varphi(t), \varphi(t)] = t_1 t_2 + (\text{higher terms}),$$

for an appropriate choice of coordinates $(t_1, t_2, \cdots, t_{41})$ on D.

On the other hand, applying an improved form of Theorem 2′ of [3] to the holomorphic map $g: S \to \boldsymbol{P}^1$ in Corollary to Theorem 1, we can construct a 40-dimensional effectively parametrized family $p_1: \mathcal{S}_1 \rightarrow M_1$ of deformations of $S = p_1^{-1}(0)$ with $0 \in M$ (see [6], Definition 6.4).

It follows that $\boldsymbol{H}[\varphi(t), \varphi(t)]$ decomposes into a product $q(t)r(t)$ with $q(t) = t_2 +$(higher terms), $r(t) = t_1 +$(higher terms). This proves the assertion i).

Other assertions can be proved by applying the general theory on deformations of holomorphic maps [4].

It seems difficult to study the deformations of a surface of which the canonical bundle is not ample. However, applying a result of Brieskorn ([1], [2]), we can prove

Theorem 3. *Every minimal algebraic surface with $p_g = 4$, $q = 0$, and $c_1^2 = 5$, is a deformation of a non-singular quintic surface.*

References

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